ON CRITERIA FOR EXTREMALITY OF TEICHMÜLLER MAPPINGS

GUOWU YAO

(Communicated by Juha M. Heinonen)

Abstract. Let $f$ be a Teichmüller self-mapping of the unit disk $\Delta$ corresponding to a holomorphic quadratic differential $\psi$. If $\psi$ satisfies the growth condition $A(r, \psi) = \int_{|z|<r} |\psi| dx dy = O((1-r)^{-s})$ (as $r \to 1$), for any given $s>0$, then $f$ is extremal, and for any given $a \in (0,1)$, there exists a subsequence $(n_k)$ of $N$ such that
$$\left\{ \frac{\psi(a^{1/2^k} z)}{\int_{\Delta} |\psi(a^{1/2^k} z)| dx dy} \right\}$$
is a Hamilton sequence. In addition, it is shown that there exists $\psi$ with bounded Bers norm such that the corresponding Teichmüller mapping is not extremal, which gives a negative answer to a conjecture by Huang in 1995.

1. Introduction

Let $\Delta$ be the unit disk $\{|z|<1\}$ in the complex plane $\mathbb{C}$. Suppose $g$ is a quasiconformal self-mapping of $\Delta$. We denote by $Q(g)$ the class of all quasiconformal self-mappings of $\Delta$ that agree with $g$ on the boundary $\partial \Delta$. A quasiconformal mapping $f_0 \in Q(g)$ is said to be an extremal mapping for the boundary values corresponding to $h = g|_{\partial \Delta}$ if it minimizes the maximal dilatations of $Q(g)$, i.e.,
$$K[f_0] = \inf \{ K[f] : f \in Q(g) \},$$
where $K[f]$ is the maximal dilatation of $f$.

A quasiconformal mapping $f(z)$ of $\Delta$ is called a Teichmüller mapping if $f$ has the complex dilatation of the form
$$\mu_f(z) = \frac{f\overline{\psi}}{f\overline{z}} = k \frac{\overline{\psi(z)}}{|\psi(z)|} \quad (0 < k < 1),$$
where $\overline{\psi} \neq 0$ is a holomorphic function in $\Delta$ and $k$ is a constant. It is of interest to know whether $f$ is extremal or, in particular, uniquely extremal among $Q(f)$.

Let $B(\Delta) = \{ \phi : \text{holomorphic in } \Delta \text{ with the norm } ||\phi|| = \int_{\Delta} |\phi(z)| dx dy < \infty \}$. A necessary and sufficient condition that $f$ is extremal is that there exists
a so-called Hamilton sequence, namely, a sequence \( \{ \phi_n \in B(\Delta) : \|\phi_n\| = 1 \} \), such that

\[
\lim_{n \to \infty} \int_{\Delta} k \frac{\varphi(z)}{|\varphi(z)|} \phi_n(z) \, dx \, dy = k.
\]

For the convenience of subsequent discussion, we define

\[
m(r, \varphi) := \frac{1}{2\pi} \int_{0}^{2\pi} |\varphi(re^{i\theta})| \, d\theta
\]

for any holomorphic function \( \varphi \) in \( \Delta \).

In this paper, we pay more attention to the problem: when does \( f \) of a Teichmüller mapping \( f \) have a Hamilton sequence such as \( \varphi(R_nz)/\|\varphi(R_nz)\|, R_n \in (0, 1), \lim_{n \to \infty} R_n = 1 \), and hence \( f \) is extremal?

The problem has been investigated by many authors including Reich and Strebel \cite{7}, Hayman and Reich \cite{1}, Reich \cite{6}, Huang \cite{2}, Wu and Lai \cite{8}, and Yao \cite{10}.

For example, Reich and Strebel \cite{7} proved

**Theorem A.** If \( \varphi(z) \) satisfies the growth condition

\[
m(r, \varphi) = O\left(\frac{1}{1 - r}\right), \quad r \to 1,
\]

then the putative sequence \( \varphi(R_nz)/\|\varphi(R_nz)\| \), \( R \uparrow 1 \) is a Hamilton sequence of \( \mu_f \) and hence \( f \) is extremal. Moreover, the extremality of \( f \) is no longer implied if \( O((1 - r)^{-1}) \) is replaced by \( O((1 - r)^{-s}) \), for any \( s > 1 \).

Furthermore, Hayman and Reich \cite{1} proved that \( f \) is also uniquely extremal if \( \varphi(z) \) satisfies the growth condition \( 1.3 \).

Up to the present, the best growth condition for extremality, due to Wu and Lai \cite{8}, is as follows.

**Theorem B.** Suppose \( \varphi(z) \) satisfies the following growth condition:

\[
m(r, \varphi) = o\left(\frac{1}{(1 - r)^s}\right), \quad r \to 1 \text{ for any given } s > 1.
\]

Then there exists a sequence \( \{ R_n \}, 0 < R_n < 1, \lim_{n \to \infty} R_n = 1 \), such that \( \{ \varphi(R_nz)/\|\varphi(R_nz)\| \} \) is a Hamilton sequence, and hence \( f \) is extremal.

In \cite{3}, Lai and Wu conjectured that \( 1.4 \) is the best possible growth condition for the extremality. But our Theorem \( 1 \) below says that their conjecture is not true in the precise sense.

**Theorem 1.** Set \( \Delta_r = \{ z \in \Delta : |z| < r < 1 \} \),

\[
A(r, \varphi) = \int_{\Delta_r} |\varphi(z)| \, dx \, dy = \int_{0}^{r} \int_{0}^{2\pi} |\varphi(te^{i\theta})| \, d\theta \, dt
\]

Suppose \( \varphi \) satisfies the growth condition:

\[
A(r, \varphi) = O\left(\frac{1}{(1 - r)^s}\right), \quad r \to 1 \text{ for any given } s > 0.
\]

Then for any given \( a \in (0, 1) \), there exists a subsequence \( \{ n_k \} \) of \( \mathbb{N} \) such that \( \{ \varphi(a^{1/2^{n_k}}z)/\|\varphi(a^{1/2^{n_k}}z)\| \} \) is a Hamilton sequence of \( \mu_f \) and hence \( f \) is extremal.
It is clear that (1.4) implies (1.5), but the converse is not true, i.e., (1.5) $\nRightarrow$ (1.4).

Let $BQD(\Delta)$ denote the Banach space consisting of all $\varphi$ holomorphic in $\Delta$ with the Bers norm

\begin{equation}
\|\varphi\|_{\Delta} = \sup_{z \in \Delta} |\rho^{-1}(z)\varphi(z)| < \infty,
\end{equation}

where $\rho(z)|dz|^2 = \frac{4|dz|^2}{(1-|z|^2)^2}$ is the Poincaré metric on $\Delta$.

In [2], Huang posed the following conjecture.

**Extremal Conjecture.** If $\varphi$ belongs to $BQD(\Delta)$, then every Teichmüller mapping corresponding to $\varphi$ is extremal.

As far as we know at present, any $\varphi$ belonging to $BQD(\Delta)$ corresponds to an extremal Teichmüller mapping. There are even $\varphi$ with unbounded Bers norms corresponding to extremal Teichmüller mappings (see an example in Section 2). It seems that the conjecture is true. However, our Theorem 2 gives a negative answer to it.

**Theorem 2.** Let $\Gamma$ be the covering transformation group of a hyperbolic finite type Riemann surface. Then for any $\varphi$ in $BQD(\Delta, \Gamma) \setminus \{0\}$, the Teichmüller mapping corresponding to $\varphi$ is not extremal, where $BQD(\Delta, \Gamma) = \{\varphi \in BQD(\Delta) : \varphi(z) = \varphi(\gamma(z)) \gamma^2(z), \text{ for all } \gamma \in \Gamma\}$.

2. Proof of Theorem 1

Theorem 1 is an immediate corollary of the following theorem:

**Theorem 3.** Suppose there exists some $a \in (0, 1)$ such that

\begin{equation}
A\left(1 + \frac{a_n}{2}\right) = O((1 - a_n)^{-s}), \text{ as } n \to \infty, \text{ for any given } s > 0,
\end{equation}

where $a_n = a^{1/2^n}$.

Then there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that $\{\varphi(a_{n_k}z)/\|\varphi(a_{n_k}z)\|\}$ is a Hamilton sequence and hence $f$ is extremal.

Theorem 3 indicates that a discrete growth condition (2.1) of $\varphi$ is sufficient to induce $f$ extremal. Meanwhile, it also makes clear that a best possible growth condition on $\varphi$ for extremality can hardly be given.

The main idea of the proof of Theorem 3 comes from [8]. We need some preparation before proving it.

For $\frac{1}{2} < t^2 < t < 1$, we write

\begin{equation}
\frac{\int_{\Delta} |\varphi(z)/\varphi(z)| \varphi(tz)dx\,dy}{\int_{\Delta} |\varphi(tz)|} = t^2 + t^2 \frac{\beta(t) + \gamma(t)}{\alpha(t)},
\end{equation}

where

\begin{align*}
\alpha(t) &= A(t, \varphi), \\
\beta(t) &= \int_{|z|<1} \frac{\varphi(z)}{|\varphi(z)|} \varphi(tz)dx\,dy, \\
\gamma(t) &= \int_{\Delta} \frac{\varphi(z)}{|\varphi(z)|} [\varphi(tz) - \varphi(z)]dx\,dy.
\end{align*}
Claim 1. $m(\varphi, \varphi') = \frac{1}{2\pi} \int_0^{2\pi} |\varphi'(ge^{i\theta})| d\theta \leq \frac{R}{\pi s-\varepsilon} m(\varphi, R)$, where $R = \frac{1+n}{n}$.

Actually, by the Cauchy formula

$$\varphi'(ge^{i\theta}) = \frac{1}{2\pi i} \int_{|z|=R} \frac{\varphi(z)}{(z-ge^{i\theta})^2} dz = \frac{R}{2\pi} \int_0^{2\pi} \varphi(Re^{i(t+\theta)}) e^{i(t-\theta)} \frac{dt}{(Re^{it}-\theta)^2},$$

we obtain

$$m(\varphi, \varphi') \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{R m(R, \varphi)}{R^2 - 2R\cos t + \varepsilon^2 dt} = \frac{R m(R, \varphi)}{R^2 - \varepsilon^2}.$$

Claim 2. Set $\zeta(t) = \int_{\frac{1}{2}}^t m(r, \varphi) dr$, $\eta(t) = \int_{\frac{1}{2}}^t \frac{1+t}{1-R} m(R, \varphi) dr$, $\varepsilon(t) = \eta(t) - \eta(t^2)$. Then we have

$$\alpha(t) \geq \pi \zeta(t),$$

$$|\beta(t)| \leq 4\pi [\zeta(t) - \zeta(t^2)],$$

$$|\gamma(t)| \leq 4\pi \varepsilon(t).$$

Inequality (2.4) is obvious. Since $|\beta(t)| \leq \frac{4\pi}{t} \int_0^t m(r, \varphi) dr$, we get (2.3). Using Claim 1 and changing the order of integration, we find

$$|\gamma(t)| \leq \int_{\Delta_t} |\varphi(zt) - \varphi(z)| dx$$

$$\leq \int_0^t r dr \int_0^{2\pi} |\varphi'(ge^{i\theta})| d\theta \leq 2\pi t \int_0^t dr \int_{\theta}^{2\pi} \frac{R m(R, \varphi) d\theta}{R^2 - \varepsilon^2}$$

$$\leq 4\pi \int_0^t dr \int_{\frac{1+t}{1-R}}^{\frac{1+t+2R}{1-R}} \frac{m(R, \varphi) dR}{(3R-1)(1-R)} \leq 4\pi \int_0^t dr \int_{\frac{1+t}{1-R}}^{\frac{1+t+2R}{1-R}} \frac{m(R, \varphi) dR}{1-R}$$

$$= 4\pi \int_0^t dr \int_{\frac{1+t}{1-R}}^{\frac{1+t+2R}{1-R}} \frac{m(R, \varphi) dR}{1-R}$$

$$= 4\pi \int_0^t dr \int_{\frac{1+t}{1-R}}^{\frac{1+t+2R}{1-R}} \frac{1+t-2R}{1-R} m(R, \varphi) dR - 4\pi \int_0^t dr \int_{\frac{1+t}{1-R}}^{\frac{1+t+2R}{1-R}} \frac{1+t+2R}{1-R} m(R, \varphi) dR$$

$$\leq 4\pi |\eta(t) - \eta(t^2)| = 4\pi \varepsilon(t).$$

The second equality in (2.7) comes from changing the order of the first two integrals.

Claim 3. Suppose (2.1) holds. Then there exists a subsequence $\{a_{n_k}\}$ of $N$ such that

$$\lim_{k \to \infty} \frac{\eta(a_{n_k})}{\eta(a_{n_k}^2)} = 1.$$

In fact, it is sufficient to show that $\lim_{n \to \infty} \frac{\eta(a_n)}{\eta(a_n^2)} = 1$ holds. Otherwise, there exists some constant $c > 1$ such that $\eta(a_n) > c^n \eta(a_n^2)$, for all $n \in N$. Then we have $\eta(a_n) > c^n \eta(a_n)$, $n \in N$. By virtue of $\lim_{n \to \infty} a_n = 1$, it is obvious that

$$\eta(a_n) > c^n \eta(a) = \eta(a)(\log a / \log a_n)^{\log_2 c} \sim c(1 - a_n)^{\log_2 c}, n \to \infty,
where $\bar{c} = \eta(a)(\log \frac{1}{a})^{\log_2 c}$. However, (2.1) gives

$$
\eta(a_n) \leq 2a_n \int_{1/2}^{1+a_n} m(r, \varphi)dr \leq 4 \int_{1/2}^{1+a_n} rm(r, \varphi)dr
$$

$$
= \frac{2}{\pi} [A(1+a_n, \varphi) - A(1/2, \varphi)] \leq \frac{2}{\pi} A(1+a_n, \varphi)
$$

$$
= \frac{1}{(1-a_n)^s}, \quad n \to \infty, \quad \text{for any given } s > 0.
$$

This contradicts (2.9), proving our claim.

Claim 4. Suppose $\{a_{n_k}\}$ is obtained from Claim 3. Let $b_k = a_{n_k}$. Then

(2.10) \quad \lim_{k \to \infty} \frac{\varepsilon(b_k)}{\zeta(1+b_k^2/2)} = 0

and

(2.11) \quad \lim_{k \to \infty} \frac{\zeta(b_k^2)}{\zeta(1+b_k^2/2)} = 1.

Since $\eta(b_k^2) \leq 2b_k^2 \int_{1/2}^{1+b_k^2} m(r, \varphi)dr = 2b_k^2 \zeta(1+b_k^2/2)$, we get

$$
\frac{\varepsilon(b_k)}{\zeta(1+b_k^2/2)} = \frac{\eta(b_k^2)}{\zeta(1+b_k^2/2)} \frac{\eta(b_k)}{\eta(b_k^2)} - 1 \leq 2b_k^2 [\frac{\eta(b_k)}{\eta(b_k^2)} - 1].
$$

By (2.8), (2.10) is obtained. Notice that

$$
\frac{t}{1+t} [\zeta(1+t^2/2) - \zeta(t^2)] = \frac{t}{1+t} \int_{t^2}^{1+t^2} m(r, \varphi)dr
$$

$$
\leq \int_{1/2}^{1+t^2} \frac{t}{1-t} m(r, \varphi)dr \leq \int_{1/2}^{1+t^2} \frac{t(1-t)}{1-r} m(r, \varphi)dr
$$

$$
= \int_{1/2}^{1+t^2} \frac{1+t-2r}{1-r} m(r, \varphi)dr - \int_{1/2}^{1+t^2} \frac{1+t^2-2r}{1-r} m(r, \varphi)dr
$$

$$
\leq \int_{1/2}^{1+t^2} \frac{1+t-2r}{1-r} m(r, \varphi)dr - \int_{1/2}^{1+t^2} \frac{1+t^2-2r}{1-r} m(r, \varphi)dr
$$

$$
= \varepsilon(t).
$$

Set $t = b_k$. By virtue of (2.10), we get (2.11).

Now, we complete the proof of Theorem 3. By Claim 2, it suffices to show that

(2.12) \quad \lim_{k \to \infty} \frac{\zeta(b_k) - \zeta(b_k^2)}{\zeta(b_k)} = 0

and

(2.13) \quad \lim_{k \to \infty} \frac{\varepsilon(b_k)}{\zeta(b_k)} = 0.
In view of Claim 4, they can be deduced from
\[
\frac{\zeta(b_k)}{\zeta(b_k)} \leq \frac{\zeta\left(1 + \frac{b_k^2}{2}\right)}{\zeta\left(1 + \frac{b_k^2}{2}\right)} = 1 - \frac{\zeta\left(1 + \frac{b_k^2}{2}\right)}{\zeta\left(b_k^2\right)}
\]
and
\[
\frac{\varepsilon(b_k)}{\zeta(b_k)} \leq \frac{\varepsilon(b_k)}{\zeta\left(1 + \frac{b_k^2}{2}\right)} \cdot \frac{\zeta\left(1 + \frac{b_k^2}{2}\right)}{\zeta(b_k^2)}.
\]
Thus, Theorem 3 follows.

In particular, if \( A(r, \varphi) = O(\log^q \frac{1}{1 - r}) \) as \( r \to 1 \) for some \( q > 0 \), then \( \varphi(z) \) is associated with extremal Teichmüller mappings.

**Example.** Let \( \varphi(z) = \frac{\log^q(1 - z)}{(1 - z)^2}, \quad q > 0 \). The function \( \varphi \) corresponds to extremal Teichmüller mappings since
\[
A(r, \varphi) = \int_0^r t dt \int_0^{2\pi} |\varphi(t e^{i\theta})| d\theta = \int_0^r t dt \int_0^{2\pi} \left| \frac{\log \frac{1 - t e^{i\theta}}{1 - t e^{i\theta}}}{|1 - t e^{i\theta}|^2} \right| d\theta \leq 2\pi \log^{q+1} \frac{1}{1 - r} = o\left(\frac{1}{(1 - r)^s}\right), \quad r \to 1,
\]
for any given \( s > 0 \).

Here, we have chosen a suitable univalent branch for \( \varphi \) in \( \Delta \). Obviously, the Bers norm \( \| \varphi \|_\Delta \) of \( \varphi \) is finite, i.e., \( \varphi \notin BQD(\Delta) \).

**Remark.** Note that \( \varepsilon(t) = \eta(t) - \eta(t^2) \leq \eta(t) - \eta(t^p) \) for \( p \geq 2 \). The same reasoning allows us to take \( a_n = a_1^{1/p} \) in Theorem 3. So, the Hamilton sequence in Theorem 1 can be replaced by \( \{ \varphi(a_1^{1/p} z)/\| \varphi(a_1^{1/p} z)\| \} \) (\( p \geq 2 \)).

Finally, we end this section with the following problem.

**Problem.** Let \( \varphi \) be holomorphic in \( \Delta \). If \( \varphi \) corresponds to an extremal Teichmüller mapping \( f \), can we say that \( \mu_f \) has a Hamilton sequence such as \( \{ \varphi(t_n z)/\| \varphi(t_n z)\| : \lim_{n \to \infty} t_n = 1, \ t_n \in (0, 1) \} \)?

### 3. Proof of Theorem 2

A Riemann surface \( M \) is said to be of finite analytic type \((g, n)\) if and only if \( M \) is obtained from a closed Riemann surface of finite genus \( g \) by deleting \( n \) points, \( n \in \mathbb{N} \). A surface of finite analytic type is hyperbolic if and only if the inequality
\[
3g - 3 + n > 0
\]
holds.

First, we state a result by McMullen in [5]:

**Theorem C.** Let \( f : X \to X' \) be a Teichmüller mapping between Riemann surfaces of hyperbolic finite type. Then the mapping \( \tilde{f} : \Delta \to \Delta \) obtained by lifting \( f \) to the universal covers of \( X \) and \( X' \) is not extremal among quasiconformal mappings with the same boundary values (unless \( f \) is conformal).

---

1 Added in proof. The problem has a negative answer for which the counterexample will be given in [11].
Let $\Gamma$ be the covering transformation group of a hyperbolic Riemann surface $X = \Delta/\Gamma$ of finite type $(g, n)$. Suppose $\tilde{f} : \Delta \to \Delta$ is a Teichmüller mapping with $\mu_{\tilde{f}} = k \frac{v}{|v|}$, where $v \in BQD(\Delta, \Gamma) \setminus \{0\}$. Then $\tilde{f}$ induces a new covering transformation group $\Gamma' = \tilde{f} \circ \Gamma \circ \tilde{f}^{-1}$ which produces a new hyperbolic finite-type Riemann surface $X' = \Delta/\Gamma'$. Therefore, $\tilde{f}$ can be projected to a Teichmüller mapping $f : \Delta/\Gamma \to \Delta/\Gamma'$.

Recall that the Bers space $BQD(X)$ is the space of all holomorphic quadratic differentials $\varphi(z)dz^2$ on $X$ that are bounded in the following sense:

$$||\varphi||_X = \sup_{p \in X} |\varphi(p)|\sigma^{-1}(p) < \infty,$$

where $\sigma(p)$ denotes the Poincaré metric density on $X$. It is well known that $BQD(X)$ is canonically identified with $BQD(\Delta, \Gamma)$. From the Riemann-Roch Theorem it readily follows that $BQD(\Delta, \Gamma)$ is a complex Banach space of $3g - 3 + n$ dimensions. The results on harmonic maps ([4], [9]) also show that there is a proper homeomorphism of $BQD(X)$ onto the Teichmüller space $T(X)$ of $X$. In the sense of not distinguishing $BQD(\Delta, \Gamma)$ from $BQD(X)$, we have $||\varphi||_\Delta = ||\varphi||_X$.

Now, we can conclude that $\tilde{f}$ is not extremal from Theorem C. This completes the proof of Theorem 2.

Combining Theorems 1 and 2, it is not difficult to see that $\varphi \in BQD(\Delta, \Gamma)$ has the property:

Corollary. Let $\Gamma$ be the covering transformation group of a hyperbolic finite-type Riemann surface. Suppose $\varphi(z)$ is in $BQD(\Delta, \Gamma) \setminus \{0\}$. Then

$$\lim_{r \to 1} \frac{A(r, \varphi)}{\log^s(1/(1-r))} = \infty, \text{ for any given } s > 0. \tag{3.1}$$

On the other hand, it is evident that $A(r, \varphi) = O\left(\frac{1}{1-r}\right)$ (as $r \to 1$) for all $\varphi$ in $BQD(\Delta)$.

ACKNOWLEDGMENT

The author would like to thank the referee for valuable comments and suggestions.

REFERENCES


School of Mathematical Sciences, Peking University, Beijing, 100871, People’s Republic of China

E-mail address: wallgreat@lycos.com

Current address: Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100080, People’s Republic of China

E-mail address: gwyao@mail.amss.ac.cn