

ON CRITERIA FOR EXTREMALITY OF TEICHMÜLLER MAPPINGS

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ABSTRACT. Let f be a Teichmüller self-mapping of the unit disk Δ corresponding to a holomorphic quadratic differential φ . If φ satisfies the growth condition $A(r, \varphi) = \iint_{|z| < r} |\varphi| dx dy = O((1-r)^{-s})$ (as $r \rightarrow 1$), for any given $s > 0$, then f is extremal, and for any given $a \in (0, 1)$, there exists a subsequence $\{n_k\}$ of \mathbb{N} such that

$$\left\{ \frac{\varphi(a^{1/2^{n_k}} z)}{\iint_{\Delta} |\varphi(a^{1/2^{n_k}} z)| dx dy} \right\}$$

is a Hamilton sequence. In addition, it is shown that there exists φ with bounded Bers norm such that the corresponding Teichmüller mapping is not extremal, which gives a negative answer to a conjecture by Huang in 1995.

1. INTRODUCTION

Let Δ be the unit disk $\{|z| < 1\}$ in the complex plane \mathbb{C} . Suppose g is a quasiconformal self-mapping of Δ . We denote by $Q(g)$ the class of all quasiconformal self-mappings of Δ that agree with g on the boundary $\partial\Delta$. A quasiconformal mapping $f_0 \in Q(g)$ is said to be an extremal mapping for the boundary values corresponding to $h = g|_{\partial\Delta}$ if it minimizes the maximal dilatations of $Q(g)$, i.e.,

$$K[f_0] = \inf\{K[f] : f \in Q(g)\},$$

where $K[f]$ is the maximal dilatation of f .

A quasiconformal mapping $f(z)$ of Δ is called a Teichmüller mapping if f has the complex dilatation of the form

$$(1.1) \quad \mu_f(z) = \frac{f_{\bar{z}}}{f_z} = k \frac{\overline{\varphi(z)}}{|\varphi(z)|} \quad (0 < k < 1),$$

where $\varphi \neq 0$ is a holomorphic function in Δ and k is a constant. It is of interest to know whether f is extremal or, in particular, uniquely extremal among $Q(f)$.

Let $B(\Delta) = \{\phi : \text{holomorphic in } \Delta \text{ with the norm } \|\phi\| = \iint_{\Delta} |\phi(z)| dx dy < \infty\}$. A necessary and sufficient condition that f is extremal is that [7] there exists

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a so-called Hamilton sequence, namely, a sequence $\{\phi_n \in B(\Delta) : \|\phi_n\| = 1\}$, such that

$$(1.2) \quad \lim_{n \rightarrow \infty} \iint_{\Delta} k \frac{\overline{\varphi(z)}}{|\varphi(z)|} \phi_n(z) dx dy = k.$$

For the convenience of subsequent discussion, we define

$$m(r, \varphi) := \frac{1}{2\pi} \int_0^{2\pi} |\varphi(re^{i\theta})| d\theta$$

for any holomorphic function φ in Δ .

In this paper, we pay more attention to the problem: when does μ_f of a Teichmüller mapping f have a Hamilton sequence such as $\{\varphi(R_n z)/\|\varphi(R_n z)\|\}$, $R_n \in (0, 1)$, $\lim_{n \rightarrow \infty} R_n = 1$, and hence f is extremal?

The problem has been investigated by many authors including Reich and Strebel [7], Hayman and Reich [1], Reich [6], Huang [2], Wu and Lai [8], and Yao [10].

For example, Reich and Strebel [7] proved

Theorem A. *If $\varphi(z)$ satisfies the growth condition*

$$(1.3) \quad m(r, \varphi) = O\left(\frac{1}{1-r}\right), \quad r \rightarrow 1,$$

then the putative sequence $\varphi(Rz)/\|\varphi(Rz)\|$, $R \uparrow 1$ is a Hamilton sequence of μ_f and hence f is extremal. Moreover, the extremality of f is no longer implied if $O((1-r)^{-1})$ is replaced by $O((1-r)^{-s})$, for any $s > 1$.

Furthermore, Hayman and Reich [1] proved that f is also uniquely extremal if $\varphi(z)$ satisfies the growth condition (1.3).

Up to the present, the best growth condition for extremality, due to Wu and Lai [8], is as follows.

Theorem B. *Suppose $\varphi(z)$ satisfies the following growth condition:*

$$(1.4) \quad m(r, \varphi) = o\left(\frac{1}{(1-r)^s}\right), \quad r \rightarrow 1 \text{ for any given } s > 1.$$

Then there exists a sequence $\{R_n\}$, $0 < R_n < 1$, $\lim_{n \rightarrow \infty} R_n = 1$, such that $\{\frac{\varphi(R_n z)}{\|\varphi(R_n z)\|}\}$ is a Hamilton sequence, and hence f is extremal.

In [3], Lai and Wu conjectured that (1.4) is the best possible growth condition for the extremality. But our Theorem 1 below says that their conjecture is not true in the precise sense.

Theorem 1. *Set $\Delta_r = \{z \in \Delta : |z| < r < 1\}$,*

$$A(r, \varphi) = \iint_{\Delta_r} |\varphi(z)| dx dy = \int_0^r t dt \int_0^{2\pi} |\varphi(te^{i\theta})| d\theta.$$

Suppose φ satisfies the growth condition:

$$(1.5) \quad A(r, \varphi) = O\left(\frac{1}{(1-r)^s}\right), \quad r \rightarrow 1 \text{ for any given } s > 0.$$

Then for any given $a \in (0, 1)$, there exists a subsequence $\{n_k\}$ of \mathbb{N} such that $\{\frac{\varphi(a^{1/2^{n_k}} z)}{\|\varphi(a^{1/2^{n_k}} z)\|}\}$ is a Hamilton sequence of μ_f and hence f is extremal.

It is clear that (1.4) implies (1.5), but the converse is not true, i.e., (1.5) $\not\Rightarrow$ (1.4).

Let $BQD(\Delta)$ denote the Banach space consisting of all φ holomorphic in Δ with the Bers norm

$$(1.6) \quad \|\varphi\|_{\Delta} = \sup_{z \in \Delta} |\rho^{-1}(z)\varphi(z)| < \infty,$$

where $\rho(z)|dz|^2 = \frac{4|dz|^2}{(1-|z|^2)^2}$ is the Poincaré metric on Δ .

In [2], Huang posed the following conjecture.

Extremal Conjecture. *If φ belongs to $BQD(\Delta)$, then every Teichmüller mapping corresponding to φ is extremal.*

As far as we know at present, any φ belonging to $BQD(\Delta)$ corresponds to an extremal Teichmüller mapping. There are even φ with unbounded Bers norms corresponding to extremal Teichmüller mappings (see an example in Section 2). It seems that the conjecture is true. However, our Theorem 2 gives a negative answer to it.

Theorem 2. *Let Γ be the covering transformation group of a hyperbolic finite type Riemann surface. Then for any φ in $BQD(\Delta, \Gamma) \setminus \{0\}$, the Teichmüller mapping corresponding to φ is not extremal, where $BQD(\Delta, \Gamma) = \{\varphi \in BQD(\Delta) : \varphi(z) = \varphi(\gamma(z))\gamma'^2(z), \text{ for all } \gamma \in \Gamma\}$.*

2. PROOF OF THEOREM 1

Theorem 1 is an immediate corollary of the following theorem:

Theorem 3. *Suppose there exists some $a \in (0, 1)$ such that*

$$(2.1) \quad A\left(\frac{1+a_n}{2}, \varphi\right) = O((1-a_n)^{-s}), \text{ as } n \rightarrow \infty, \text{ for any given } s > 0,$$

where $a_n = a^{1/2^n}$. Then there exists a subsequence $\{n_k\}$ of \mathbb{N} such that $\{\varphi(a_{n_k}z)/\|\varphi(a_{n_k}z)\|\}$ is a Hamilton sequence and hence f is extremal.

Theorem 3 indicates that a discrete growth condition (2.1) of φ is sufficient to induce f extremal. Meanwhile, it also makes clear that a best possible growth condition on φ for extremality can hardly be given.

The main idea of the proof of Theorem 3 comes from [8]. We need some preparation before proving it.

For $\frac{1}{2} < t^2 < t < 1$, we write

$$(2.2) \quad \frac{\iint_{\Delta} (\overline{\varphi(z)}/|\varphi(z)|)\varphi(tz)dx dy}{\iint_{\Delta} |\varphi(tz)|} = t^2 + t^2 \frac{\beta(t) + \gamma(t)}{\alpha(t)},$$

where

$$\alpha(t) = A(t, \varphi), \quad \beta(t) = \iint_{t < |z| < 1} \frac{\overline{\varphi(z)}}{|\varphi(z)|} \varphi(tz) dx dy,$$

$$\gamma(t) = \iint_{\Delta_t} \frac{\overline{\varphi(z)}}{|\varphi(z)|} [\varphi(tz) - \varphi(z)] dx dy.$$

Claim 1. $m(\varrho, \varphi') = \frac{1}{2\pi} \int_0^{2\pi} |\varphi'(\varrho e^{i\theta})| d\theta \leq \frac{R}{R^2 - \varrho^2} m(\varphi, R)$, where $R = \frac{1+\varrho}{2}$.

Actually, by the Cauchy formula

$$\varphi'(\varrho e^{i\theta}) = \frac{1}{2\pi i} \int_{|z|=R} \frac{\varphi(z)}{(z - \varrho e^{i\theta})^2} dz = \frac{R}{2\pi} \int_0^{2\pi} \frac{\varphi(Re^{i(t+\theta)})e^{i(t-\theta)}}{(Re^{it} - \varrho)^2} dt,$$

we obtain

$$(2.3) \quad m(\varrho, \varphi') \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{R m(R, \varphi)}{R^2 - 2R\varrho \cos t + \varrho^2} dt = \frac{R m(R, \varphi)}{R^2 - \varrho^2}.$$

Claim 2. Set $\zeta(t) = \int_{\frac{1}{2}}^t m(r, \varphi) dr$, $\eta(t) = \int_{\frac{1}{2}}^{\frac{1+t}{2}} \frac{1+t-2R}{1-R} m(R, \varphi) dR$, $\varepsilon(t) = \eta(t) - \eta(t^2)$. Then we have

$$(2.4) \quad \alpha(t) \geq \pi\zeta(t),$$

$$(2.5) \quad |\beta(t)| \leq 4\pi[\zeta(t) - \zeta(t^2)],$$

$$(2.6) \quad |\gamma(t)| \leq 4\pi\varepsilon(t).$$

Inequality (2.4) is obvious. Since $|\beta(t)| \leq \frac{2\pi}{t^2} \int_{t^2}^t r m(r, \varphi) dr$, we get (2.5). Using Claim 1 and changing the order of integration, we find

$$(2.7) \quad \begin{aligned} |\gamma(t)| &\leq \iint_{\Delta_t} |\varphi(tz) - \varphi(z)| dx dy \\ &\leq \int_0^t r dr \int_0^{2\pi} d\theta \int_{tr}^r |\varphi'(\varrho e^{i\theta})| d\varrho \leq 2\pi t \int_0^t dr \int_{tr}^r \frac{R}{R^2 - \varrho^2} m(R, \varphi) d\varrho \\ &\leq 4\pi \int_0^t dr \int_{\frac{1+tr}{2}}^{\frac{1+r}{2}} \frac{R}{(3R-1)(1-R)} m(R, \varphi) dR \leq 4\pi \int_0^t dr \int_{\frac{1+tr}{2}}^{\frac{1+r}{2}} \frac{m(R, \varphi)}{1-R} dR \\ &= 4\pi \int_0^t dr \int_{\frac{1}{2}}^{\frac{1+r}{2}} \frac{m(R, \varphi)}{1-R} dR - 4\pi \int_0^t dr \int_{\frac{1}{2}}^{\frac{1+tr}{2}} \frac{m(R, \varphi)}{1-R} dR \\ &= 4\pi \int_{\frac{1}{2}}^{\frac{1+t}{2}} \frac{1+t-2R}{1-R} m(R, \varphi) dR - 4\pi \int_{\frac{1}{2}}^{\frac{1+t^2}{2}} \frac{1+t^2-2R}{t(1-R)} m(R, \varphi) dR \\ &\leq 4\pi[\eta(t) - \eta(t^2)] = 4\pi\varepsilon(t). \end{aligned}$$

The second equality in (2.7) comes from changing the order of the first two integrals.

Claim 3. Suppose (2.1) holds. Then there exists a subsequence $\{n_k\}$ of \mathbb{N} such that

$$(2.8) \quad \lim_{k \rightarrow \infty} \frac{\eta(a_{n_k})}{\eta(a_{n_k}^2)} = 1.$$

In fact, it is sufficient to show that $\lim_{n \rightarrow \infty} \frac{\eta(a_n)}{\eta(a_n^2)} = 1$ holds. Otherwise, there exists some constant $c > 1$ such that $\eta(a_n) > c\eta(a_n^2)$, for all $n \in \mathbb{N}$. Then we have $\eta(a_n) > c^n \eta(a)$, $n \in \mathbb{N}$. By virtue of $\lim_{n \rightarrow \infty} a_n = 1$, it is obvious that

$$(2.9) \quad \eta(a_n) > c^n \eta(a) = \eta(a) \left(\frac{\log a}{\log a_n}\right)^{\log_2 c} \sim \tilde{c} \left(\frac{1}{1-a_n}\right)^{\log_2 c}, \quad n \rightarrow \infty,$$

where $\tilde{c} = \eta(a)(\log \frac{1}{a})^{\log_2 c}$. However, (2.1) gives

$$\begin{aligned} \eta(a_n) &\leq 2a_n \int_{\frac{1}{2}}^{\frac{1+a_n}{2}} m(r, \varphi) dr \leq 4 \int_{\frac{1}{2}}^{\frac{1+a_n}{2}} rm(r, \varphi) dr \\ &= \frac{2}{\pi} [A(\frac{1+a_n}{2}, \varphi) - A(\frac{1}{2}, \varphi)] \leq \frac{2}{\pi} A(\frac{1+a_n}{2}, \varphi) \\ &= O(\frac{1}{(1-a_n)^s}), \quad n \rightarrow \infty, \text{ for any given } s > 0. \end{aligned}$$

This contradicts (2.9), proving our claim.

Claim 4. Suppose $\{a_{n_k}\}$ is obtained from Claim 3. Let $b_k = a_{n_k}$. Then

$$(2.10) \quad \lim_{k \rightarrow \infty} \frac{\varepsilon(b_k)}{\zeta(\frac{1+b_k^2}{2})} = 0$$

and

$$(2.11) \quad \lim_{k \rightarrow \infty} \frac{\zeta(b_k^2)}{\zeta(\frac{1+b_k^2}{2})} = 1.$$

Since $\eta(b_k^2) \leq 2b_k^2 \int_{\frac{1}{2}}^{\frac{1+b_k^2}{2}} m(r, \varphi) dr = 2b_k^2 \zeta(\frac{1+b_k^2}{2})$, we get

$$\frac{\varepsilon(b_k)}{\zeta(\frac{1+b_k^2}{2})} = \frac{\eta(b_k^2)}{\zeta(1+b_k^2/2)} [\frac{\eta(b_k)}{\eta(b_k^2)} - 1] \leq 2b_k^2 [\frac{\eta(b_k)}{\eta(b_k^2)} - 1].$$

By (2.8), (2.10) is obtained. Notice that

$$\begin{aligned} \frac{t}{1+t} [\zeta(\frac{1+t^2}{2}) - \zeta(t^2)] &= \frac{t}{1+t} \int_{t^2}^{\frac{1+t^2}{2}} m(r, \varphi) dr \\ &\leq \int_{t^2}^{\frac{1+t^2}{2}} \frac{t(1-t)}{1-r} m(r, \varphi) dr \leq \int_{\frac{1}{2}}^{\frac{1+t^2}{2}} \frac{t(1-t)}{1-r} m(r, \varphi) dr \\ &= \int_{\frac{1}{2}}^{\frac{1+t^2}{2}} \frac{1+t-2r}{1-r} m(r, \varphi) dr - \int_{\frac{1}{2}}^{\frac{1+t^2}{2}} \frac{1+t^2-2r}{1-r} m(r, \varphi) dr \\ &\leq \int_{\frac{1}{2}}^{\frac{1+t}{2}} \frac{1+t-2r}{1-r} m(r, \varphi) dr - \int_{\frac{1}{2}}^{\frac{1+t^2}{2}} \frac{1+t^2-2r}{1-r} m(r, \varphi) dr \\ &= \varepsilon(t). \end{aligned}$$

Set $t = b_k$. By virtue of (2.10), we get (2.11).

Now, we complete the proof of Theorem 3. By Claim 2, it suffices to show that

$$(2.12) \quad \lim_{k \rightarrow \infty} \frac{\zeta(b_k) - \zeta(b_k^2)}{\zeta(b_k)} = 0$$

and

$$(2.13) \quad \lim_{k \rightarrow \infty} \frac{\varepsilon(b_k)}{\zeta(b_k)} = 0.$$

In view of Claim 4, they can be deduced from

$$\frac{\zeta(b_k) - \zeta(b_k^2)}{\zeta(b_k)} \leq \frac{\zeta(\frac{1+b_k^2}{2}) - \zeta(b_k^2)}{\zeta(\frac{1+b_k^2}{2})} = 1 - \frac{\zeta(b_k^2)}{\zeta(\frac{1+b_k^2}{2})}$$

and

$$\frac{\varepsilon(b_k)}{\zeta(b_k)} \leq \frac{\varepsilon(b_k)}{\zeta(\frac{1+b_k^2}{2})} \cdot \frac{\zeta(\frac{1+b_k^2}{2})}{\zeta(b_k^2)}.$$

Thus, Theorem 3 follows.

In particular, if $A(r, \varphi) = O(\log^q \frac{1}{1-r})$ as $r \rightarrow 1$ for some $q > 0$, then $\varphi(z)$ is associated with extremal Teichmüller mappings.

Example. Let $\varphi(z) = \frac{\log^q(1-z)}{(1-z)^2}$, $q > 0$. The function φ corresponds to extremal Teichmüller mappings since

$$\begin{aligned} A(r, \varphi) &= \int_0^r t dt \int_0^{2\pi} |\varphi(te^{i\theta})| d\theta \\ &= \int_0^r t dt \int_0^{2\pi} \frac{|\log \frac{1}{1-te^{i\theta}}|^q}{|1-te^{i\theta}|^2} d\theta \leq 2\pi \log^{q+1} \frac{1}{1-r} \\ &= o(\frac{1}{(1-r)^s}), \quad r \rightarrow 1, \text{ for any given } s > 0. \end{aligned}$$

Here, we have chosen a suitable univalent branch for φ in Δ . Obviously, the Bers norm $\|\varphi\|_\Delta$ of φ is infinite, i.e., $\varphi \notin BQD(\Delta)$.

Remark. Note that $\varepsilon(t) = \eta(t) - \eta(t^2) \leq \eta(t) - \eta(t^p)$ for $p \geq 2$. The same reasoning allows us to take $a_n = a^{1/p^n}$ in Theorem 3. So, the Hamilton sequence in Theorem 1 can be replaced by $\{\varphi(a^{1/p^{n_k}} z) / \|\varphi(a^{1/p^{n_k}} z)\|\}$ ($p \geq 2$).

Finally, we end this section with the following problem.¹

Problem. Let φ be holomorphic in Δ . If φ corresponds to an extremal Teichmüller mapping f , can we say that μ_f has a Hamilton sequence such as $\{\varphi(t_n z) / \|\varphi(t_n z)\| : \lim_{n \rightarrow \infty} t_n = 1, t_n \in (0, 1)\}$?

3. PROOF OF THEOREM 2

A Riemann surface M is said to be of finite analytic type (g, n) if and only if M is obtained from a closed Riemann surface of finite genus g by deleting n points, $n \in \mathbb{N}$. A surface of finite analytic type is hyperbolic if and only if the inequality

$$3g - 3 + n > 0$$

holds.

First, we state a result by McMullen in [5]:

Theorem C. *Let $f : X \rightarrow X'$ be a Teichmüller mapping between Riemann surfaces of hyperbolic finite type. Then the mapping $\tilde{f} : \Delta \rightarrow \Delta$ obtained by lifting f to the universal covers of X and X' is not extremal among quasiconformal mappings with the same boundary values (unless f is conformal).*

¹Added in proof. The problem has a negative answer for which the counterexample will be given in [11].

Let Γ be the covering transformation group of a hyperbolic Riemann surface $X = \Delta/\Gamma$ of finite type (g, n) . Suppose $f : \Delta \rightarrow \Delta$ is a Teichmüller mapping with $\mu_{\tilde{f}} = k \frac{\bar{z}}{|z|}$, where $\varphi \in BQD(\Delta, \Gamma) \setminus \{0\}$. Then \tilde{f} induces a new covering transformation group $\Gamma' = \tilde{f} \circ \Gamma \circ \tilde{f}^{-1}$ which produces a new hyperbolic finite-type Riemann surface $X' = \Delta/\Gamma'$. Therefore, \tilde{f} can be projected to a Teichmüller mapping $f : \Delta/\Gamma \rightarrow \Delta/\Gamma'$.

Recall that the Bers space $BQD(X)$ is the space of all holomorphic quadratic differentials $\varphi(z)dz^2$ on X that are bounded in the following sense:

$$\|\varphi\|_X = \sup_{p \in X} |\varphi(p)|\sigma^{-1}(p) < \infty,$$

where $\sigma(p)$ denotes the Poincaré metric density on X . It is well known that $BQD(X)$ is canonically identified with $BQD(\Delta, \Gamma)$. From the Riemann-Roch Theorem it readily follows that $BQD(\Delta, \Gamma)$ is a complex Banach space of $3g - 3 + n$ dimensions. The results on harmonic maps ([4], [9]) also show that there is a proper homeomorphism of $BQD(X)$ onto the Teichmüller space $T(X)$ of X . In the sense of not distinguishing $BQD(X)$ from $BQD(\Delta, \Gamma)$, we have $\|\varphi\|_\Delta = \|\varphi\|_X$.

Now, we can conclude that f is not extremal from Theorem C. This completes the proof of Theorem 2.

Combining Theorems 1 and 2, it is not difficult to see that $\varphi \in BQD(\Delta, \Gamma)$ has the property:

Corollary. *Let Γ be the covering transformation group of a hyperbolic finite-type Riemann surface. Suppose $\varphi(z)$ is in $BQD(\Delta, \Gamma) \setminus \{0\}$. Then*

$$(3.1) \quad \overline{\lim}_{r \rightarrow 1} \frac{A(r, \varphi)}{\log^s(1/(1-r))} = \infty, \text{ for any given } s > 0.$$

On the other hand, it is evident that $A(r, \varphi) = O(\frac{1}{1-r})$ (as $r \rightarrow 1$) for all φ in $BQD(\Delta)$.

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REFERENCES

- [1] W. K. Hayman and E. Reich, *On the Teichmüller mappings of the disk*, Complex Variables Theory Appl. **1** (1982), 1–12. MR **84e**:30031
- [2] X. Z. Huang, *On the extremality for Teichmüller mappings*, J. Math. Kyoto Univ. **35** (1995), 115–132. MR **97a**:30029
- [3] W. C. Lai and Z. M. Wu, *On extremality and unique extremality of Teichmüller mappings*, Chinese Ann. Math. (Ser. B) **16** (1995), 399–406. MR **97b**:30027
- [4] V. Marković, *Harmonic diffeomorphism of noncompact surfaces and Teichmüller spaces*, J. London Math. Soc. **65** (2002), 103–114. MR **2002k**:32015
- [5] C. McMullen, *Amenability, Poincaré series and quasiconformal maps*, Invent. Math. **97** (1989), 95–127. MR **90e**:30048
- [6] E. Reich, *Construction of Hamilton sequences for certain Teichmüller mappings*, Proc. Amer. Math. Soc. **103** (1988), 789–796. MR **89m**:30045
- [7] E. Reich and K. Strebel, *Extremal quasiconformal mappings with given boundary values*, Contributions to Analysis, A Collection of Papers Dedicated to Lipman Bers, Academic Press, New York, 1974, pp. 375–392. MR **50**:13511

- [8] Z. M. Wu and W. C. Lai, *On criterion of the extremality and construction of Hamilton sequences for a class of Teichmüller mappings*, Chinese Ann. Math. (Ser. B) **21** (2000), 339–342. MR **2002c**:30031
- [9] M. Wolf, *The Teichmüller theory of harmonic maps*, J. Differential Geom. **29** (1989), 449–479. MR **90h**:58023
- [10] G. W. Yao, *Hamilton sequences and extremality for certain Teichmüller mappings*, Ann. Acad. Sci. Fenn. Math. **29** (2004), 185–194.
- [11] G. W. Yao, *A remark on Hamilton sequences of extremal Teichmüller mappings*, to appear.

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