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HILBERT-SCHMIDT HANKEL OPERATORS ON THE SEGAL-BARGMANN SPACE

WOLFRAM BAUER

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Abstract. This paper considers Hankel operators on the Segal-Bargmann space of holomorphic functions on $\mathbb{C}^n$ that are square integrable with respect to the Gaussian measure. It is shown that in the case of a bounded symbol $g \in L^\infty(\mathbb{C}^n)$ the Hankel operator H_g is of the Hilbert-Schmidt class if and only if $H_{\bar{g}}$ is Hilbert-Schmidt. In the case where the symbol is square integrable with respect to the Lebesgue measure it is known that the Hilbert-Schmidt norms of the Hankel operators H_g and $H_{\bar{g}}$ coincide. But, in general, if we deal with bounded symbols, only the inequality $\|H_g\|_{HS} \leq 2\|H_{\bar{g}}\|_{HS}$ can be proved. The results have a close connection with the well-known fact that for bounded symbols the compactness of H_g implies the compactness of $H_{\bar{g}}$.

Introduction

Let $n \in \mathbb{N}$ be fixed and let μ denote the Gaussian measure on the complex space $\mathbb{C}^n$ defined by $d\mu(z) = \exp(-|z|^2)dV(z)\pi^{-n}$, where V is the usual Lebesgue measure on $\mathbb{C}^n$.

The *Segal-Bargmann space* $\mathcal{F}$ is the closed subspace of the Hilbert space $L^2(\mathbb{C}^n, \mu)$ of all square integrable holomorphic functions on $\mathbb{C}^n$. Let P denote the orthogonal projection from $L^2(\mathbb{C}^n, \mu)$ onto $\mathcal{F}$. Now, for a function $g \in \mathcal{T}(\mathbb{C}^n)$ (for definition see section 1), the *Toeplitz operator* $T_g : \mathcal{D}(T_g) \subset \mathcal{F} \longrightarrow \mathcal{F}$ and the *Hankel operator* $H_g : \mathcal{D}(H_g) \subset \mathcal{F} \longrightarrow \mathcal{F}^\perp$ are the (in general, unbounded) operators defined by

$$T_g f = P(gf), \qquad H_g f = (I-P)(gf), \qquad f \in \mathcal{D}(T_g).$$

C. Berger and L. Coburn proved ([BC1]) that, when $g \in L^\infty(\mathbb{C}^n)$, the *Hankel operator* H_g is compact if and only if $H_{\bar{g}}$ is compact. Stroethoff ([S1]) gave a necessary and sufficient condition for the compactness of the operators T_g and H_g, and his methods led to an alternate approach to Berger and Coburn's work. It seems to be natural to examine the question if, in general, all *Schatten class Hankel operators* are stable under conjugation of a bounded symbol. J. Xia and D. Zheng proved this in the case of Hilbert-Schmidt *Hankel operators* on the complex plane ([XZ]). Furthermore, they gave a necessary and sufficient condition for *Hankel operators* H_g and $H_{\bar{g}}$ on the *Segal-Bargmann space* to belong simultaneously to the *Schatten class* $\mathcal{C}_p$ for $1 \leq p < \infty$.

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This article examines the case of Hilbert-Schmidt *Hankel operators* in the case of all dimensions and it gives a necessary and sufficient condition under which the *Hankel operator* H_g is of Hilbert-Schmidt type. The main theorem is an analogue to the above-mentioned case of compact *Hankel operators.* Namely, with a symbol $g \in L^\infty(\mathbb{C}^n)$ the operator H_g is of Hilbert-Schmidt type if and only if $H_{\bar{g}}$ is of Hilbert-Schmidt type.

1. Preliminaries

Let $\langle \cdot, \cdot \rangle$ [resp. $|\cdot|$] denote the Euclidean sesquilinear form [resp. the Euclidean norm] on $\mathbb{C}^n$. In the following, $\|\cdot\|_2$ [resp. $\|\cdot\|_{L^p(\mathbb{C}^n,V)}$] means the usual $L^2(\mathbb{C}^n,\mu)$ [resp. $L^p(\mathbb{C}^n,V)$] norm with $1 \leq p \leq \infty$. Furthermore, write $\langle \cdot,\cdot \rangle_2$ for the $L^2(\mathbb{C}^n,\mu)$-scalar product.

Because each *point evaluation* is a continuous functional on $\mathcal{F}$, the *Segal-Bargmann space* is a Hilbert space with *reproducing kernel function* K. For $z, \lambda \in \mathbb{C}^n$ it is easy to check that $K(z,\lambda) = \exp(\langle z, \lambda\rangle)$ and for $f \in \mathcal{F}$ it yields the equality $f(\lambda) = \langle f, K(\cdot,\lambda)\rangle_2$. In the following we also use the *normalized kernel function*

$$k_\lambda(z) := \frac{K(z,\lambda)}{\|K(\cdot,\lambda)\|_2} = \exp\left(\langle z,\lambda\rangle - \frac{1}{2}|\lambda|^2\right).$$

For $j = (j_1,\dots,j_n) \in \mathbb{N}_0^n$ define $z^j := z_1^{j_1}\cdots z_n^{j_n}$ and $j! := j_1!\cdots j_n!$. Consider the monomials $e_j(z) := \frac{z^j}{\sqrt{j!}}$. Then the system $[e_j : j \in \mathbb{N}_0^n]$ forms an orthonormal basis in $\mathcal{F}$. For $\lambda \in \mathbb{C}^n$ define the shift $\tau_\lambda : \mathbb{C}^n \to \mathbb{C}^n$ by $\tau_\lambda(z) := z + \lambda$. Lemma 1.1 shows a connection between the *normalized kernel functions* k_λ and the shifts τ_λ, and it follows with a direct computation.

Lemma 1.1. *Let* $g \in L^1(\mathbb{C}^n,\mu)$. *Then* $(g\circ\tau_\lambda)|k_{-\lambda}|^2$ *is* μ*-integrable and we have*

$$\int_{\mathbb{C}^n} g(z)d\mu(z) = \int_{\mathbb{C}^n} g\circ\tau_\lambda(z)|k_{-\lambda}(z)|^2 d\mu(z)$$

for all $\lambda \in \mathbb{C}^n$.

For $\lambda \in \mathbb{C}^n$ consider the operator $W_\lambda : L^2(\mathbb{C}^n,\mu) \ni f \mapsto k_\lambda[f\circ\tau_{-\lambda}] \in L^2(\mathbb{C}^n,\mu)$. According to Lemma 1.1 the operators W_λ are unitary and they are called *Weyl operators.*

If we define $\mathcal{T}(\mathbb{C}^n) := \{g \in L^2(\mathbb{C}^n,\mu) : g\circ\tau_\lambda \in L^2(\mathbb{C}^n,\mu) \text{ for every } \lambda \in \mathbb{C}^n\}$, then from Lemma 1.1 it follows that a measurable function g on $\mathbb{C}^n$ belongs to $\mathcal{T}(\mathbb{C}^n)$ if and only if the functions $w \mapsto g(w)K(w,\lambda)$ belong to $L^2(\mathbb{C}^n,\mu)$ for every $\lambda \in \mathbb{C}^n$.

Because the linear span of the set of all kernel functions $\{K(\cdot,\lambda) : \lambda \in \mathbb{C}^n\}$ is dense in $\mathcal{F}$ the linear space $\mathcal{D}(T_g) = \mathcal{D}(H_g) := \{h \in \mathcal{F} : gh \in L^2(\mathbb{C}^n,\mu)\}$ is dense in $\mathcal{F}$ if $g \in \mathcal{T}(\mathbb{C}^n)$. Lemma 1.2 below can be found as Proposition 1 in [S1].

Lemma 1.2. *Let* $g \in \mathcal{T}(\mathbb{C}^n)$. *Then we have for all* $\lambda \in \mathbb{C}^n$

(i) $P(gk_\lambda) = [P(g\circ\tau_\lambda)\circ\tau_{-\lambda}]\,k_\lambda$,

(ii) $(I-P)(gk_\lambda) = [(I-P)(g\circ\tau_\lambda)\circ\tau_{-\lambda}]\,k_\lambda$.

Using the notation of the *Weyl operator*, the identities *(i)* and *(ii)* in Lemma 1.2 can be written as $T_g k_\lambda = W_\lambda T_{g\circ\tau_\lambda}1$ and $H_g k_\lambda = W_\lambda H_{g\circ\tau_\lambda}1$. Because W_λ is unitary, we immediately conclude the following corollary.

Corollary 1.1. *Let $g \in \mathcal{T}(\mathbb{C}^n)$. Then we have for all $\lambda \in \mathbb{C}^n$*
(*i*) $\|P(g \circ \tau_\lambda)\|_2 = \|P(gk_\lambda)\|_2$,
(*ii*) $\|(I-P)[g \circ \tau_\lambda]\|_2 = \|(I-P)[gk_\lambda]\|_2$.

2. Proof of the theorems

Theorems 2.1 and 2.2 below correspond to Theorems 5 and 6 in Stroethoff ([S1]). If A is a bounded operator, then denote by $\|A\|$ the operator norm of A. If, in addition, A is a *Hilbert-Schmidt operator*, then we write $\|A\|_{HS}$ for its *Hilbert-Schmidt norm.*

Theorem 2.1. *Let $g \in \mathcal{T}(\mathbb{C}^n)$. Then the following statements (a) and (b) are equivalent:*

(*a*) *H_g is a Hilbert-Schmidt operator,*
(*b*) $\int_{\mathbb{C}^n} \|(I-P)[g \circ \tau_\lambda]\|_2^2 dV(\lambda) < \infty$.

If (a) and (b) are valid, then $\|H_g\|_{HS}^2 = \frac{1}{\pi^n}\int_{\mathbb{C}^n} \|(I-P)[g \circ \tau_\lambda]\|_2^2 dV(\lambda)$.

Theorem 2.2. *Let $g \in \mathcal{T}(\mathbb{C}^n)$. Then the following statements (a) and (b) are equivalent:*

(*a*) *T_g is a Hilbert-Schmidt operator,*
(*b*) $\int_{\mathbb{C}^n} \|P[g \circ \tau_\lambda]\|_2^2 dV(\lambda) < \infty$.

If (a) and (b) are valid, then $\|T_g\|_{HS}^2 = \frac{1}{\pi^n}\int_{\mathbb{C}^n} \|P[g \circ \tau_\lambda]\|_2^2 dV(\lambda)$.

Proof of Theorems 2.1 and 2.2. This follows from $\|(I-P)[g\circ\tau_\lambda]\|_2^2 = \langle H_g^* H_g k_\lambda, k_\lambda\rangle_2$ and $\|P[g \circ \tau_\lambda]\|_2^2 = \langle T_g^* T_g k_\lambda, k_\lambda\rangle_2$ (see Corollary 1.1) and the well-known trace formula for any positive operator T on $\mathcal{F}$:

$$\mathrm{tr}(T) = \frac{1}{\pi^n}\int_{\mathbb{C}^n} \langle Tk_\lambda, k_\lambda\rangle_2 dV(\lambda).$$

(See, for example, Proposition 6.3.2 in [Z1].) □

For each function $g \in \mathcal{T}(\mathbb{C}^n)$ and $\lambda \in \mathbb{C}^n$ define the *Berezin symbol* $\tilde{g}$ of g by the formula

$$\tilde{g}(\lambda) = \frac{1}{\pi^n}\int_{\mathbb{C}^n} g(z)\exp(-|\lambda - z|^2)dV(z) = \langle gk_\lambda, k_\lambda\rangle_2. \tag{1}$$

As an immediate consequence of the change-of-variable formula (see Lemma 1.1) the *Berezin symbol* of g has the form $\tilde{g}(\lambda) = \langle g \circ \tau_\lambda, 1\rangle_2$. Formula (1) shows that $\widetilde{g \circ \tau_\lambda} = \tilde{g} \circ \tau_\lambda$ and $\tilde{\bar{g}} = \bar{\tilde{g}}$.

Lemma 2.1. *Let $g \in \mathcal{T}(\mathbb{C}^n)$ with $g \geq 0$ a.e. Then $g \in L^1(\mathbb{C}^n, V)$ if and only if its Berezin transform $\tilde{g} \in L^1(\mathbb{C}^n, V)$. In this case, $\|g\|_{L^1(\mathbb{C}^n,V)} = \|\tilde{g}\|_{L^1(\mathbb{C}^n,V)}$.*

Proof. Because $g \geq 0$ a.e., then it is obvious that $\tilde{g} \geq 0$. By *Fubini's Theorem*, it follows that, if either g or $\tilde{g}$ are Lebesgue integrable, then

$$\begin{aligned}\infty > \int_{\mathbb{C}^n} \tilde{g}(\lambda)dV(\lambda) &= \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} g(\lambda + z)d\mu(z)dV(\lambda) = \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} g(\lambda+z)dV(\lambda)d\mu(z)\\ &= \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} g(\lambda)dV(\lambda)d\mu(z) = \int_{\mathbb{C}^n} g(\lambda)dV(\lambda).\end{aligned}$$

Now, the desired result follows. □

The next lemma as well as Lemma 2.3 are essential for the proof of our main theorem. We thank the reviewer for some simplifications of the original proofs.

Lemma 2.2. *For $g \in \mathcal{T}(\mathbb{C}^n)$,*

$$\int_{\mathbb{C}^n} \|(I-P)[g \circ \tau_\lambda]\|_2^2 dV(\lambda) = \int_{\mathbb{C}^n} \left(\|P[\bar{g} \circ \tau_z] - \bar{\tilde{g}}(z)\|_2^2 + |g(z) - \tilde{g}(z)|^2 \right) dV(z).$$

The integral on the right-hand side exists if and only if the integral on the left-hand side exists.

Proof. Using Corollary 1.1 and the definition of the *normalized kernel function*, we obtain

$$\|(I-P)[g \circ \tau_\lambda]\|_2^2 = \|(I-P)[gk_\lambda]\|_2^2 = \|(I-P)[gK(\cdot,\lambda)]\|_2^2 \exp(-|\lambda|^2). \tag{2}$$

If we integrate equation (2) with respect to the Lebesgue measure and use *Fubini's Theorem*, we obtain

$$\begin{aligned} \int_{\mathbb{C}^n} &\|(I-P)[g \circ \tau_\lambda]\|_2^2 \frac{dV(\lambda)}{\pi^n} \\ &= \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} |g(z)K(z,\lambda) - P[gK(\cdot,\lambda)](z)|^2 d\mu(z) d\mu(\lambda) \\ &= \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} |\bar{g}(z)K(\lambda,z) - P[\bar{g}K(\cdot,z)](\lambda)|^2 d\mu(\lambda) d\mu(z). \end{aligned} \tag{3}$$

Here we used $\overline{K(z,\lambda)} = K(\lambda,z)$ and $\overline{P[gK(\cdot,\lambda)](z)} = P[\bar{g}K(\cdot,z)](\lambda)$. For each $z \in \mathbb{C}^n$ we have

$$\langle P[\bar{g}K(\cdot,z)], K(\cdot,z)\rangle_2 = \langle \bar{g}K(\cdot,z), K(\cdot,z)\rangle_2 = \bar{\tilde{g}}(z)K(z,z) = \langle \bar{\tilde{g}}(z)K(\cdot,z), K(\cdot,z)\rangle_2;$$

so it follows that $\langle P[\bar{g}K(\cdot,z)] - \bar{\tilde{g}}(z)K(\cdot,z), K(\cdot,z)\rangle_2 = 0$. So, using the *Pythagorean Theorem* and Corollary 1.1, we obtain for the inner integral in (3),

$$\begin{aligned} \int_{\mathbb{C}^n} &|\bar{g}(z)K(\lambda,z) - P[\bar{g}K(\cdot,z)](\lambda)|^2 d\mu(\lambda) \\ &= \int_{\mathbb{C}^n} |\{\bar{g}(z) - \bar{\tilde{g}}(z)\}K(\lambda,z) - \{P[\bar{g}K(\cdot,z)](\lambda) - \bar{\tilde{g}}(z)K(\lambda,z)\}|^2 d\mu(\lambda) \\ &= \int_{\mathbb{C}^n} |\left[\bar{g}(z) - \bar{\tilde{g}}(z)\right] K(\lambda,z)|^2 d\mu(\lambda) \\ &\qquad + \int_{\mathbb{C}^n} |P[\bar{g}K(\cdot,z)](\lambda) - \bar{\tilde{g}}(z)K(\lambda,z)|^2 d\mu(\lambda) \\ &= (|g(z) - \tilde{g}(z)|^2 + \|P[\bar{g} \circ \tau_z] - \bar{\tilde{g}}(z)\|_2^2) \exp(|z|^2). \end{aligned}$$

From this together with (3) and $d\mu(z) = \frac{1}{\pi^n}\exp(-|z|^2)dV(z)$ the assertion follows. □

Corollary 2.1. *Let $g \in \mathcal{T}(\mathbb{C}^n)$ and assume that H_g is a Hilbert-Schmidt operator, then $g - \tilde{g} \in L^2(\mathbb{C}^n, V)$.*

Proof. This follows from Theorem 2.1 and Lemma 2.2. □

Lemma 2.3. *For $g \in \mathcal{T}(\mathbb{C}^n)$,*

$$\int_{\mathbb{C}^n} \|\tilde{g}(\lambda) - P[g \circ \tau_\lambda]\|_2^2 dV(\lambda) = \int_{\mathbb{C}^n} \|\bar{\tilde{g}} \circ \tau_z - P[\bar{g} \circ \tau_z]\|_2^2 dV(z).$$

The integral on the right-hand side exists if and only if the integral on the left-hand side exists.

Proof. Using Corollary 1.1 we obtain

$$\begin{aligned}\|\tilde{g}(\lambda) - P[g \circ \tau_\lambda]\|_2^2 &= \|P[(\tilde{g}(\lambda) - g) \circ \tau_\lambda]\|_2^2 = \|P[(\tilde{g}(\lambda) - g)k_\lambda]\|_2^2 \\ &= \|\tilde{g}(\lambda)k_\lambda - P[gk_\lambda]\|_2^2 = \|\tilde{g}(\lambda)K(\cdot,\lambda) - P[gK(\cdot,\lambda)]\|_2^2 \exp(-|\lambda|^2).\end{aligned}$$

This together with *Fubini's Theorem* and the definition of the *normalized kernel function* imply

$$\begin{aligned}\int_{\mathbb{C}^n} \|\tilde{g}(\lambda) - P[g \circ \tau_\lambda]\|_2^2 \frac{dV(\lambda)}{\pi^n} &= \int_{\mathbb{C}^n} \|\tilde{g}(\lambda)K(\cdot,\lambda) - P[gK(\cdot,\lambda)]\|_2^2 d\mu(\lambda) \\ &= \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} |\tilde{g}(\lambda)K(z,\lambda) - P[gK(\cdot,\lambda)](z)|^2 d\mu(\lambda)d\mu(z) \\ &= \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} |\tilde{\bar{g}}(\lambda)K(\lambda,z) - P[\bar{g}K(\cdot,z)](\lambda)|^2 d\mu(\lambda)d\mu(z) \\ &= \int_{\mathbb{C}^n} \|\tilde{\bar{g}}k_z - P[\bar{g}k_z]\|_2^2 \frac{dV(z)}{\pi^n}.\end{aligned} \tag{4}$$

Here we used $\overline{K(z,\lambda)} = K(\lambda,z)$ and $\overline{P[gK(\cdot,\lambda)](z)} = P[\bar{g}K(\cdot,z)](\lambda)$. Finally, using Corollary 1.1, we observe that

$$\begin{aligned}\|\tilde{\bar{g}}k_z - P[\bar{g}k_z]\|_2^2 &= \|P[(\tilde{\bar{g}} - \bar{g})\, k_z]\|_2^2 + \|(I-P)[\tilde{\bar{g}}k_z]\|_2^2 \\ &= \|P[(\tilde{\bar{g}} - \bar{g}) \circ \tau_z]\|_2^2 + \|(I-P)\,[\tilde{\bar{g}} \circ \tau_z]\,\|_2^2 = \|\tilde{\bar{g}} \circ \tau_z - P[\bar{g} \circ \tau_z]\|_2^2.\end{aligned} \tag{5}$$

The equalities (4) and (5) prove the assertion. □

The proof of Theorem 2.3 below can be found in [XZ], Proposition 1.4. Here we give a different proof, which is more elementary and uses the results above.

Theorem 2.3. *For any $g \in L^2(\mathbb{C}^n, V)$ the Hankel operator H_g is a Hilbert Schmidt operator and we have $\|H_g\|_{HS} = \|H_{\bar{g}}\|_{HS} \leq \pi^{-\frac{n}{2}}\|g\|_{L^2(\mathbb{C}^n,V)}$.*

Proof. Let $g \in L^2(\mathbb{C}^n, V)$. Then using Lemma 2.1 it follows that

$$\begin{aligned}\int_{\mathbb{C}^n} \|(I-P)[g \circ \tau_\lambda]\|_2^2 dV(\lambda) &\leq \int_{\mathbb{C}^n} \|g \circ \tau_\lambda\|_2^2 dV(\lambda) \\ &= \int_{\mathbb{C}^n} \widetilde{|g|^2}(\lambda)dV(\lambda) = \|g\|_{L^2(\mathbb{C}^n,V)}^2 < \infty,\end{aligned}$$

and using Theorem 2.1 we conclude that H_g is of *Hilbert-Schmidt type*. Moreover, we have the inequality $\|H_g\|_{HS} \leq \pi^{-\frac{n}{2}}\|g\|_{L^2(\mathbb{C}^n,V)}$. An analogous computation in connection with Theorem 2.2 shows that T_g is also a *Hilbert-Schmidt operator*.

We conclude that $T_{|g|^2} = H_g^*H_g + T_g^*T_g$ is a *trace class*. Finally, with $T_{\bar{g}} = T_g^*$ we observe that

$$\begin{aligned}\|H_g\|_{HS}^2 &= \operatorname{tr}(H_g^*H_g) = \operatorname{tr}(T_{|g|^2} - T_{\bar{g}}T_g) = \operatorname{tr}(T_{|g|^2}) - \operatorname{tr}(T_{\bar{g}}T_g) \\ &= \operatorname{tr}(T_{|g|^2}) - \operatorname{tr}(T_gT_{\bar{g}}) = \operatorname{tr}(T_{|g|^2} - T_gT_{\bar{g}}) = \operatorname{tr}(H_{\bar{g}}^*H_{\bar{g}}) = \|H_{\bar{g}}\|_{HS}^2.\end{aligned} \tag{6}$$

From this Theorem 2.3 follows. □

For $g \in L^\infty(\mathbb{C}^n)$ define inductively $g^{(1)} := \tilde{g}$ and $g^{(m)} := \widetilde{g^{(m-1)}}$ where $m \in \mathbb{N}$ and $m > 1$. The following lemma is due to Stroethoff ([S1]), and its proof makes use of the boundedness of the symbol g. Lemma 2.4 is essential for the proof of

Theorem 2.4. So Theorem 2.4 is valid only in the case of bounded symbols. At the end of this article we will give an easy example of a symbol $g \in \mathcal{T}(\mathbb{C}^n)$ for which Theorem 2.4 fails.

Lemma 2.4. *Let* $g \in L^\infty(\mathbb{C}^n)$*. Then* $\|H_{g^{(m)}}\| \to 0$ *as* $m \to \infty$*.*

Proof. See [S1], Corollary 9. □

Lemma 2.5. *If* $g \in \mathcal{T}(\mathbb{C}^n)$ *and* H_g *is a Hilbert-Schmidt operator, then the operators* $H_{\tilde{g}}$*,* $T_{g-\tilde{g}}$ *and* $H_{g-\tilde{g}}$ *are Hilbert-Schmidt operators and we have*

$$\|H_g\|^2_{HS} = \|T_{g-\tilde{g}}\|^2_{HS} + \|H_{\tilde{g}}\|^2_{HS} + \frac{1}{\pi^n}\|g-\tilde{g}\|^2_{L^2(\mathbb{C}^n,V)} \tag{7}$$

$$= 2\|T_{g-\tilde{g}}\|^2_{HS} + \|H_{g-\tilde{g}}\|^2_{HS} + \|H_{\tilde{g}}\|^2_{HS}. \tag{8}$$

In particular, we obtain $\|H_{g-\tilde{g}}\|_{HS} \leq \|H_g\|_{HS}$ *and* $\|T_{g-\tilde{g}}\|_{HS} \leq \frac{1}{\sqrt{2}}\|H_g\|_{HS}$*.*

Proof. By Theorem 2.1, the function $\lambda \mapsto \|(I-P)[g\circ\tau_\lambda]\|_2^2$ is integrable with respect to the Lebesgue measure. From Lemma 2.2 and Lemma 2.3, it follows that

$$\begin{aligned}\infty &> \int_{\mathbb{C}^n} \|(I-P)[g\circ\tau_\lambda]\|_2^2 dV(\lambda)\\ &= \int_{\mathbb{C}^n} \|P[\overline{g}\circ\tau_\lambda] - \overline{\tilde{g}}(\lambda)\|_2^2 dV(\lambda) + \int_{\mathbb{C}^n} |g(\lambda)-\tilde{g}(\lambda)|^2 dV(\lambda)\\ &= \int_{\mathbb{C}^n} \|\tilde{g}\circ\tau_\lambda - P[g\circ\tau_\lambda]\|_2^2 dV(\lambda) + \|g-\tilde{g}\|^2_{L^2(\mathbb{C}^n,V)} = (*). \end{aligned} \tag{9}$$

Because $\|\tilde{g}\circ\tau_\lambda - P[g\circ\tau_\lambda]\|_2^2 = \|(I-P)[\tilde{g}\circ\tau_\lambda]\|_2^2 + \|P[(g-\tilde{g})\circ\tau_\lambda]\|_2^2$, Theorems 2.1 and 2.2 in connection with $(*)$ imply

$$\begin{aligned}&\|H_g\|^2_{HS} - \frac{1}{\pi^n}\|g-\tilde{g}\|^2_{L^2(\mathbb{C}^n,V)}\\ &\quad= \frac{1}{\pi^n}\int_{\mathbb{C}^n}\left\{\|(I-P)[\tilde{g}\circ\tau_\lambda]\|_2^2 + \|P[(g-\tilde{g})\circ\tau_\lambda]\|_2^2\right\} dV(\lambda)\\ &\quad= \|H_{\tilde{g}}\|^2_{HS} + \|T_{g-\tilde{g}}\|^2_{HS} = (**).\end{aligned}$$

This proves (7). Now, using Lemma 2.1 and again Theorems 2.1 and 2.2, we obtain

$$\begin{aligned}\|g-\tilde{g}\|^2_{L^2(\mathbb{C}^n,V)} &= \int_{\mathbb{C}^n} \widetilde{|g-\tilde{g}|^2}(\lambda) dV(\lambda) = \int_{\mathbb{C}^n} \|[g-\tilde{g}]\circ\tau_\lambda\|_2^2 dV(\lambda)\\ &= \int_{\mathbb{C}^n} \|P([g-\tilde{g}]\circ\tau_\lambda)\|_2^2 + \|(I-P)([g-\tilde{g}]\circ\tau_\lambda)\|_2^2 dV(\lambda)\\ &= \pi^n\left[\|T_{g-\tilde{g}}\|^2_{HS} + \|H_{g-\tilde{g}}\|^2_{HS}\right].\end{aligned}$$

Formula (8) follows from this and $(**)$. Now, (8) shows that $H_{\tilde{g}}$, $T_{g-\tilde{g}}$ and $H_{g-\tilde{g}}$ are Hilbert-Schmidt operators and the inequalities above follow. □

Corollary 2.2. *If* $g \in L^\infty(\mathbb{C}^n)$ *and* H_g *is a Hilbert-Schmidt operator, then the operators* $H_{g^{(m)}}$*,* $m \in \mathbb{N}$ *are Hilbert-Schmidt operators with* $\|H_{g^{(m)}}\|_{HS} \leq \|H_g\|_{HS}$*.*

Proof. Using Lemma 2.5 m times we obtain with $g^{(0)} := g$,

$$\begin{aligned}\|H_g\|_{HS}^2 &= \frac{1}{\pi^n}\sum_{j=0}^{m-1}\|g^{(j)}-g^{(j+1)}\|^2_{L^2(\mathbb{C}^n,V)}\\ &\quad+\sum_{j=0}^{m-1}\|T_{g^{(j)}-g^{(j+1)}}\|^2_{HS}+\|H_{g^{(m)}}\|^2_{HS}.\end{aligned}\tag{10}$$

This implies Corollary 2.2. □

Remark. I thank Jingbo Xia who has pointed out that the inequality $\|H_{\tilde f}\|_{HS} \leq \|H_f\|_{HS}$ also can be proved by the identity

$$H_{\tilde f} = \int_{\mathbb{C}^n} W_{-z}H_fW_z\,d\mu(z),\tag{11}$$

which follows from Theorem 6 in [BC1]. Now, standard estimates together with formula (11) lead to

$$\|H_{\tilde f}\|_{HS} \leq \int_{\mathbb{C}^n}\|W_{-z}H_fW_z\|_{HS}d\mu(z) \leq \int_{\mathbb{C}^n}\|H_f\|_{HS}d\mu(z) = \|H_f\|_{HS}.$$

Theorem 2.4. *Let $g \in L^\infty(\mathbb{C}^n)$ be such that H_g is a Hilbert-Schmidt operator. Then $H_{\bar g}$ is a Hilbert-Schmidt operator and $\|H_{\bar g}\|_{HS} \leq 2\|H_g\|_{HS}$.*

Proof. If H_g is a Hilbert-Schmidt operator, then by Corollary 2.2 the operators $H_{g^{(m)}}$, $m \in \mathbb{N}$, are Hilbert-Schmidt operators. Hence the operators $H_{g-g^{(m)}} = H_g - H_{g^{(m)}}$ are also Hilbert-Schmidt. Now, Corollary 2.1 shows that $g^{(m-1)}-g^{(m)} \in L^2(\mathbb{C}^n, V)$, where $m \in \mathbb{N}$ and $g^{(0)} := g$. It follows that

$$g - g^{(m)} = (g-g^{(1)}) + (g^{(1)}-g^{(2)}) + \cdots + (g^{(m-1)}-g^{(m)}) \in L^2(\mathbb{C}^n, V).$$

By Theorem 2.3 and Corollary 2.2, we have

$$\|H_{\bar g-\bar g^{(m)}}\|_{HS} = \|H_{g-g^{(m)}}\|_{HS} \leq \|H_g\|_{HS} + \|H_{g^{(m)}}\|_{HS} \leq 2\|H_g\|_{HS}.\tag{12}$$

The boundedness of the numerical sequence $(\|H_{\bar g-\bar g^{(m)}}\|_{HS})_m$ and the convergence $H_{\bar g^{(m)}} \to 0$ (see Lemma 2.4) now imply that $H_{\bar g}$ is also a Hilbert-Schmidt operator. Also, (12) implies that $\|H_{\bar g}\|_{HS} \leq 2\|H_g\|_{HS}$. □

Remark. Let $g \in L^\infty(\mathbb{C}^n)$ and let H_g be a Hilbert-Schmidt operator. Then

$$\|T_{g^{(j)}-g^{(j+1)}}\|_{HS} = \|T^*_{g^{(j)}-g^{(j+1)}}\|_{HS} = \|T_{\bar g^{(j)}-\bar g^{(j+1)}}\|_{HS}, \quad j \in \mathbb{N}_0.$$

This and (10) together imply that

$$\|H_{\bar g}\|^2_{HS} + \lim_{m\to\infty}\|H_{g^{(m)}}\|^2_{HS} = \|H_g\|^2_{HS} + \lim_{m\to\infty}\|H_{\bar g^{(m)}}\|^2_{HS}.$$

The following easy example shows that Theorem 2.4 fails in general for symbols $g \in \mathcal{T}(\mathbb{C}^n)$.

Example. Consider the symbol $g \in \mathcal{T}(\mathbb{C})$ defined by $g(z) = z$. Then obviously $H_g = 0$. Let $e_j := \frac{z^j}{\sqrt{j!}}$, $j \in \mathbb{N}_0$. Then

$$P[\bar g e_j](w) = \frac{1}{\sqrt{j!}}\sum_{l=0}^{\infty}\frac{1}{l!}\langle z^j, z^{l+1}\rangle_2 w^l = \begin{cases}0 & \text{if } j = 0,\\ \sqrt{j}e_{j-1}(w) & \text{if } j > 0.\end{cases}$$

Therefore, we obtain for all $j \in \mathbb{N}_0$,

$$\|H_{\bar{g}} e_j\|_2^2 = \|\bar{g} e_j\|_2^2 - \|P[\bar{g} e_j]\|_2^2 = \|\frac{z^{j+1}}{\sqrt{j!}}\|_2^2 - j = 1.$$

This computation shows that $H_{\bar{g}} \notin \mathcal{L}^{(2)}(\mathcal{F}, \mathcal{F}^{\perp})$.

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DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK, BUFFALO, NEW YORK 14260

E-mail address: `BauerWolfram@web.de`

Current address: Johannes Gutenberg Universität Mainz, Fachbereich Mathematik und Informatik, Staudinger Weg 9, 55128 Mainz, Germany