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THE STABLE RANK OF FULL CORNERS IN C*-ALGEBRAS

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Abstract. We give a treatment of Rieffel's theory of stable rank for C*-algebras in terms of left invertibility of generalized nonsquare matrices, and prove that if p is a full projection in a unital C*-algebra A, then the stable rank of the corner pAp is at least as large as the stable rank of A.

1. Introduction

An algebraic theory of stable rank in rings was developed by H. Bass [Bas68], primarily to handle cancellation problems in algebraic K-theory. M. Rieffel adapted the theory to C*-algebras (and more general topological algebras) [Rie83]. This theory was formally modeled on dimension theory for compact Hausdorff spaces, but it was quickly realized that stable rank does not resemble a dimension theory very closely in the noncommutative case. The theory has nonetheless proved interesting and useful, particularly with regard to nonstable K-theory questions (cf. [War80].)

Several variations of the theory of stable rank have been developed, such as the theory of real rank of [BP91]. Another variation, which behaves more like a true dimension theory for noncommutative C*-algebras, is the completely positive rank of [Win01].

The principal reason that stable rank does not behave like a dimension theory in the noncommutative case, and also the reason it gives nonstable K-theory information, is its behavior under forming matrix algebras ([Vn71], [Rie83, 6.1]): the stable rank $sr(M_n(A))$ is roughly $sr(A)$ divided by n (see 4.4 for the precise formula). In particular, $sr(M_n(A)) \leq sr(A)$. Since $A \cong pM_n(A)p$ for a "rank-one" projection p (such a subalgebra is called a *corner*), this result suggests that the stable rank of any full corner (not contained in a proper two-sided ideal) pAp in a unital C*-algebra A satisfies $sr(pAp) \geq sr(A)$. This relationship was conjectured at the time of Rieffel's paper nearly twenty years ago, but only partial results have been known (see e.g. [Bla83, A6]).

In this paper, we give an exposition of the basics of the theory of stable rank which differs somewhat from that of [Rie83], and which leads to a positive resolution (4.5) of this "loose end" from the theory, the main new result of the paper. The proofs are for the most part straightforward adaptations of arguments from [Rie83].

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2. Basic definitions

For the convenience of the reader, we recall some standard definitions and facts about stable rank from [Rie83].

Both stable rank and real rank are motivated by the following fact from topology ([HW41, VI.1], [Pea75, 3.3.2]; cf. [Kat50] and [GJ60] for related results).

Theorem 2.1. *Let X be a compact metrizable space. Then all dimension theories coincide on X, and $dim(X)$ can be characterized as the smallest n with the following property: any continuous function $f : X \to \mathbb{R}^{n+1}$ can be uniformly approximated arbitrarily closely by $g : X \to \mathbb{R}^{n+1}$ such that $g(X)$ does not contain the origin in $\mathbb{R}^{n+1}$.*

Definition 2.2. Let A be a unital C*-algebra. Let $Lg_n(A)$ be the set of $(x_1, \ldots, x_n)$ in A^n such that there exists $(y_1, \ldots, y_n) \in A^n$ with $\sum_{i=1}^n y_i x_i = 1$.

The *stable rank* of A, denoted $sr(A)$, is the smallest n such that $Lg_n(A)$ is dense in A^n. If there is no such n, set $sr(A) = \infty$.

If A is nonunital, then $sr(A)$ is defined to be $sr(\tilde{A})$.

The elements of $Lg_n(A)$ are the n-tuples which generate A as a left A-module. It is easily seen that $(x_1, \ldots, x_n) \in Lg_n(A)$ if and only if $\sum_{i=1}^n x_i^* x_i$ is invertible. In particular, if $A = C(X)$, then $(f_1, \ldots, f_n) \in Lg_n(A)$ if and only if for every $t \in X$ there is an i such that $f_i(t) \neq 0$. Thus, since $(f_1, \ldots, f_n) \in A^n$ can be regarded as a continuous function from X to $\mathbb{C}^n \cong \mathbb{R}^{2n}$, we obtain from 2.1 that $sr(C(X)) = \left\lfloor \frac{dim(X)}{2} \right\rfloor + 1$, where $\lfloor \cdot \rfloor$ denotes "integer part of."

The number $sr(A)$ defined in Definition 2.2 is properly called the *left topological stable rank* of A, denoted $ltsr(A)$ in [Rie83]; $Rg_n(A)$ and $rtsr(A)$ can be defined analogously. Because of the involution, there is an obvious correspondence between $Lg_n(A)$ and $Rg_n(A)$, and $ltsr(A) = rtsr(A)$; this number is called $tsr(A)$ in [Rie83] to distinguish it from the Bass stable rank $Bsr(A)$. The topological stable rank of a C*-algebra was shown to coincide with the Bass stable rank in [HV84]; thus we may use the term "stable rank" and the notation $sr(A)$ unambiguously.

3. The generalized matrix picture

Elements of A^n may be regarded as $n \times 1$ matrices over A, and $Lg_n(A)$ becomes the set of left invertible $n \times 1$ matrices, so stable rank can be defined in terms of density of left invertible matrices.

If p and q are projections in a C*-algebra A, it is useful to think of the subspace pAq of A as a space of "nonsquare matrices" with "p rows" and "q columns". If r and s are other projections orthogonal to p and q respectively, then a "$(p+r)\times(q+s)$ matrix" (an element of $(p + r)A(q + s)$) may be symbolically written as a 2×2 "block matrix": write $x \in (p + r)A(q + s)$ as

$$\begin{bmatrix} pxq & pxs \\ rxq & rxs \end{bmatrix}.$$

The algebraic operations in these sets (as subsets of A) can be calculated by formal matrix algebra.

It is convenient to generalize the notion of left invertibility:

Definition 3.1. Let A be a C*-algebra, and let p and q be projections in A. An element $x \in pAq$ is *left invertible* (in pAq or with respect to (p, q)) if there is $y \in qAp$

with $yx = q$. We write $Lg_{(p,q)}(A)$ for the set of left invertible elements of pAq. If p and q are projections in $M_n(A)$, define $Lg_{(p,q)}(A)$ to be $Lg_{(p,q)}(M_n(A))$.

It is easily seen that $x \in pAq$ is left invertible in pAq if and only if x^*x is invertible in qAq. Thus left invertibility with respect to (p,q) really depends only on the q, and if $r \perp p$ and $Lg_{(p,q)}(A)$ is dense in pAq, then $Lg_{(p+r,q)}(A)$ is dense in $(p+r)Aq$.

Recall that if p and q are projections in A, then p and q are *(Murray-von Neumann) equivalent* (in A), written $p \sim q$, if there is a $u \in A$ with $u^*u = p$, $uu^* = q$; q is *subordinate* to p (in A), written $q \precsim p$, if $q \sim q' \leq p$, i.e. if there is a $u \in A$ with $u^*u = q$ and $uu^* \leq p$. We will write $n \cdot q \precsim p$ if there are mutually orthogonal subprojections $p_1, \dots, p_n$ of p, each equivalent to q.

If $p \sim p'$, $q \sim q'$, then there is an obvious isometric isomorphism from pAq onto $p'Aq'$ sending $Lg_{(p,q)}(A)$ onto $Lg_{(p',q')}(A)$, given by $x \mapsto uxv$, where $u^*u = p$, $uu^* = p'$, $vv^* = q$, $v^*v = q'$.

Proposition 3.2. *Let A be a C*-algebra, and p, q projections in A.*

(i) *There exists a left invertible element in pAq if and only if $q \precsim p$.*
(ii) *If $sr(qAq) = n < \infty$ and $n \cdot q \precsim p$, then the left invertible elements of pAq are dense in pAq.*
(iii) *If p is equivalent to a proper subprojection of q, then the left invertible elements of pAq are not dense in pAq.*

Proof. (i) If $u^*u = q$ and $uu^* \leq p$, then $u \in pAq$ and is left invertible. Conversely, if $x \in pAq$ is left invertible, with $y \in qAp$ with $yx = q$, then $q = x^*y^*yx \leq \|y\|^2 x^*x$, so x^*x is invertible in qAq. If $r \in qAq$ with $x^*xr = q$, set $u = xr^{1/2}$. Then $u^*u = q$ and uu^* is a projection in pAp.

(ii) If $p_1, \dots, p_n$ are mutually orthogonal subprojections of p, and u_i satisfies $u_i^*u_i = p_i$ and $u_iu_i^* = q$ for $1 \leq i \leq n$, set $r = p - \sum p_i$. If $x \in pAq$, write $x_i = p_ix$ for $1 \leq i \leq n$ and $x_{n+1} = rx$. Set $y_i = u_ix_i \in qAq$ for $1 \leq i \leq n$. By assumption, $(y_1, \dots, y_n)$ can be approximated by $(z_1, \dots, z_n) \in Lg_n(qAq)$, i.e. with $\sum_{i=1}^n z_i^*z_i$ invertible in qAq. Set $w_i = u_i^*z_i$ for $1 \leq i \leq n$, and $w = \sum_{i=1}^n w_i + x_{n+1}$. Then w closely approximates x, and $w^*w = \sum_{i=1}^n z_i^*z_i + x_{n+1}^*x_{n+1}$ is invertible in qAq, so w is left invertible in pAq.

(iii) We may assume $p \lneqq q$. Suppose $x = pxq \in pAq$ is left invertible and approximates p closely enough that $xp = pxp$ is invertible in pAp. Let $y = qyp \in qAp$ with $yx = q$. If $r = q - p$, then $[ryp][pxp] = ryxp = (q-p)p = 0$, and since pxp is invertible in pAp, $ryp = 0$. But $[ryp][pxr] = ryxr = rqr = r \neq 0$, a contradiction. □

Part (iii) is a version of the well-known fact that a proper isometry in a (unital) C*-algebra cannot be a limit of invertible elements. See [Rør88] for more detailed results along this line.

Recall that a projection p in a C*-algebra A is *infinite* if p is equivalent to a proper subprojection of p; otherwise p is *finite*. The projection p is *properly infinite* if there are subprojections p_1, p_2 of p with $p_1 \perp p_2$ and $p_1 \sim p_2 \sim p$. A unital C*-algebra A is finite [infinite, properly infinite] if 1_A is finite [infinite, properly infinite]. A is finite if and only if every left invertible element of A is invertible.

Corollary 3.3 ([Rie83, 3.1,6.5], [Rob80])**.** *Let A be a unital C*-algebra.*

(i) *If $sr(A) = 1$, then A is finite.*
(ii) *If A is properly infinite, then $sr(A) = \infty$.*

Part (i) follows from 3.2(iii), applied to $p = q = 1_A$; and (ii) follows from 3.2(ii)–(iii), since if A is properly infinite, $n \cdot 1$ is equivalent to a proper subprojection of 1 for every n.

Although a C*-algebra of stable rank 1 must be finite (in fact, stably finite by 4.4), an infinite (unital) C*-algebra need not have infinite stable rank: if T is the Toeplitz algebra, then it is not difficult to show that $sr(T) = 2$ [Rie83, 4.13]. And any stable C*-algebra has stable rank ≤ 2 [Rie83, 6.4].

4. Expanding and contracting matrices

The next two propositions can be used to establish the behavior of stable rank for full corners, as well as obtaining Rieffel's results on the stable rank of matrix algebras. The proofs are adaptations of Rieffel's arguments for the matrix case.

Note that the elements of pAp act on pAq by left multiplication, and left multiplication by an invertible element sends $Lg_{(p,q)}(A)$ onto itself.

Recall that an element in a unital ring of the form $1 + x$, x nilpotent, is called *unipotent*. A unipotent element is invertible.

Proposition 4.1 (cf. [Rie83, 3.4]). *Let A be a C*-algebra, and p, q, and r projections in A with $p \perp r$, $q \perp r$. If $Lg_{(p+r,q+r)}(A)$ is dense in $(p + r)A(q + r)$, then $Lg_{(p,q)}(A)$ is dense in pAq.*

Proof. Let $x \in pAq$. Let $0 < \epsilon < 1$, and approximate $x + r$ within ϵ by an element $y \in Lg_{(p+r,q+r)}(A)$. Then $\|ryq\| < \epsilon$, $\|pyr\| < \epsilon$, and $\|r - ryr\| < \epsilon$. Thus there is an $a \in rAr$ with $\|a\| < (1 - \epsilon)^{-1}$ and $a(ryr) = r$. We then have

$$(p + r - pyra)y(q + r - aryq) = pyq - (pyr)a(ryq) + ryr.$$

Also, $p+r-pyra$ and $q+r-aryq$ are unipotent and hence invertible in $(p+r)A(p+r)$ and $(q + r)A(q + r)$ respectively, and y is left invertible in $(p + r)A(q + r)$, so $pyq-(pyr)a(ryq)+ryr$ is left invertible in $(p+r)A(q+r)$, and hence $pyq-(pyr)a(ryq)$ is left invertible in pAq. But

$$\begin{aligned}\|x - [pyq - (pyr)a(ryq)]\| &\leq \|x - pyq\| + \|pyr\|\|a\|\|ryq\| \\ &< \epsilon + \epsilon^2(1 - \epsilon)^{-1} = \epsilon(1 - \epsilon)^{-1}.\end{aligned}$$

□

Proposition 4.2 (cf. [Rie83, 6.1]). *Let A be a C*-algebra, and p, q, and r projections in A with $p \perp r$, $q \perp r$, $r \precsim n \cdot q$ for some n. If $Lg_{(p,q)}(A)$ is dense in pAq, then $Lg_{(p+r,q+r)}(A)$ is dense in $(p + r)A(q + r)$.*

Proof. We may assume $q \leq p$ by 3.2(i). It suffices to prove the result for $r \sim q$, for then the case where $r \sim (2^n - 1) \cdot q$ follows by induction, and 4.1 then gives the case $r \precsim n \cdot q$. Since $s = p - q + r \sim p$, $Lg_{(s,q)}(A)$ and $Lg_{(s,r)}(A)$ are dense in sAq and sAr respectively. If $x \in (p + r)A(q + r)$ and $\epsilon > 0$, then x can be approximated within $\epsilon/2$ by $y \in (p + r)A(q + r)$ such that $a = syq \in Lg_{(s,q)}(A)$. We will show there is an invertible $z \in (p + r)A(p + r)$ such that $zyq = q$. This will suffice to prove the statement, since by hypothesis there will be $w \in Lg_{(s,r)}(A)$ approximating $szyr$ within $\epsilon/2\|z^{-1}\|$, and then $q + w + qzyr$ will be an element of $Lg_{(p+r,q+r)}(A)$ approximating $q + szyr + qzyr = zyq + (p + r)zyr = zy$ within $\epsilon/2\|z^{-1}\|$, so $z^{-1}(q + w + qzyr) \in Lg_{(p+r,q+r)}(A)$ approximates y within $\epsilon/2$ and therefore approximates x within ϵ.

To find z, first note that if $b \in qAs$ with $ba = q$, then $u = p + r + (q - qyq)b$ is unipotent and hence invertible in $(p+r)A(p+r)$; if $v = p + r - syq$, then v is unipotent in $(p+r)A(p+r)$ and $vuyq = q$. Set $z = vu$. □

The proofs of Propositions 4.1 and 4.2 are easier to follow and understand if elements are written symbolically as matrices, as described earlier.

Note that Proposition 4.2 is not true in general if r is "too large" compared to q: let $\mathcal{H}$ be an infinite-dimensional Hilbert space, $A = \mathcal{B}(\mathcal{H})$, $p = q$ a finite-rank projection, and $r = 1 - p$.

Corollary 4.3. *Let A be a unital C*-algebra and $m \geq 0$. If the left invertible $(m+1) \times 1$ matrices are dense in the $(m+1) \times 1$ matrices over A, then the left invertible $(m+k) \times k$ matrices are dense in the $(m+k) \times k$ matrices over A for all k. Conversely, if the left invertible $(m+k) \times k$ matrices are dense for some k, then the left invertible $(m+1) \times 1$ matrices over A are dense in A^{m+1}.*

Proof. Apply the previous two propositions to $M_{m+k}(A)$ with p, q, and r diagonal projections of rank $m+1$, 1, and $k-1$ respectively. □

Corollary 4.4 ([Rie83, 6.1])**.** *Let A be a C*-algebra. Then, for any n, we have $sr(M_n(A)) = \left\lceil \frac{sr(A)-1}{n} \right\rceil + 1$, where $\lceil\cdot\rceil$ denotes "least integer $\geq$." In particular, $sr(M_n(A)) = 1$ [resp. ∞] if and only if $sr(A) = 1$ [resp. ∞].*

Indeed, the $r \times 1$ matrices over $M_n(A)$ can be identified with the $nr \times n$ matrices over A. Thus, if $sr(A) = m+1$, $Lg_r(M_n(A))$ is dense in $(M_n(A))^r$ if and only if $m + n \leq nr$. (In the nonunital case, one must also show that $sr(M_n(\tilde{A})) = sr(\widetilde{M_n(A)})$; cf. [Rie83, 6.1].)

Recall that a projection p in a C*-algebra A is *full* if it is not contained in any proper closed, two-sided ideal of A. If A is unital and p is full in A, then $1 \precsim n \cdot p$ for some n, in some matrix algebra over A [Cun77].

The main result of this article is another consequence of Proposition 4.2:

Theorem 4.5. *Let A be a unital C*-algebra and p a full projection in A. Then $sr(pAp) \geq sr(A)$. If $sr(A) < \infty$, then $sr(pAp) < \infty$; and $sr(pAp) = 1$ if and only if $sr(A) = 1$.*

Proof. We have that $1 \precsim n \cdot p$ for some n, and hence $1 - p \precsim n \cdot p$. If $m = sr(pAp) < \infty$, then $Lg_{(m\cdot p,p)}(A)$ is dense in $(pAp)^m$, and hence by Proposition 4.2 $Lg_{(m\cdot p+1-p,p+1-p)}(A)$ is dense in the corresponding column space, which may be regarded as a subspace of either A^m or $M_m(A)$ (1 denotes the identity of A). The same is then true for $Lg_{(m\cdot p+m\cdot(1-p),1)}(A) \cong Lg_m(A)$ in A^m, i.e. $sr(A) \leq m$. On the other hand, 1 is equivalent to a full projection in $M_n(pAp)$, so $sr(A) \geq sr(M_n(pAp)) = \left\lceil \frac{sr(pAp)-1}{n} \right\rceil + 1$; in particular, if $sr(pAp) = \infty$, then $sr(A) = \infty$. Also, if $sr(pAp) = 1$, then $sr(A) \leq sr(pAp) = 1$, and conversely if $sr(A) = 1$, then $sr(M_n(pAp)) \leq sr(A) = 1$, and hence $sr(pAp) = 1$ by Corollary 4.4. □

Simple counterexamples show that the hypothesis that p be full is necessary: for example, $A = \mathcal{B}(\mathcal{H})$ for infinite-dimensional $\mathcal{H}$, p a finite-rank projection; or $A = \mathbb{C} \oplus O_2$, $p = (1, 0)$.

What about the nonunital case? It is very plausible that if p is a full projection in A, then $sr(pAp) \geq sr(A)$ even if A is nonunital; however, this does not seem to follow from our arguments except in a special case:

Corollary 4.6. *Let A be a C*-algebra with an approximate unit (q_i) of projections, and let p be a full projection in A. Then $sr(pAp) \geq sr(A)$.*

Proof. We have $sr(A) \leq \liminf sr(q_i A q_i)$, and there is an i_0 such that $p \precsim q_i$ for all $i \geq i_0$, so $sr(pAp) \geq sr(q_i A q_i)$ for $i \geq i_0$. $\square$

It is less clear whether a general full corner in a nonunital C*-algebra A (a hereditary C*-subalgebra of the form pAp for a projection p in the multiplier algebra $M(A)$, with $span(ApA)$ dense in A) also satisfies the inequality. As pointed out by N. Elhage Hassan, the inequality fails for general full hereditary C*-subalgebras: if A is a purely infinite simple unital C*-algebra, e.g. O_2, and B is a nonunital hereditary C*-subalgebra, then B is stable, so $2 = sr(B) < sr(A) = \infty$.

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