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THE NUMBER OF HALL π-SUBGROUPS OF A π-SEPARABLE GROUP

ALEXANDRE TURULL

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Abstract. We observe a simple formula to compute the number $\nu_\pi(G)$ of Hall π-subgroups of a π-separable finite group G in terms of only the action of a fixed Hall π-subgroup of G on a set of normal π'-sections of G. As a consequence, we obtain that $\nu_\pi(K)$ divides $\nu_\pi(G)$ whenever K is a subgroup of a finite π-separable group G. This generalizes a recent result of Navarro. In addition, our method gives an alternative proof of Navarro's result.

1. Introduction

Let G be a finite group, and let π be a set of primes. We denote by $\nu_\pi(G)$ the number of Hall π-subgroups of G. In the case when G is π-separable, we know that $\nu_\pi(G) > 0$, and $\nu_\pi(G) = [G : \mathrm{N}_G(H)]$, where H is any Hall π-subgroup of G. This formula involves the calculation of a normalizer, and it does not immediately allow us to compare $\nu_\pi(G)$ to $\nu_\pi(K)$ for a subgroup K of G. We propose the following formula, which uses only the calculation of some centralizers, and provides an obvious way to relate $\nu_\pi(G)$ and $\nu_\pi(K)$. It appears not to have been explicitly observed before.

Theorem 1.1. *Let π be a set of primes, and let G be a finite π-separable group. Let H be a Hall π-subgroup of G, and let*

$$1 = N_0 \lhd N_1 \lhd \cdots \lhd N_\ell = G$$

be a sequence of normal subgroups of G such that N_i/N_{i-1} is either a π-group or a π'-group for $i = 1, \ldots, \ell$. Let $\mathcal{F}$ be the set of N_i/N_{i-1} that are π'-groups. Then,

$$\nu_\pi(G) = \prod_{F \in \mathcal{F}} [F : \mathrm{C}_F(H)].$$

As a consequence, we obtain the following corollary, which is a generalization of the main result of Navarro [1]. Our proof of the corollary is based on our theorem, and provides an alternate proof of Navarro's result.

Corollary 1.2. *Let π be a set of primes, let G be a finite π-separable group, and let K be a subgroup of G. Then $\nu_\pi(K)$ divides $\nu_\pi(G)$.*

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2. Proofs

Proof of Theorem 1.1. Assume the theorem is false. Among all counterexamples, choose one with ℓ as small as possible. Since the result holds for $\ell = 0$, and it is well known if $\ell \leq 2$, we have $\ell > 2$. Set $N = N_1$. Let $\mathcal{H}$ be the set of Hall π-subgroups of G. Define an equivalence relation on $\mathcal{H}$ by, for $H_1, H_2 \in \mathcal{H}$, $H_1 \sim H_2$ if and only if $H_1N = H_2N$. The equivalence classes are in one-to-one correspondence with the groups H_1N/N, that is, the Hall π-subgroups of G/N. Hence, the number of equivalence classes in $\mathcal{H}$ is exactly $\nu_\pi(G/N)$. Since the elements of $\mathcal{H}$ are all conjugate in G, and N is a normal subgroup of G, the groups H_1N for $H_1 \in \mathcal{H}$ are all isomorphic to HN. In particular, $\nu_\pi(H_1N) = \nu_\pi(HN)$ does not depend on $H_1 \in \mathcal{H}$. It follows that

$$\nu_\pi(G) = |\mathcal{H}| = \nu_\pi(G/N)\nu_\pi(HN).$$

By our choice of ℓ, the theorem holds for both G/N and HN, and this implies that the theorem holds for G. This contradiction completes the proof of the theorem. □

The following lemma appears in slightly different form in [1]. We include a proof for completeness.

Lemma 2.1. *Suppose H is a finite group, acting on the finite group F, and assume that $|H|$ and $|F|$ are relatively prime. Suppose that K is an H-invariant subgroup of F. Then $[K : \mathrm{C}_K(H)]$ divides $[F : \mathrm{C}_F(H)]$.*

Proof. Assume the lemma is false, and choose a counterexample with $|F|$ as small as possible, and among all such, with $[F : K]$ as small as possible. Since $|F|$ and $|H|$ are relatively prime, for each prime p, the H-invariant Sylow p-subgroups of F (respectively of K) are the maximal H-invariant p-subgroups of F (respectively of K), and they are conjugate by elements of $\mathrm{C}_F(H)$ (respectively by elements of $\mathrm{C}_K(H)$). Hence, for each prime p, we may take some H-invariant Sylow p-subgroup P of F that contains an H-invariant Sylow p-subgroup of K. This implies that P also contains a Sylow p-subgroup of $\mathrm{C}_F(H)$, and a Sylow p-subgroup of $\mathrm{C}_K(H)$. Hence, P, $\mathrm{C}_P(H)$, $P \cap K$ and $K \cap \mathrm{C}_P(H)$ are respectively Sylow p-subgroups of F, $\mathrm{C}_F(H)$, K and $\mathrm{C}_K(H)$. It follows that the p-part of $[K : \mathrm{C}_K(H)]$ is $[P \cap K : P \cap \mathrm{C}_K(H)]$, and the p-part of $[F : \mathrm{C}_F(H)]$ is $[P : P \cap \mathrm{C}_F(H)]$. Hence, the minimality of our counterexample implies that F is a p-group for some prime p. The minimality of $[F : K]$ in our counterexample implies that K is maximal among the H-invariant subgroups of F. Let Φ be the Frattini subgroup of F. Since K is a proper subgroup, $K\Phi$ is a proper H-invariant subgroup of F, and it follows that $\Phi \subseteq K$. Hence, K is a normal subgroup of F. It follows that H acts on the group F/K and that $\mathrm{C}_F(H)/\mathrm{C}_K(H)$ is isomorphic to the centralizer of the action of H on F/K. This implies that $[\mathrm{C}_F(H) : \mathrm{C}_K(H)]$ divides $[F : K]$. Hence the conclusion of the lemma holds. This final contradiction completes the proof of the lemma. □

Proof of Corollary 1.2. Let H be a Hall π-subgroup of G that contains a Hall π-subgroup of K. Hence, $K \cap H$ is a Hall π-subgroup of K. Let

$$1 = N_0 \lhd N_1 \lhd \cdots \lhd N_\ell = G$$

be a sequence of normal subgroups of G such that N_i/N_{i-1} is either a π-group or a π'-group for $i = 1, \ldots, \ell$. Let $\mathcal{F}$ be the set of N_i/N_{i-1} that are π'-groups. Then,

by Theorem 1.1, we have

$$\nu_\pi(G) = \prod_{F \in \mathcal{F}} [F : \mathrm{C}_F(H)]. \tag{2.1}$$

Furthermore, $\nu_\pi(K)$ can be computed in a similar way by considering the action of $H \cap K$ on the appropriate quotients of the sequence of normal subgroups of K,

$$1 = N_0 \cap K \lhd N_1 \cap K \lhd \cdots \lhd N_\ell \cap K = K.$$

Notice that $(N_i \cap K)/(N_{i-1} \cap K)$ is $(H \cap K)$-isomorphic to $(N_i \cap K)N_{i-1}/N_{i-1}$, an $(H \cap K)$-invariant subgroup of N_i/N_{i-1}. For each $F \in \mathcal{F}$, we set $K(F)$ to be $(N_i \cap K)N_{i-1}/N_{i-1}$. Then we have

$$\nu_\pi(K) = \prod_{F \in \mathcal{F}} [K(F) : \mathrm{C}_{K(F)}(H \cap K)].$$

It then follows from Lemma 2.1 that $\nu_\pi(K)$ divides

$$\prod_{F \in \mathcal{F}} [F : \mathrm{C}_F(H \cap K)].$$

Since this clearly divides our expression (2.1) for $\nu_\pi(G)$ above, the corollary follows. $\square$

References

[1] G. Navarro, *Number of Sylow subgroups in p-solvable groups*, Proc. Amer. Math. Soc. **131** (2003), 3019–3020. MR **2004d:**20020

Department of Mathematics, University of Florida, Gainesville, Florida 32611-8105
E-mail address: `turull@math.ufl.edu`