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$(n+1, m+1)$-HYPERGEOMETRIC FUNCTIONS ASSOCIATED TO CHARACTER ALGEBRAS

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Abstract. In this paper, we obtain certain discrete orthogonal polynomials expressed in terms of the $(d+1, 2(d+1))$-hypergeometric functions, from the eigenmatrices of character algebras.

Introduction

In 1942, Kawada [8] introduced the notion of character algebras, motivated by the Tannaka duality for noncompact topological groups (see §1 below for the definition of character algebras). Later, Bannai [5] showed that the concept of character algebras is closely related to that of fusion algebras (at the algebraic level), which appears in conformal field theory in mathematical physics. This concept also provides an algebraic framework for the theory of association schemes (in the sense that the Bose-Mesner algebras of commutative association schemes turn out to be character algebras), while the concept of association schemes can in turn be considered as a purely combinatorial interpretation of that of transitive permutation groups. The purpose of the present paper is to obtain "many" discrete orthogonal polynomials which are expressed in terms of certain hypergeometric functions defined below, called the $(n+1, m+1)$-hypergeometric functions [2, 13], from the eigenmatrices of character algebras. Our main result is given in Theorem 1.1.

Throughout this paper, we denote the set of the nonnegative integers by $\mathbb{N}_0$:

$$\mathbb{N}_0 = \{0, 1, 2, \dots\}.$$

Also let

$$M_{m,n}(\mathbb{N}_0) = \{A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}} \mid a_{ij} \in \mathbb{N}_0\}.$$

When $m = n$, we simply write M_n instead of $M_{n,n}$. Let

$$(x)_r = \begin{cases} x(x+1)(x+2)\dots(x+r-1) & \text{if} \quad r = 1, 2, \dots, \\ 1 & \text{if} \quad r = 0. \end{cases}$$

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If N is a positive integer, then we define the $(n+1, m+1)$-*hypergeometric function* $F(\alpha, \beta; -N; X)$ by

$$F(\alpha, \beta; -N; X) = \sum_{\substack{A \in M_{n,m-n-1}(\mathbb{N}_0) \\ \sum_{i,j} a_{ij} \leq N}} \frac{\prod_{i=1}^{n} (\alpha_i)_{\sum_j a_{ij}} \prod_{j=1}^{m-n-1} (\beta_j)_{\sum_i a_{ij}}}{(-N)_{\sum_{i,j} a_{ij}}} \frac{\prod_{i,j} x_{ij}^{a_{ij}}}{\prod_{i,j} a_{ij}!},$$

for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ and $\beta = (\beta_1, \ldots, \beta_{m-n-1}) \in \mathbb{C}^{m-n-1}$, where X stands for the variables x_{ij} $(1 \leq i \leq n, 1 \leq j \leq m-n-1)$. This definition is originally due to K. Aomoto and I. M. Gelfand.

In [10], the first author showed that the Gelfand pair $(G(r,1,n), \mathfrak{S}_n)$ gives rise to orthogonal polynomials expressed by $(r, 2r)$-hypergeometric functions, where $\mathfrak{S}_n$ and $G(r,1,n) \cong (\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$ denote the symmetric group of degree n and the imprimitive complex reflection group, respectively. Also, Akazawa and Mizukawa [1] obtained a similar result for the pair $(D(r,n), D(1,n))$, where $D(r,n) = D_r \wr \mathfrak{S}_n$ is the wreath product of the dihedral group of order $2r$ and $\mathfrak{S}_n$. Since the Hecke algebra of any Gelfand pair is a character algebra, our result includes all the cases dealt with in [10, 1] (see Remark 2.1).

The class of P- and Q-polynomial association schemes plays a central role in the theory of commutative association schemes (see [6, 3]). It is often considered as a combinatorial analogue of compact symmetric spaces of rank one. As is well known, the zonal spherical functions of compact symmetric spaces of rank l are described by generalized Jacobi polynomials in l variables (see e.g. [12]). On the other hand, a famous theorem of Leonard [9] (see also [6, §3.5]) states that the P- and Q-polynomial association schemes, or more precisely the character algebras of P- and Q-polynomial type, characterize the Askey-Wilson orthogonal polynomials in one variable (including certain limiting cases). For instance, Krawtchouk polynomials are obtained as entries of the eigenmatrices of Hamming schemes $H(n,q)$, one of the most important families of P- and Q-polynomial association schemes.

In the proof of our main theorem, we consider symmetric tensor spaces of a character algebra. If the character algebra is of class one, then they essentially coincide with the Bose-Mesner algebras of Hamming schemes (see also Remark 2.1). We prove Theorem 1.1 by explicitly expressing their eigenmatrices. The idea is quite simple and natural, but it seems that this is in fact the first example of a systematic construction of discrete orthogonal polynomials in several variables having hypergeometric series expressions from character algebras. Although the notion of association schemes (or character algebras) of "higher ranks" has not yet been established, our result gives a nice generalization of the relation between Hamming schemes and Krawtchouk polynomials. In this sense, it seems possible to say that this paper represents a kind of an attempt to extend Leonard's theorem mentioned above.

1. Main result

A *character algebra of class d* is an associative and commutative algebra $\mathfrak{A}$ over the complex number field $\mathbb{C}$ with a distinguished basis $x_0, x_1, \ldots, x_d$ that satisfies the following conditions (see [6, 5]):

(i) x_0 is the identity element of $\mathfrak{A}$;

(ii) the structure constants $\{p_{ij}^k\}_{0 \leq i,j,k \leq d}$ (i.e., $x_i x_j = \sum_{k=0}^{d} p_{ij}^k x_k$) are real numbers;

(iii) there exists a permutation $i \mapsto i'$ of $\{0, 1, \dots, d\}$ such that $(i')' = i$ and $p^k_{ij} = p^{k'}_{i'j'}$;

(iv) for each $i = 0, 1, \dots, d$, there exists $k_i > 0$ such that $p^0_{ij} = \delta_{ij'} k_i$ for all j, and the map $x_i \mapsto k_i$ $(i = 0, 1, \dots, d)$ is a linear representation of $\mathfrak{A}$.

The Hecke algebras of Gelfand pairs, or more generally, the Bose-Mesner algebras of commutative association schemes, are typical examples of character algebras.

The character algebra $\mathfrak{A}$ is semi-simple, and therefore has the basis $e_0, e_1, \dots, e_d$ of the primitive idempotents, where e_0 is the idempotent of $\mathfrak{A}$ corresponding to the linear representation $x_i \mapsto k_i$. We define the *first* and the *second eigenmatrices* $P = (P_{ij})_{0 \le i,j \le d}$ and $Q = (Q_{ij})_{0 \le i,j \le d}$ of $\mathfrak{A}$ by

$$x_j = \sum_{i=0}^{d} P_{ij} e_i \quad \text{and} \quad e_j = \frac{1}{N} \sum_{i=0}^{d} Q_{ij} x_i$$

for $0 \le j \le d$, where $N = k_0 + \cdots + k_d$. In particular, we have $k_j = P_{0j}$ for all j. Note that in the case of the Hecke algebra of a Gelfand pair, the eigenmatrices are essentially equivalent to the zonal spherical functions (cf. [6, §2.11]). The eigenmatrices P and Q satisfy the orthogonality relations (cf. [6, p. 94, Theorem 5.5])

$$Q_{ij}/m_j = \overline{P_{ji}}/k_i, \tag{1}$$

where $m_j = Q_{0j}$ (> 0),

$$\sum_{l=0}^{d} \frac{1}{k_l} P_{il} \overline{P_{jl}} = \frac{N}{m_i} \delta_{ij}, \tag{2}$$

$$\sum_{l=0}^{d} \frac{1}{m_l} Q_{il} \overline{Q_{jl}} = \frac{N}{k_i} \delta_{ij}. \tag{3}$$

Now, for any $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}_0^r$, we define $|\alpha| = \alpha_1 + \cdots + \alpha_r$. The following is the main result in this paper:

Theorem 1.1. *We use the same notation as above. Let* $\tilde{\Omega} = (1 - \omega_{ij})_{1 \le i,j \le d}$ *where* $\omega_{ij} = P_{ij}/k_j$ *for* $1 \le i, j \le d$. *Then, for any positive integer* n *we have*

$$\text{(i)} \quad \sum_{\substack{\beta \in \mathbb{N}_0^d \\ |\beta| \le n}} k_0^{\beta_0} \dots k_d^{\beta_d} \binom{n}{\beta_0, \dots, \beta_d} \cdot F((-\alpha), (-\beta); -n; \tilde{\Omega}) \overline{F((-\hat{\alpha}), (-\beta); -n; \tilde{\Omega})}$$
$$= \frac{N^n}{m_0^{\alpha_0} \dots m_d^{\alpha_d}} \binom{n}{\alpha_0, \dots, \alpha_d}^{-1} \delta_{\alpha\hat{\alpha}},$$

for all α, $\hat{\alpha}$ *in* $\mathbb{N}_0^d$ *with* $|\alpha|, |\hat{\alpha}| \le n$.

$$\text{(ii)} \quad \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \le n}} m_0^{\alpha_0} \dots m_d^{\alpha_d} \binom{n}{\alpha_0, \dots, \alpha_d} F((-\alpha), (-\beta); -n; \tilde{\Omega}) \overline{F((-\alpha), (-\hat{\beta}); -n; \tilde{\Omega})}$$
$$= \frac{N^n}{k_0^{\beta_0} \dots k_d^{\beta_d}} \binom{n}{\beta_0, \dots, \beta_d}^{-1} \delta_{\beta\hat{\beta}},$$

for all β, $\hat{\beta}$ *in* $\mathbb{N}_0^d$ *with* $|\beta|, |\hat{\beta}| \le n$.

In the above equations, conventionally we put $\beta_0 = n - \sum_{j=1}^{d} \beta_j$ *for any* $\beta \in \mathbb{N}_0^d$ *with* $|\beta| \leq n$.

Remark 1.2. In fact, in the process of proving Theorem 1.1, we give an explicit formula (see (6)) for the entries of the first eigenmatrix of the character algebra of the n-th symmetric tensor space of $\mathfrak{A}$, in terms of the $(d+1, 2(d+1))$-hypergeometric functions appearing in the theorem. See also Remark 2.1.

Remark 1.3. It is known that there exists a character algebra $\mathfrak{A}^*$ with eigenmatrices $P^* = Q$ and $Q^* = P$ (cf. [6, §2.5]). Therefore, Theorem 1.1 (ii) can also be obtained by applying Theorem 1.1 (i) to the dual algebra $\mathfrak{A}^*$.

Remark 1.4. If $d = 1$, then by (2) we have $\omega_{11} = -\frac{1}{N-1}$. In this case, the polynomials $F(-k, -j; -n; \tilde{\Omega})$ $(0 \leq k, j \leq n)$ become ${}_2F_1(-k, -j; -n; \frac{N}{N-1})$ in the standard notation of hypergeometric series in one variable. Thus, Theorem 1.1 gives the orthogonality relation of Krawtchouk polynomials $K_k(x; n, N) = \binom{n}{k}(N-1)^k {}_2F_1(-k, -x; -n; \frac{N}{N-1})$.

2. Proof of Theorem 1.1

Let $T^n = T^n(\mathfrak{A}) = \mathfrak{A} \otimes \cdots \otimes \mathfrak{A}$ (n times). Then it is easy to see that T^n is a character algebra with a distinguished basis $x_{j_1} \otimes \cdots \otimes x_{j_n}$ $(j_1, \ldots, j_n \in \{0, 1, \ldots, d\})$ and the primitive idempotents $e_{i_1} \otimes \cdots \otimes e_{i_n}$ $(i_1, \ldots, i_n \in \{0, 1, \ldots, d\})$. Clearly, the $(i_1, \ldots, i_n; j_1, \ldots, j_n)$-entry of the first eigenmatrix of T^n is given by $P_{i_1 j_1} \ldots P_{i_n j_n}$.

Now, let $S^n = S^n(\mathfrak{A})$ be the symmetric tensor subspace of T^n, that is, S^n consists of the vectors that are invariant under the action $z_1 \otimes \cdots \otimes z_n \mapsto z_{w(1)} \otimes \cdots \otimes z_{w(n)}$ $(w \in \mathfrak{S}_n)$ of the symmetric group $\mathfrak{S}_n$ on T^n. For each $\beta = (\beta_0, \beta_1, \ldots, \beta_d) \in \mathbb{N}_0^{d+1}$ with $|\beta| = n$, define

$$x_\beta = \frac{1}{\beta_0! \ldots \beta_d!} \cdot \sum_{w \in \mathfrak{S}_n} x_{\mu_\beta(w(1))} \otimes \cdots \otimes x_{\mu_\beta(w(n))},$$

where

$$\mu_\beta = (\mu_\beta(1), \ldots, \mu_\beta(n)) = (\underbrace{d, \ldots, d}_{\beta_d}, \ldots, \underbrace{1, \ldots, 1}_{\beta_1}, \underbrace{0, \ldots, 0}_{\beta_0}) \in \mathbb{N}_0^n.$$

Note that the vectors x_β $(|\beta| = n)$ are $(0,1)$-combinations of the $x_{j_1} \otimes \cdots \otimes x_{j_n}$ and they form a basis of S^n. We also let

$$e_{\boldsymbol{\alpha}} = \frac{1}{\alpha_0! \ldots \alpha_d!} \cdot \sum_{w \in \mathfrak{S}_n} e_{\mu_\alpha(w(1))} \otimes \cdots \otimes e_{\mu_\alpha(w(n))}$$

for each $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^{d+1}$ with $|\alpha| = n$. Then, the e_α are pairwise orthogonal idempotents, and we have

$$x_\beta = \sum_{|\alpha| = n} P_{\alpha\beta} e_\alpha,$$

where

$$P_{\alpha\beta} = \frac{1}{\beta_0! \ldots \beta_d!} \cdot \sum_{w \in \mathfrak{S}_n} P_{\mu_\alpha(1)\mu_\beta(w(1))} \ldots P_{\mu_\alpha(n)\mu_\beta(w(n))}. \tag{4}$$

This implies that the subspace S^n is closed under the multiplication in T^n. Therefore, S^n is also a character algebra with a distinguished basis x_β ($|\beta| = n$) (see also [4, Lemma 1]).

Remark 2.1. If $\mathfrak{A}$ is the Hecke algebra of a Gelfand pair (G, H), then $S^n(\mathfrak{A})$ coincides with that of the Gelfand pair $(G \wr \mathfrak{S}_n, H \wr \mathfrak{S}_n)$ (see [10, 1]). In particular, when $G = \mathfrak{S}_q$ and $H = \mathfrak{S}_{q-1}$, $S^n(\mathfrak{A})$ is the Bose-Mesner algebra of the Hamming association scheme $H(n, q)$. More generally, if $\mathfrak{A}$ is the Bose-Mesner algebra of a commutative association scheme $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$, then $S^n(\mathfrak{A})$ gives that of the "extension of $\mathfrak{X}$ of length n" considered in [7, §2.5] (see also [11]).

In what follows, we evaluate

$$\frac{P_{\alpha\beta}}{P_{(0^n)\beta}} = \frac{1}{n!} \cdot \sum_{w \in \mathfrak{S}_n} \omega_{\mu_\alpha(1)\mu_\beta(w(1))} \cdots \omega_{\mu_\alpha(n)\mu_\beta(w(n))}, \tag{5}$$

where (0^n) stands for $(n, 0, \dots, 0) \in \mathbb{N}_0^{d+1}$. First, we have

$$\frac{P_{\alpha\beta}}{P_{(0^n)\beta}} = \frac{\prod_{i=0}^d \alpha_i! \prod_{j=0}^d \beta_j!}{n!} \cdot \sum_{A \in \mathcal{A}} \prod_{1 \le i,j \le d+1} \frac{(\omega_{i-1,j-1})^{a_{ij}}}{a_{ij}!},$$

where $\mathcal{A} = \{A = (a_{ij}) \in M_{d+1}(\mathbb{N}_0) \mid \sum_{i=1}^{d+1} a_{ij} = \beta_{j-1}, \sum_{j=1}^{d+1} a_{ij} = \alpha_{i-1}\}$, which is equal to

$$\begin{aligned}
&\frac{\prod_{i=0}^d \alpha_i! \prod_{j=0}^d \beta_j!}{n!} \cdot \sum_{A \in \mathcal{A}} \prod_{1 \le i,j \le d+1} \frac{(1 + (\omega_{i-1,j-1} - 1))^{a_{ij}}}{a_{ij}!} \\
&= \frac{\prod_{i=0}^d \alpha_i! \prod_{j=0}^d \beta_j!}{n!} \cdot \sum_{A \in \mathcal{A}} \left(\prod_{1 \le i,j \le d+1} a_{ij}! \right)^{-1} \sum_{B \in \mathcal{B}_A} \prod_{1 \le i,j \le d} \binom{a_{i+1,j+1}}{b_{ij}} (\omega_{ij} - 1)^{b_{ij}},
\end{aligned}$$

where $\mathcal{B}_A = \{B = (b_{ij}) \in M_d(\mathbb{N}_0) \mid b_{ij} \le a_{i+1,j+1} \text{ for } 1 \le i,j \le d\}$, since $\omega_{i0} = \omega_{0j} = 1$ for all $0 \le i,j \le d$. Now, for each $B \in M_d(\mathbb{N}_0)$, define

$$\mathcal{C}_B = \left\{ C = (c_{ij}) \in M_{d+1}(\mathbb{N}_0) \;\middle|\; \begin{array}{l} \sum_{i=1}^{d+1} c_{ij} = \beta_{j-1} - \sum_{i=1}^d b_{i,j-1}, \\ \sum_{j=1}^{d+1} c_{ij} = \alpha_{i-1} - \sum_{j=1}^d b_{i-1,j} \end{array} \right\},$$

where we put $b_{i0} = b_{0j} = 0$ for all i,j. Then,

$$\begin{aligned}
\frac{P_{\alpha\beta}}{P_{(0^n)\beta}} &= \frac{\prod_{i=0}^d \alpha_i! \prod_{j=0}^d \beta_j!}{n!} \cdot \sum_B \sum_{C \in \mathcal{C}_B} \left(\prod_{1 \le i,j \le d+1} c_{ij}! \right)^{-1} \prod_{1 \le i,j \le d} \frac{(\omega_{ij} - 1)^{b_{ij}}}{b_{ij}!} \\
&= \frac{1}{n!} \cdot \sum_B \frac{\prod_{i=1}^d \alpha_i! \prod_{j=1}^d \beta_j! (n - \sum_{i,j} b_{ij})!}{\prod_{i=1}^d (\alpha_i - \sum_{j=1}^d b_{ij})! \prod_{j=1}^d (\beta_j - \sum_{i=1}^d b_{ij})!} \prod_{i,j} \frac{(\omega_{ij} - 1)^{b_{ij}}}{b_{ij}!} \\
&= \sum_B \frac{\prod_{i=1}^d (-\alpha_i)_{\sum_j b_{ij}} \prod_{j=1}^d (-\beta_j)_{\sum_i b_{ij}}}{(-n)_{\sum_{i,j} b_{ij}}} \prod_{i,j} \frac{(1 - \omega_{ij})^{b_{ij}}}{b_{ij}!} \\
&= F((-\alpha_1, \dots, -\alpha_d), (-\beta_1, \dots, -\beta_d); -n; \tilde{\Omega}),
\end{aligned} \tag{6}$$

where in the above equations, the first sums are over all $B \in M_d(\mathbb{N}_0)$ such that $\sum_{i=1}^d b_{ij} \le \beta_j$ and $\sum_{j=1}^d b_{ij} \le \alpha_i$.

Therefore, the result in Theorem 1.1 immediately follows by applying the orthogonality relations (2) and (3) to the character algebra S^n.

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