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# SPECTRAL GAP FOR HYPERBOUNDED OPERATORS

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ABSTRACT. Let $(E, \mathcal{F}, \mu)$ be a probability space, and $P$ a symmetric linear contraction operator on $L^2(\mu)$ with $P1 = 1$ and $\|P\|_{L^2(\mu)\to L^4(\mu)} < \infty$. We prove that $\|P\|^4_{L^2(\mu)\to L^4(\mu)} < 2$ is the optimal sufficient condition for $P$ to have a spectral gap. Moreover, the optimal sufficient conditions are obtained, respectively, for the defective log-Sobolev and for the defective Poincaré inequality to imply the existence of a spectral gap. Finally, we construct a symmetric, hyperbounded, ergodic contraction $C_0$-semigroup without a spectral gap.

## 1. Introduction

Let $(E, \mathcal{F}, \mu)$ be a probability space, and $P$ a symmetric contraction linear operator on $L^2(\mu)$ with $P1 = 1$. Then the spectral radius of $P$ is $R(P) = 1$. We say that $P$ has a spectral gap if

$$R_1(P) := \sup\{\|Pf\|_{L^2(\mu)} : \mu(f^2) = 1, \mu(f) = 0\} < 1.$$

According to Simon and Høegh-Krohn [15], $P$ is called hyperbounded if

$$\delta(P) := \|P\|^4_{L^2(\mu)\to L^4(\mu)} < \infty.$$

We consider the following question raised in [15]: *does a hyperbounded operator possess a spectral gap?*

It turns out that the answer is NO for nonergodic positive definite operators. Recall that $P$ is called positive definite if $\mu(fPf) \geq 0$ for all $f \in L^2(\mu)$, and is called ergodic if (see [20])

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P^k f = \mu(f), \quad f \in L^2(\mu);$$

here and in the sequel the limit is taken in $L^2(\mu)$. Let $\mathcal{H} := \{f : Pf = f\}$, and let $\pi : L^2(\mu) \to \mathcal{H}$ be the orthogonal projection. For positive definite $P$ it follows from the spectral theorem that

$$\lim_{n\to\infty} P^n f = \pi f + \lim_{n\to\infty} \int_0^{1-} \lambda^n \mathrm{d}E_\lambda(f) = \pi f, \quad f \in L^2(\mu),$$

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where $\{E_\lambda : \lambda \in [0,1]\}$ is the resolution of the identity of $P$. Therefore, it is easy to see that the ergodicity is equivalent to each of the following two statements (we write $f = g$ if these two functions are equal $\mu$-a.e.):
(a) *for any* $f \in L^2(\mu)$, $P^n f \to \mu(f)$ *as* $n \to \infty$*;*
(b) *for any* $f \in L^2(\mu)$*, if* $Pf = f$*, then* $f$ *is constant.*
Obviously, (a) and hence the ergodicity implies the positivity-improving property (or the $\mu$-essential irreducibility [11, 18]):
(c) *for any* $A, B$ *with* $\mu(A)\mu(B) > 0$*, there exists* $n \geq 1$ *such that* $\mu(1_A P^n 1_B) > 0$.
Conversely, if $P$ is positive (or positivity-preserving), i.e., $Pf \geq 0$ for all $f \geq 0$, then (c) implies (b) and hence is equivalent to the ergodicity. Indeed, for $Pf = f$ with $\mu(f) = 0$, one has $Pf^+ \geq (Pf)^+ = f^+$, where $f^+ := f \vee 0$. But $\mu(Pf^+) = \mu(f^+)$, and so one has $Pf^+ = f^+$. Thus, for any $n \geq 1$ and any $\varepsilon > 0$,

$$0 = \mu(f^+ f^-) = \mu(f^- P^n f^+) \geq \varepsilon^2 \mu(1_{\{f^- > \varepsilon\}} P^n 1_{\{f^+ > \varepsilon\}}).$$

By (c) one has $\mu(f^- > \varepsilon)\mu(f^+ > \varepsilon) = 0$ for all $\varepsilon > 0$. Hence, either $\mu(f^+) = 0$ or $\mu(f^-) = 0$. But $\mu(f) = 0$ implies $\mu(f^-) = \mu(f^+)$, and one concludes that $f = 0$. Therefore, (c) implies (b).

Let us recall some progress concerning this question. In the context of a symmetric contraction $C_0$-semigroup $(P_t)_{t\geq 0}$ with $P_t 1 = 1$, the Rothaus-Simon mass gap theorem says that the hypercontractivity (i.e., there exists $t > 0$ such that $\delta(P_t) = 1$) implies the existence of a spectral gap; see [13, 14] and also [8]. More precisely, if $\delta(P_T) = 1$, then (see e.g. [4, Lemma 6.1.5])

$$R_1(P_t) \leq 3^{-[t/T]/2}, \quad t \geq T,$$

where $[r] := \sup\{n \in \mathbb{Z} : n \leq r\}$. According to [12, Theorem 2.3], this implies the following lower bound estimate for the spectral gap of the generator $(L, \mathcal{D}(L))$:

$$\text{gap}(L) := \inf\{-\mu(fLf) : f \in \mathcal{D}(L), \mu(f^2) = 1, \mu(f) = 0\} \geq \frac{\log 3}{2T}. \tag{1.1}$$

Therefore, $\delta(P_T) = 1$ implies $R_1(P_t) \leq 3^{-t/(2T)}$ for all $t \geq 0$.

Next, in 1998, Aida [1] proved that $\text{gap}(L) > 0$ provided there is $t > 0$ such that $P_t$ is hyperbounded and is uniformly positivity-improving (Kusuoka [10]):

$$\inf\{\mu(1_A P_t 1_B) : \mu(A), \mu(B) \geq \varepsilon\} > 0, \quad \varepsilon > 0.$$

Obviously, the uniformly positivity-improving property is stronger than the ergodicity. In 2000, Wu [18] introduced the notion of uniform integrability for linear operators and studied the above question for ergodic, uniformly integrable, positive operators (see [18, Problem 3.10]). Moreover, Aida's above-mentioned result was generalized or extended in [9, 6] for general positive operators. Very recently, the essential spectral radius of Markov operators was estimated by Wu [19].

Our first observation in this note is that the hyperbound estimate $\delta(P) < 2$ is the minimal sufficient condition for the existence of a spectral gap.

**Theorem 1.1.** (1) *If* $\delta(P) < 2$*, then* $P$ *has a spectral gap with*

$$R_1(P) \leq \left\{ \frac{\sqrt{2\delta(P)}}{3\sqrt{2\delta(P)} - 4\sqrt{\delta(P) - 1}} \right\}^{1/2} < 1. \tag{1.2}$$

(2) *There exists a Markov operator* $P$ *with* $\delta(P) = 2$ *and* $R_1(P) = 1$ (*i.e.,* $P$ *does not have a spectral gap*).

(3) *If $P$ is positive definite and $\delta := \delta(P) < 2$, then $\|P^r\|_{L^2(\mu)\to L^4(\mu)} = 1$ for all $r \geq r_0$, where*

$$r_0 := \frac{\log\left\{3 + 3\left(17\delta - 8 - 12\sqrt{2\delta(\delta-1)}\right)/16\right\}}{\log\left(3 - 4\sqrt{\delta - 1}/\sqrt{2\delta}\right)} < \infty.$$

Theorem 1.1 (1) generalizes [4, Lemma 5.5.11] where $\delta(P) = 1$ is considered, and the proof of [4, Lemma 5.5.11] does not work for $\delta(P) > 1$. Below is a direct consequence of Theorem 1.1 in the context of contraction $C_0$-semigroups, which improves the corresponding known result (see e.g. [4, Corollary 6.1.17]).

**Corollary 1.2.** *Let $(P_t)_{t\geq 0}$ be the symmetric contraction $C_0$-semigroup generated by a negative definite selfadjoint operator $(L, \mathcal{D}(L))$ on $L^2(\mu)$ with $L1 = 0$.*
(1) *If there exists $t > 0$ such that $\delta(P_t) < 2$, then*

$$\text{gap}(L) \geq -\frac{1}{2t}\log\frac{\sqrt{2\delta(P_t)}}{3\sqrt{2\delta(P_t)} - 4\sqrt{\delta(P_t) - 1}} > 0. \tag{1.3}$$

(2) *If $P_t$ is Markovian and the defective log-Sobolev inequality holds,*

$$\mu(f^2\log f^2) \leq -C_1\mu(fLf) + C_2, \quad f \in \mathcal{D}(L),\ \mu(f^2) = 1, \tag{1.4}$$

*for some $C_1 > 0$ and $C_2 \in [0, \log 2)$, then*

$$\text{gap}(L) \geq -\frac{2}{C_1\log 3}\log\frac{\sqrt{2\mathrm{e}^{C_2}}}{3\sqrt{2e^{C_2}} - 4\sqrt{\mathrm{e}^{C_2} - 1}} > 0. \tag{1.5}$$

*In particular, if (1.4) holds for $C_2 = 0$, then $\text{gap}(L) \geq \frac{2}{C_1}$, as is well known.*
(3) *There exists an example such that (1.4) holds for $C_1 = 0$ and $C_2 = \log 2$, but $\text{gap}(L) = 0$.*

Corollary 1.2 shows that $C_2 < \log 2$ is the optimal sufficient condition for the defective log-Sobolev inequality (1.4) to imply $\text{gap}(L) > 0$. Below we present the optimal sufficient condition for the defective Poincaré inequality to imply the existence of a spectral gap.

**Theorem 1.3.** *Let $(L, \mathcal{D}(L))$ be the generator associated with a conservative Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. If there exist $C_1 > 0$ and $C_2 \in [1, 2)$ such that*

$$\mu(f^2) \leq C_1\mathcal{E}(f,f) + C_2\mu(|f|)^2, \quad f \in \mathcal{D}(\mathcal{E}), \tag{1.6}$$

*then*

$$\text{gap}(L) \geq \frac{1 - \sqrt{C_2(C_2-1)/2}}{2C_1} > 0.$$

*On the other hand, there exists a conservative Dirichlet form such that (1.6) holds for $C_2 = 2$ but $\text{gap}(L) = 0$.*

Finally, a counterexample is constructed in the paper to show that even in the ergodic case the hyperboundedness is insufficient for the existence of a spectral gap. This example enables us to state the following result.

**Theorem 1.4.** *There exist $(E, \mathcal{F}, \mu)$ and a symmetric, hyperbounded, ergodic contraction $C_0$-semigroup $(P_t)_{t\geq 0}$ on $L^2(\mu)$ without a spectral gap.*

We remark that Theorem 1.4 does not provide any negative answer to Wu's problem 3.10 in [18], since the operator constructed in the counterexample is not positivity-preserving. Wu ([18, Example 1.8]) provided an example of a hyper-bounded (but non-$C_0$) semigroup without a spectral gap.

## 2. Proofs of Theorem 1.1 and Corollary 1.2

*Proof of Theorem 1.1.* (1) For $\varepsilon \in (0,1)$ and $f \in L^2(\mu)$ with $\mu(f^2)=1$ and $\mu(f)=0$, let $g := \sqrt{\varepsilon}+\sqrt{1-\varepsilon}f$. We assume that $\mu((Pf)^3)\geq 0$; otherwise simply replace $f$ by $-f$. Since $\mu(Pf)=\mu(fP1)=\mu(f)=0$ and since $\mu((Pf)^4)\geq \mu((Pf)^2)^2$, it follows that

$$\begin{aligned}\delta(P) &:= \|P\|^4_{L^2(\mu)\to L^4(\mu)} \geq \mu((Pg)^4)\\ &= \varepsilon^2 + (1-\varepsilon)^2\mu((Pf)^4) + 6\varepsilon(1-\varepsilon)\mu((Pf)^2)\\ &\qquad + 4\varepsilon^{3/2}\sqrt{1-\varepsilon}\mu(Pf) + 4\sqrt{\varepsilon}(1-\varepsilon)^{3/2}\mu((Pf)^3)\\ &\geq (1-\varepsilon)^2\mu((Pf)^2)^2 + 6\varepsilon(1-\varepsilon)\mu((Pf)^2)+\varepsilon^2.\end{aligned}$$

Thus,

$$\mu((Pf)^2) \leq \frac{\sqrt{8\varepsilon^2+\delta(P)}-3\varepsilon}{1-\varepsilon} =: h(\varepsilon), \quad \mu(f)=0, \mu(f^2)=1, \varepsilon\in(0,1).$$

Therefore,

$$R_1(P)^2 \leq \xi(\delta(P)) := \inf_{\varepsilon\in(0,1)} h(\varepsilon). \tag{2.1}$$

It is easy to see that when $\delta(P)=1$ one has

$$\xi(1) = \lim_{\varepsilon\uparrow 1} \frac{\sqrt{8\varepsilon^2+1}-3\varepsilon}{1-\varepsilon} = \frac{1}{3}.$$

Next, for $1<\delta(P)<2$, the minimum of $h(\varepsilon)$ over $\varepsilon\in(0,1)$ is reached at

$$\varepsilon(\delta(P)) := \delta(P) - \frac{3}{4}\sqrt{2\delta(P)(\delta(P)-1)}.$$

Indeed, one has (where "$\Longleftrightarrow$" means "is equivalent to")

$$\begin{aligned}h'(\varepsilon)=0 &\Longleftrightarrow (1-\varepsilon)\left(\frac{8\varepsilon}{\sqrt{8\varepsilon^2+\delta(P)}}-3\right)+\sqrt{8\varepsilon^2+\delta(P)}-3\varepsilon=0\\ &\Longleftrightarrow 8\varepsilon+\delta(P) = 3\sqrt{8\varepsilon^2+\delta(P)}\\ &\Longleftrightarrow 8\varepsilon^2 - 16\delta(P)\varepsilon + 9\delta(P)-\delta(P)^2=0\\ &\Longleftrightarrow \varepsilon = \delta(P)-\frac{1}{4}\sqrt{16\delta(P)^2-2(9\delta(P)-\delta(P)^2)} =: \varepsilon(\delta(P)).\end{aligned}$$

Thus, $\xi(\delta(P)) = h(\varepsilon(\delta(P)))$. It remains to calculate $h(\varepsilon(\delta))$ for $\delta\geq 1$. Observing that

$$\begin{aligned}&8\Big(\delta-\frac{3}{4}\sqrt{2\delta(\delta-1)}\Big)^2+\delta = 8\delta^2+9\delta(\delta-1)-12\delta\sqrt{2\delta(\delta-1)}+\delta\\ &= 8\delta(\delta-1)-12\delta\sqrt{2\delta(\delta-1)}+9\delta^2 = \Big(3\delta-2\sqrt{2\delta(\delta-1)}\Big)^2,\end{aligned}$$

and noting that $3\delta > 2\sqrt{2\delta(\delta-1)}$ for $\delta \geq 1$, one has

$$\xi(\delta) = h(\varepsilon(\delta)) = \frac{\sqrt{2\delta(\delta-1)}}{3\sqrt{2\delta(\delta-1)} - 4(\delta-1)} = \frac{\sqrt{2\delta}}{3\sqrt{2\delta} - 4\sqrt{\delta-1}}.$$

Therefore, (1.2) follows from (2.1).

(2) Let $E = \{0,1\}, \mu(0) = \mu(1) = \frac{1}{2}$ and $P = I$ (the identity operator). Then $P$ does not have any spectral gap. Next, let $f_1(0) = -1$ and $f_1(1) = 1$. For any $f \in L^2(\mu)$ with $\mu(f^2) = 1$, there exist $c_1, c_2 \in \mathbb{R}$ such that $c_1^2 + c_2^2 = 1$ and $f = c_1 + c_2 f_1$. Then

$$\begin{aligned}\mu((Pf)^4) &= \mu(f^4) = c_1^4 + c_2^4 + 6c_1^2c_2^2 = (c_1^2+c_2^2)^2 + 4c_1^2c_2^2\\ &\leq \sup_{r\in(0,1)} \{1 + 4r(1-r)\} = 2,\end{aligned}$$

with equality holding when $c_1^2 = c_2^2 = \frac{1}{2}$. Thus, $\delta(P) = 2$.

(3) For $f \in L^2(\mu)$ with $\mu(f^2) = 1$, let $\hat{f} := f - \mu(f)$. Let $R$ denote the upper bound of $R_1(P)$ given in (1), and simply denote $\delta = \delta(P)$. One has $R < 1$, since $\delta < 2$. Since $P$ is positive definite, $P^r$ is well-defined for $r \geq 1$ and we have $\mu(P^r\hat{f}) = \mu(\hat{f}) = 0$. Hence

$$\mu((P^r f)^4) = \mu(f)^4 + 4\mu(f)\mu((P^r\hat{f})^3) + 6\mu(f)^2\mu((P^r\hat{f})^2) + \mu((P^r\hat{f})^4).$$

Noting that

$$\begin{aligned}&4\mu(f)\mu((P^r\hat{f})^3) \leq 4|\mu(f)|\sqrt{\mu((P^r\hat{f})^4)\mu((P^r\hat{f})^2)}\\ &\leq 2t\mu(f)^2\mu((P^r\hat{f})^2) + 2t^{-1}\mu((P^r\hat{f})^4), \quad t>0,\\ &\mu((P^r\hat{f})^2) \leq R^{2r}\mu(\hat{f}^2),\\ &\mu((P^r\hat{f})^4) \leq \delta\mu((P^{r-1}\hat{f})^2)^2 \leq \delta R^{4(r-1)}\mu(\hat{f}^2)^2,\end{aligned}$$

we obtain

$$\mu((P^r f)^4) \leq \mu(f)^4 + 2R^{2r}(3+t)\mu(f)^2\mu(\hat{f}^2) + (1+2/t)\delta R^{4(r-1)}\mu(\hat{f}^2)^2, \quad t>0. \tag{2.2}$$

Letting

$$t = \frac{1}{2}\Big\{\sqrt{9 + 2\delta R^{2r-4} + \delta^2 R^{4r-8}} + \delta R^{2r-4} - 3\Big\},$$

we have

$$\begin{aligned}R^{2r}(3+t) &= \delta R^{4(r-1)}(1+2/t)\\ &= \frac{1}{2}R^{2r}\Big\{3 + \delta R^{2r-4} + \sqrt{9 + 2\delta R^{2r-4} + \delta^2 R^{4r-8}}\Big\} =: h(r).\end{aligned}$$

It suffices to prove that $h(r_0) \leq 1$. Indeed, if $h(r_0) \leq 1$, then it follows from (2.2) that

$$\mu((P^{r_0}f)^4) \leq \{\mu(f)^2 + \mu(\hat{f}^2)\}^2 = 1$$

for all $f \in L^2(\mu)$ with $\mu(f^2) = 1$.

Let $s = 3\delta/(16R^4)$; then $r_0 = \frac{\log(3+s)}{\log R^{-2}}$ and

$$3 + 2s - \frac{\delta}{(3+s)R^4} \geq 3 + \frac{3\delta}{8R^4} - \frac{\delta}{3R^4} > 0.$$

Moreover, $R^{2r_0} = \frac{1}{3+s}$. Then

$$
\begin{aligned}
h(r_0) \le 1 &\iff \frac{1}{6+2s}\left\{3 + \frac{\delta R^{-4}}{3+s} + \left(9 + \frac{2\delta R^{-4}}{3+s} + \frac{\delta^2 R^{-8}}{(3+s)^2}\right)^{1/2}\right\} \le 1 \\
&\iff \left(3 + 2s - \frac{\delta R^{-4}}{3+s}\right)^2 \ge 9 + \frac{2\delta R^{-4}}{3+s} + \frac{\delta^2 R^{-8}}{(3+s)^2} \\
&\iff 4s^2 + 12s \ge \frac{6\delta R^{-4}(1+s)}{3+s} \\
&\iff \frac{12s^2 + 2s^3 + 18s}{1+s} \ge 3\delta R^{-4}.
\end{aligned}
$$

Since $s^2 \ge 2s - 1$, we have

$$12s^2 + 2s^3 + 18s \ge 12s^2 + 2s(2s-1) + 18s = 16s(1+s).$$

Therefore, $h(r_0) \le 1$ follows from the fact that

$$\frac{16s(1+s)}{1+s} = 16s = 3\delta R^{-4}.$$

□

*Proof of Corollary 1.2.* (1) If $\delta(P_t) < 2$ for some $t > 0$, then by Theorem 1.1 one has

$$\mu((P_t f)^2) \le \frac{\sqrt{2\delta(P_t)}}{3\sqrt{2\delta(P_t)} - 4\sqrt{\delta(P_t) - 1}} =: \varepsilon_t < 1, \quad \mu(f) = 0, \quad \mu(f^2) = 1.$$

Thus, it follows from [16, Lemma 2.1] (see also [12, Lemma 2.2]) that

$$\mu((P_s f)^2) \le \mu((P_t f)^2)^{s/t} \mu(f^2)^{1-s/t} \le \varepsilon_t^{s/t}, \quad s \in [0,t], \quad \mu(f) = 0,\ \mu(f^2) = 1.$$

Since all three terms in the above formula are equal when $s = 0$, it follows that

$$-2\mu(fLf) = \frac{\mathrm{d}}{\mathrm{d}s}\mu((P_s f)^2)|_{s=0} \le \frac{1}{t}\log\varepsilon_t, \quad f \in \mathcal{D}(L), \quad \mu(f) = 0, \quad \mu(f^2) = 1.$$

Thus, we arrive at the Poincaré inequality

$$\mu(f^2) - \mu(f)^2 \le -\frac{2t}{\log\varepsilon_t}\mu(fLf), \quad f \in \mathcal{D}(L),$$

and hence (1.3) holds.

(2) If (1.4) holds, then by Gross' theorem one has $\|P_t\|_{L^2(\mu)\to L^4(\mu)} \le \mathrm{e}^{C_2/4}$ for $t := \frac{C_1}{4}\log 3$, see e.g. [2, Theorem 1.5]. Hence (1.5) follows from (1.3). Finally, let $E = \{0,1\}$ with $\mu(0) = \mu(1) = \frac{1}{2}$. Let $L = 0$ (hence the corresponding form $\mathcal{E} = 0$), which does not have a spectral gap. We have, for any function $f$ with $\mu(f^2) = 1$,

$$
\begin{aligned}
\mu(f^2\log f^2) &= \frac{1}{2}\big(f(0)^2\log f(0)^2 + f(1)^2\log f(1)^2\big) \\
&\le \frac{1}{2}\sup_{r\in[0,2]}\big\{r\log r + (2-r)\log(2-r)\big\} = \log 2.
\end{aligned}
$$

Therefore, (1.4) holds for $C_1 = 0$ and $C_2 = \log 2$. □

## 3. Proof of Theorem 1.3

To prove Theorem 1.3 we need the following lemma, which is a dual version of [17, Proposition 3.2] in the setting of infinite reference measures.

**Lemma 3.1.** *Let $(L, \mathcal{D}(L))$ be associated with a Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Assume that there are four constants $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ such that*

$$\mu(f^2) \le \alpha_1 \mathcal{E}(f,f) + \beta_1 \big\{(\sup f)^2 \wedge (\inf f)^2\big\}, \quad f \in \mathcal{D}(L),\ \mu(f) = 0, \tag{3.1}$$

*and*

$$\mu(f^2) \le \alpha_2 \mathcal{E}(f,f) + \beta_2 \mu(|f|)^2, \quad f \in \mathcal{D}(\mathcal{E}). \tag{3.2}$$

*If $\beta_1\beta_2 < 2$, then*

$$\mathrm{gap}(L) \ge \frac{1 - \sqrt{\beta_1\beta_2/2}}{2(\alpha_1 \vee \alpha_2)} > 0.$$

*Proof.* Let $f \in \mathcal{D}(\mathcal{E})$ with $\mu(f) = 0$ and $\mu(f^2) = 1$. For any $r > 0$, let $r' \in [r, \infty]$ be such that either $\mu((f \wedge r) \vee (-r')) = 0$ or $\mu((f \wedge r') \vee (-r)) = 0$. Indeed, if $\mu((f \wedge r) \vee (-r)) \ge 0$, then there exists $R' \in [r, \infty]$ such that $\mu((f \wedge r) \vee (-r')) = 0$ (note that $\mu(f \wedge r) \le 0$ since $\mu(f) = 0$, while if $\mu((f \wedge r) \vee (-r)) < 0$, then there exists $r' \in [r, \infty]$ such that $\mu((f \wedge r') \vee (-r)) = 0$). Without loss of generality, assume that $\mu((f \wedge r) \vee (-r')) = 0$ for some $r' \ge r$. Let $f_r := (f \wedge r) \vee (-r')$. It follows from (3.1) that

$$\mu(f_r^2) \le \alpha_1 \mathcal{E}(f_r, f_r) + \beta_1 r^2. \tag{3.3}$$

Next, one has

$$\begin{aligned} \mu(|f - f_r|) &\le \mu((|f| - r)^+) = \mu((|f| - r)1_{\{|f|>r\}}) \\ &\le \sqrt{\mu(|f| > r)} - r\mu(|f| > r) \le \sup_{s \in (0, r^{-1})} (s - s^2 r) = \frac{1}{4r}. \end{aligned} \tag{3.4}$$

Moreover, one has $\mu((f + r')^-) = \mu((f - r)^+) = \frac{1}{2}\mu(|f - f_r|)$, since $\mu(f - f_r) = 0$. Then it follows from (3.2) and (3.4) that

$$\begin{aligned} \mu((f - f_r)^2) &= \mu({(f - r)^+}^2) + \mu({(f + r')^-}^2) \\ &\le \alpha_2 \big\{\mathcal{E}((f - r)^+, (f - r)^+) + \mathcal{E}((f + r')^-, (f + r')^-)\big\} + \frac{\beta_2}{2}\mu(|f - f_r|)^2 \\ &\le \alpha_2 \big\{\mathcal{E}(f,f) - \mathcal{E}(f_r, f_r)\big\} + \frac{\beta_2}{32 r^2}, \end{aligned} \tag{3.5}$$

where, in the last step, we have used the fact that (see e.g. [5, Lemma 2.2])

$$\mathcal{E}(f_r, f_r) + \mathcal{E}((f - r)^+, (f - r)^+) + \mathcal{E}((f + r')^-, (f + r')^-) \le \mathcal{E}(f,f).$$

Combining (3.3), (3.4) and (3.5), we obtain

$$\begin{aligned} 1 = \mu(f^2) &\le 2\mu(f_r^2) + 2\mu((f - f_r)^2) \\ &\le 2(\alpha_1 \vee \alpha_2)\mathcal{E}(f,f) + 2\beta_1 r^2 + \frac{\beta_2}{16 r^2}. \end{aligned}$$

Letting $r^2 = \frac{1}{4}\sqrt{\beta_2/(2\beta_1)}$, we arrive at

$$\mathcal{E}(f,f) \geq \frac{1-\sqrt{\beta_1\beta_2/2}}{2(\alpha_1 \vee \alpha_2)},$$

and hence the proof is finished. □

*Proof of Theorem 1.3.* Let $f \in L^2(\mu)$ with $\mu(f) = 0$. Applying (1.6) to $f - \inf f$, we obtain

$$\mu(f^2) \leq C_1\mathcal{E}(f,f) + (C_2-1)(\inf f)^2.$$

Similarly, the same holds for $(\sup f)^2$ in place of $(\inf f)^2$. Thus,

$$\mu(f^2) \leq C_1\mathcal{E}(f,f) + (C_2-1)\big\{(\sup f)^2 \wedge (\inf f)^2\big\}, \quad f \in \mathcal{D}(\mathcal{E}),\ \mu(f) = 0. \tag{3.6}$$

Then the first assertion follows from Lemma 3.1.

On the other hand, let $E = \{0,1\}$ with $\mu(0) = \mu(1) = \frac{1}{2}$. Let $L = 0$ (hence the corresponding form $\mathcal{E} = 0$), which does not have a spectral gap. We have, for any function $f$,

$$\mu(f^2) = \frac{1}{2}\big\{f(0)^2 + f(1)^2\big\} \leq \frac{1}{2}(|f(0)| + |f(1)|)^2 = 2\mu(|f|)^2.$$

Hence (1.6) holds for $C_1 = 0$ and $C_2 = 2$. □

## 4. A counterexample: Proof of Theorem 1.4

Consider, for instance, $E_0 := \{0,1\}$ and

$$L_0 f(i) := \frac{1}{2}[f(j) - f(i)], \quad i,j \in E_0,\ j \neq i.$$

Let $\mu_0(0) = \mu_0(1) = \frac{1}{2}$. Then $L_0$ is selfadjoint in $L^2(\mu_0)$ with spectrum $\{0,-1\}$, where the nontrivial eigenfunction is given by $u(0) = -1, u(1) = 1$. Moreover, the corresponding Markov semigroup is determined by

$$P_t^{(0)} f := \mu(f) + \mathrm{e}^{-t}\mu(fu)u, \quad t \geq 0,\ f \in L^2(\mu_0).$$

Now, let $E = E_0^{\mathbb{N}} := \{(x_n)_{n\geq 1} : x_n \in E_0, n \geq 1\}$ and let $\mu = \mu_0^{\mathbb{N}}$ be the product measure. Consider the semigroup

$$P_t := \prod_{n\geq 1} P_t^{(n)}, \quad t \geq 0,$$

where $P_t^{(n)} := P_t^{(0)}$ acting on the $n$-th space. Therefore, $P_t$ is a symmetric $C_0$ Markov semigroup on $L^2(\mu)$ generated by

$$L = \sum_{n=1}^{\infty} L_0^{(n)}$$

with domain $\mathcal{D}(L)$ containing all functions $f \in L^2(\mu)$ such that $\sum_{n=1}^{\infty} L_0^{(n)} f$ converges in $L^2(\mu)$, where $L_0^{(n)}$ stands for the operator $L_0$ acting on the $n$-th variable. It is clear that the spectrum of $L$ contains merely eigenvalues $\{0,-1,-2,\cdots\}$, and the multiplicity of each eigenvalue is infinite. Let $\mathcal{H}_i$ denote the eigenspace w.r.t. the eigenvalue $-i$. One has $\mathcal{H}_0 = \mathbb{R}$ and

$$\mathcal{H}_1 = \mathrm{span}\{f_n : f_n(x) := u(x_n), x \in E, n \geq 1\}.$$

Let $\pi_i : L^2(\mu^{\mathbb{N}}) \to \mathcal{H}_i$ be the orthogonal projection, $i \geq 1$. Then the spectral representation gives

$$Lf = -\sum_{i=1}^{\infty} i\pi_i(f), \quad f \in \mathcal{D}(L).$$

We now define a new selfadjoint operator on $L^2(\mu)$ via the spectral representation

$$\tilde{L}f := -\sum_{i\neq 1} i\pi_i(f) - \sum_{n=1}^{\infty} \frac{1}{n}\mu(ff_n)f_n, \quad f \in \mathcal{D}(\tilde{L}), \tag{4.1}$$

where $\mathcal{D}(\tilde{L})$ contains all $f \in L^2(\mu)$ such that the right-hand side of (4.1) makes sense in $L^2(\mu)$. Let $\tilde{P}_t$ be the corresponding $C_0$-semigroup. It is clear that

$$\tilde{P}_t f = P_t f, \quad f \in \mathcal{H}_1^{\perp}, \tag{4.2}$$

where $\mathcal{H}_1^{\perp}$ denotes the orthogonal complement of $\mathcal{H}_1$.

**Proposition 4.1.** *$\tilde{P}_t$ is a hyperbounded, ergodic, symmetric contraction $C_0$-semigroup without a spectral gap.*

*Proof.* Since $Lf_n = -\frac{1}{n}f_n,\ n \geq 1$, there is no spectral gap. Next, let $f \in L^2(\mu)$ be such that $\tilde{P}_t f = f$ for some $t > 0$, and let $f''$ be the orthogonal projection of $f$ onto $\mathcal{H}_1^{\perp}$. We have $\sum_{n=1}^{\infty}(1 - \mathrm{e}^{-t/n})\mu(ff_n)f_n = 0$, and hence $\mu(ff_n) = 0$ for all $n \geq 1$. Thus, $f = f''$ and $P_t f = \tilde{P}_t f = f$. By the ergodicity of $P_t$, the function $f$ has to be constant. Therefore, $\tilde{P}_t$ is ergodic.

It remains to prove the hyperboundedness of $\tilde{P}_t$. For $f \in L^2(\mu)$ with $\mu(f^2) \leq 1$, one has

$$f = f' + f'' = \sum_{n=1}^{\infty} c_n f_n + f''$$

for some sequence $\{c_n\} \subset \mathbb{R}$ with $\sum_{n=0}^{\infty} c_n^2 = \mu({f'}^2) \leq 1$, where $f'$ and $f''$ denote, respectively, the orthogonal projections of $f$ onto $\mathcal{H}_1$ and $\mathcal{H}_1^{\perp}$. Since $\mu$ is the product measure and since $\mu(f_n) = \mu(f_n^3) = 0$ and $f_n^2 = f_n^4 = 1$ for all $n \geq 1$, it follows from Fatou's lemma that

$$\begin{aligned} \mu((\tilde{P}_t f')^4) &\leq \liminf_{N\to\infty} \int_E \left(\sum_{n=1}^{N} c_n \mathrm{e}^{-t/n} f_n\right)^4 \mathrm{d}\mu \\ &\leq \sum_{n=1}^{\infty} c_n^4 + \sum_{m,n=1}^{\infty} c_m^2 c_n^2 \leq 2\mu({f'}^2)^2. \end{aligned} \tag{4.3}$$

On the other hand, since $P_t^{(0)}$ is hypercontractive, so is $P_t$ (see [7]). Hence there exists $t > 0$ (independent of $f$) such that

$$\mu((\tilde{P}_t f'')^4) = \mu((P_t f'')^4) \leq \mu({f''}^2)^2.$$

Combining this with (4.3), we arrive at

$$\mu((\tilde{P}_t f)^4) \leq 2^3\{\mu((\tilde{P}_t f')^4) + \mu((\tilde{P}_t f'')^4)\} \leq 2^4[\mu({f'}^2) + \mu({f''}^2)]^2 = 2^4\mu(f^2)^2.$$

Therefore, $\tilde{P}_t$ is hyperbounded. □

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