

ON THE EXISTENCE OF EIGENVALUES OF TOEPLITZ OPERATORS ON PLANAR REGIONS

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ABSTRACT. A study is made of the eigenvalues of self-adjoint Toeplitz operators on multiply connected planar regions having g holes. The presence of eigenvalues is detected through an analysis of the zeros of translations of theta functions restricted to \mathbb{R}^g in \mathbb{C}^g .

INTRODUCTION

The theory of Toeplitz operators on multiply connected planar regions D having g holes acting on the Hardy spaces $H^2(dm_a)$ with respect to harmonic measures m_a (based at a fixed point a in D) has been investigated by M. B. Abrahamse [1]. Among his results is the following example: let D be the annulus $D_r = \{z : r < |z| < 1\}$ with the outer boundary b_0 , inner boundary b_1 and let $\phi \equiv 1$ on b_0 , $\phi \equiv -1$ on b_1 . Let T_ϕ be the corresponding self-adjoint Toeplitz operator on $H^2(dm_a)$. Then the essential spectrum of T_ϕ is given by $\sigma_e(T_\phi) = \{-1, 1\}$. In fact, Abrahamse shows that $-1, 1$ cannot be eigenvalues of T_ϕ and, hence, there exist two sequences $\{\lambda_n^\pm\}$ of eigenvalues in the “gap” $(-1, 1)$ in the range of ϕ such that $\lambda_n^\pm \rightarrow \pm 1$ and $\sigma(T_\phi) = \{-1, 1\} \cup \{\lambda_n^\pm : n = 1, 2, \dots\}$. The explicit determination of λ_n^\pm is in Clancey [3], $\lambda_n^\pm = \frac{q_0 + p_1 r^{2n}}{q_0 - p_1 r^{2n}}$, $n \in \mathbb{Z}$, where q_0 is the pole of the Green’s function for D_r in the interval $(r, 1)$ and p_1 is the critical value of the Green’s function for D_r with pole at q_0 . Self-adjoint Toeplitz operators on multiply connected regions have also been studied by Pincus and Xia [8]. Among other things they give the estimate $\dim \text{Ker}(T_\phi - \lambda) \leq g$, where g is the number of holes.

It is also known (see, e.g., [3]), when the symbol ϕ (not necessarily real) is Hölder continuous on ∂D , that the multiplicity of the zero of an appropriate theta function determines the dimension of $\text{Ker}(T_\phi - \lambda)$.

One interesting problem is to determine a necessary and sufficient condition for the existence of infinitely many eigenvalues of self-adjoint Toeplitz operators T_ϕ in the gaps in the range of the symbol ϕ . This paper is focused on this problem in the cases when $g = 1, 2$. The analysis here uses a resolvent formula for self-adjoint Toeplitz operators on multiply connected planar regions. This resolvent formula is expressed in terms of theta functions associated with double [4; 2]. Consequently, our investigation leads to the study of zeros of the theta functions. Moreover, when

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the genus $g = 2$, we need to give an explicit description of the loci of the zeros of the theta functions on real horizontal planes through half periods. The case when $g = 1$ is resolved completely and a sufficient condition for the case $g = 2$ is provided.

This paper is organized as follows. In Section 1 we will review some preliminaries concerning doubles of multiply connected planar regions as compact Riemann surfaces and their associated theta functions. Moreover, some introductory discussions about Toeplitz operators on the least harmonic majorant Hardy spaces and their resolvent formulae will be made in this section. In Theorem 1 of Section 2 we will present a necessary and sufficient condition for the existence of infinitely many eigenvalues in the gaps in the range of the symbol ϕ . The proof of Theorem 1 will be given in this section. In Section 3 we will give an explicit description of the zeros of the theta functions on real horizontal planes through half periods. In Section 4, when $g = 2$, we will establish a sufficient condition for the existence of infinite sequences of eigenvalues in the gaps.

1. PRELIMINARIES

Let D be a bounded multiply connected planar region having $g \geq 1$ holes with analytic boundaries b_1, \dots, b_g , and let b_0 be the boundary of the unbounded component of the complement of D . Let X be the double of D marked by the symmetric canonical homology basis $a_1, \dots, a_g : b_1, \dots, b_g$, where $a_i = \alpha_i - J\alpha_i$, and α_i is a cross cut from a fixed point p_0 on b_0 to a point on b_i , $i = 1, \dots, g$. Let $\{dw_1, \dots, dw_g\}$ be the basis for the space $\Omega(X)$ of holomorphic differentials dual ($\int_{\alpha_i} dw_j = \delta_{ij}$) to the above canonical homology basis with the “B-period” matrix $\tau = iP$. Let m_a be the harmonic measure based at a fixed point a in D . The least harmonic majorant Hardy space $H^2(dm_a)$ associated with m_a is the closure in $L^2(dm_a)$ of the functions holomorphic on the closure of D . The Hardy spaces $H^2(dm_a)$ are reproducing kernel Hilbert spaces. This means that for a given z in D , there is an element k_z in $H^2(dm_a)$ satisfying $f(z) = \langle f, k_z \rangle$, for any f in $H^2(dm_a)$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $H^2(dm_a)$.

The theta function associated with τ is the entire function on \mathbb{C}^g defined by

$$\theta(\underline{z}) = \theta(\underline{z}, \tau) = \sum_{\underline{n} \in \mathbb{Z}^g} \exp\left\{2\pi i \left(\frac{1}{2} \underline{n}^t \tau \underline{n} + \underline{n}^t \underline{z}\right)\right\}.$$

Here we consider elements $\underline{z} \in \mathbb{C}^g$ and $\underline{n} \in \mathbb{Z}^g$ as column vectors and “ t ” denotes transpose. This function is quasi-periodic in the sense that for $\underline{z} \in \mathbb{C}^g$ and $\underline{m}, \underline{n}$ in \mathbb{Z}^g ,

$$\theta(\underline{z} + \underline{m} + \underline{n}^t \tau) = \exp\left\{2\pi i \left(-\frac{1}{2} \underline{n}^t \tau \underline{n} - \underline{n}^t \underline{z}\right)\right\} \theta(\underline{z}).$$

In particular, θ is \mathbb{Z}^g periodic. The quasi-periodicity of the theta function θ implies the invariance of the zero set θ_0 of θ under translation by elements in the period lattice $\mathbb{Z}^g + \tau \mathbb{Z}^g$. Therefore, $[\theta_0] = \{[\underline{z}] : \theta(\underline{z}) = 0\}$ is a well-defined subvariety of the complex torus $\text{Jac}(X) = \mathbb{C}^g / (\mathbb{Z}^g + \tau \mathbb{Z}^g)$.

Recall that the Abel-Jacobi map Φ_0 based at a fixed point p_0 on b_0 is a map from X to the Jacobian variety $\text{Jac}(X)$ defined by

$$\Phi_0(p) = \int_{p_0}^p d\vec{w} \bmod (\mathbb{Z}^g + \tau \mathbb{Z}^g),$$

where $d\vec{w} = (dw_1, \dots, dw_g)^t$. Since the B-period matrix of the marked double X has the form $\tau = iP$, where P is a real $g \times g$ positive definite symmetric matrix, the antiholomorphic involution map J defined on the Jacobian variety by $J([\underline{z}]) = -[\underline{z}]$ is a well-defined map.

Let W_d denote the image in $\text{Jac}(X)$ under Φ_0 of the collection of non-negative divisors of degree d (≥ 1), and let $W_0 = \{0\}$. The following theorem of Riemann [6] characterizes the zero locus of the theta function associated with τ : When the genus of the marked double X is $g \geq 1$, $[\theta_0]$ satisfies $[\theta_0] = W_{g-1} + [\Delta_0]$, where Δ_0 is a constant called the Riemann constant. The explicit form of $[\Delta_0]$ is provided below.

Let P denote the orthogonal projection of $L^2(dm_a)$ onto $H^2(dm_a)$. Given ϕ in $L^\infty(dm_a)$ one defines the Toeplitz operator T_ϕ with the symbol ϕ on $H^2(dm_a)$ by

$$T_\phi f = P(\phi f), \quad f \in H^2(dm_a).$$

The Toeplitz operator T_ϕ is self-adjoint if and only if the symbol ϕ is real-valued. The resolvent formula for the self-adjoint Toeplitz operators is given in [4] as follows. Let ϕ be a real-valued element in $L^\infty(dm_a)$, and let z be given in D . Then for the reproducing kernel k_z in $H^2(dm_a)$ and for any complex number λ satisfying $\text{Im}\lambda \neq 0$, there holds

$$\begin{aligned} & \langle (T_\phi - \lambda)^{-1} k_z, k_z \rangle \\ &= C(z) \frac{\theta(\omega_z + t_a - \frac{1}{2\pi i} \int_{\partial D} \text{Log}(\phi - \lambda) d\vec{w})}{\theta(t_a - \frac{1}{2\pi i} \int_{\partial D} \text{Log}(\phi - \lambda) d\vec{w})} e^{-\int_{\partial D} \text{Log}(\phi - \lambda) dm_z}, \end{aligned}$$

where t_a is a constant real number and $C(z)$, ω_z are constants depending only on z .

We need here to mention the fact that $d\vec{w} = \frac{1}{2}d\vec{\omega} + \frac{i}{2} * d\vec{\omega} = \frac{i}{2} * d\vec{\omega}$, on ∂D , where $\omega_j = \omega_j(z)$ ($0 \leq j \leq g$) is the harmonic measure of b_j based at z . We will denote the signed measure $\frac{-1}{2} * d\omega_j$ by $d\nu_j$. In fact, the holomorphic differential $d\omega_j$ is the reflection of the holomorphic differential $\partial\omega_j dz$ from \overline{D} on the double X .

We close this section with the remark that the theta function associated with a double of genus g never vanishes in \mathbb{R}^g . This is a result of Fay [7, p. 118]. Moreover, in the case $g = 1$ the associated theta function vanishes only at the point $\frac{1}{2} + \frac{1}{2}\tau$ and its translations by $\mathbb{Z} + \tau\mathbb{Z}$.

It will be convenient to use the following terminology. Given ϕ in $L^\infty(dm_a)$ the open intervals forming the bounded components of the complement of the essential range of ϕ will be called *gaps*.

2. THEOREM 1

From the functional calculus for a self-adjoint operator it follows that the isolated singularities of the resolvent are eigenvalues. Moreover, if T is a self-adjoint operator with eigenvector f corresponding to the eigenvalue $z = a$, then whenever $\langle f, k \rangle$ is non-zero, the restriction of T to the smallest invariant subspace containing k will have eigenvalue $z = a$. Since k_z , $z \in D$ is dense in $H^2(dm_a)$, one can see from the resolvent formula that the zeros of the function

$$f(t) = \theta(t_a - \frac{1}{2\pi i} \int_{\partial D} \text{Log}(\phi - t) d\vec{w})$$

in a gap are precisely the eigenvalues of T_ϕ in the gap. We can use the work of Riemann to describe in \mathbb{C}^g the zero subvariety for the theta function θ associated with the marked double X of genus $g \geq 1$. Moreover, since the Hardy spaces are conformally invariant, to study the existence of eigenvalues of the self-adjoint Toeplitz operators T_ϕ on the Hardy spaces $H^2(dm_a)$ of the g -holed D , one can replace the region D by its conformal model, which can be chosen as a circular region obtained from the unit disc by removing g disjoint closed discs. When $g = 1$, one can choose the region D to be an annulus.

Theorem 1. *Suppose that D is the annulus $D_r = \{z : r < |z| < 1\}$, and suppose that $\phi \in L^\infty(dm_a)$, $\phi = \bar{\phi}$. Let $m_i = \text{essinf}_{t \in b_i} \phi(t)$, $M_i = \text{esssup}_{t \in b_i} \phi(t)$, $i = 0, 1$. Assume that the intervals $[m_0, M_0]$ and $[m_1, M_1]$ are disjoint. Without loss of generality, we assume that $M_1 < m_0$. Then there exist sequences $\{\lambda_n\}_{n=1}^\infty$ of eigenvalues in the gap (M_1, m_0) converging to M_1 (respectively, m_0) if and only if*

$$\int_0^{2\pi} \log|\phi(re^{i\theta}) - M_1| \frac{d\theta}{2\pi} = -\infty \text{ (respectively, } \int_0^{2\pi} \log|\phi(e^{i\theta}) - m_0| \frac{d\theta}{2\pi} = -\infty).$$

Proof. When t is in the gap (M_1, m_0) of the spectrum of T_ϕ , the function $f(t) = \theta(t_a - \frac{1}{2\pi i} \int_{\partial D} \text{Log}(\phi - t) dw)$ associated with the function in the denominator of the resolvent formula is in the form $f(t) = \theta(x(t) + \frac{1}{2}\tau)$, where $x(t) = t_a + \frac{1}{2\pi} \int_{\partial D} \log|\phi - t| dv$, $t \in (\text{gap})$. Therefore, one needs to study the intersections of zero locus of the theta function θ in the real hyperplane $P = \{z = x + \frac{1}{2}\tau : x \in \mathbb{R}\}$ with the path $x(t) + \frac{1}{2}\tau$. The zero locus of θ in P is the set of $m + \frac{1}{2} + \frac{1}{2}\tau$, $m \in \mathbb{Z}$. Thus, the proof reduces to the study of a necessary and sufficient condition for the validity of the equation $x(t) = m + \frac{1}{2}$ infinitely often. One can see that $\int_{b_1} \log|\phi - M_1| dv = -\infty$ if and only if $\int_0^{2\pi} \log|\phi(re^{i\theta}) - M_1| \frac{d\theta}{2\pi} = -\infty$. Continuity of the path $x(t)$ implies that as t approaches M_1 from the right, the equation $x(t) = m + \frac{1}{2}$ will hold infinitely often if and only if $\int_0^{2\pi} \log|\phi(re^{i\theta}) - M_1| \frac{d\theta}{2\pi} = -\infty$. Similarly, as t approaches m_0 from the left, the path $x(t)$ will hit the zero locus of θ infinitely often. The proof of the theorem is complete. \square

3. PROPOSITION 1

If the region D is obtained from the unit disc by removing g disjoint closed discs centered on the real axis, the double X of such a region has the extra anti-conformal map $Q : X \rightarrow X$ defined by reflection in the real axis. In such a case the Riemann constant Δ_0 for X based at $p_0 = -1$ has the explicit form

$$\Delta_0 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \frac{1}{2}\tau \begin{bmatrix} g \\ g-1 \\ \vdots \\ 1 \end{bmatrix}.$$

See [5, p. 310], or [2, p. 31] for a detailed computation. When g is 2 it is always possible to conformally map D to such a ‘‘conjugate symmetric’’ circular region. In particular, the Riemann constant Δ_0 for the double X of genus two can be assumed to have the form

$$(*) \quad \Delta_0 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\tau \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

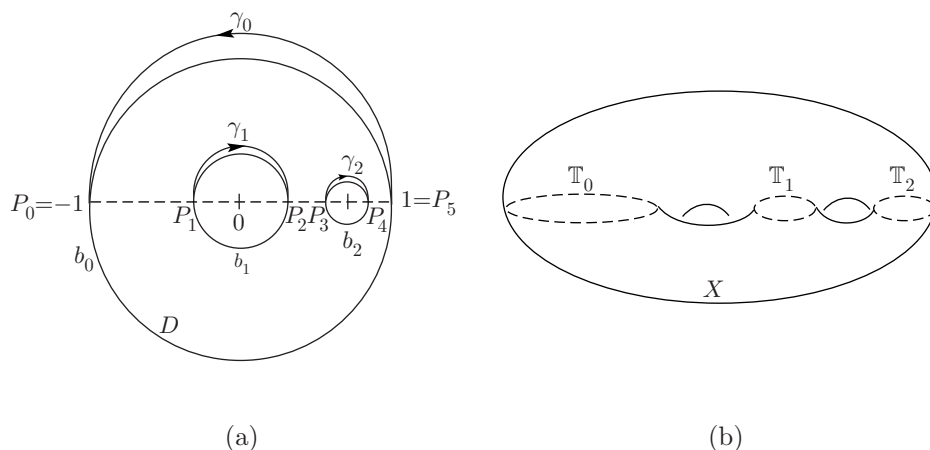


FIGURE 1.

The result we will describe here is for genus $g = 2$. We choose D to be as in Figure 1(a). The notation $\mathbb{T}_0, \mathbb{T}_1$ and \mathbb{T}_2 will be used for the “circles” in X formed by doubling the intervals $I_0 = (p_0, p_1), I_1 = (p_2, p_3)$ and $I_2 = (p_4, p_5)$, respectively. See Figure 1(b).

We will see that we need only to look for zeros of theta functions in the real horizontal planes $P_{\underline{\lambda}} = \{z = \underline{x} + \frac{1}{2}\tau\underline{\lambda} : \underline{x} \in \mathbb{R}^2\}$ through the half periods $\frac{1}{2}\tau\underline{\lambda}, \underline{\lambda} \in \mathbb{Z}^2/2\mathbb{Z}^2$. Since $\underline{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \in \mathbb{Z}^2/2\mathbb{Z}^2$, there are 2^2 of these planes. Moreover, θ maps \mathbb{R}^2 into \mathbb{R}^* . Therefore, the zero locus corresponding to $\underline{\lambda} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the empty set. Thus, it remains to study 3 of these planes. Since theta functions are quasi-periodic, this leads to the study of the zeros of only the following functions in \mathbb{R}^2 :

- (I) $F(\underline{x}) = \theta(\underline{x} + \frac{1}{2}\tau \begin{bmatrix} 0 \\ 1 \end{bmatrix}), \quad \underline{x} \in \mathbb{R}^2;$
- (II) $G(\underline{x}) = \theta(\underline{x} + \frac{1}{2}\tau \begin{bmatrix} 1 \\ 0 \end{bmatrix}), \quad \underline{x} \in \mathbb{R}^2;$
- (III) $H(\underline{x}) = \theta(\underline{x} + \frac{1}{2}\tau \begin{bmatrix} 1 \\ 1 \end{bmatrix}), \quad \underline{x} \in \mathbb{R}^2.$

The discussion below will show that the zero loci of the above functions $F, G,$ and H are as in the following Figures 2(a), 2(b), and 2(c), respectively.

One determines the zero set of F using the following:

Proposition 1. *The complete zero set of the function F given by (I) is the union of the arcs with the following parameterization:*

$$\begin{bmatrix} \frac{1}{2}\omega_1(p) + k + \frac{1}{2} \\ \frac{1}{2}\omega_2(p) + \frac{1}{2} \end{bmatrix}, \quad k \in \mathbb{Z},$$

where p varies over \mathbb{T}_0 .

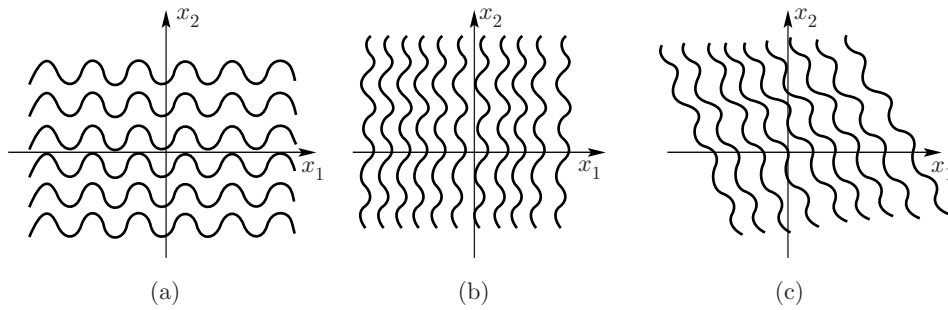


FIGURE 2.

Proof. Referring to (*), $F(\underline{x}) = \theta(\underline{x} + \frac{1}{2}\tau \begin{bmatrix} 0 \\ 1 \end{bmatrix}) = 0 \Leftrightarrow \theta(\underline{x} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}) + \Delta_0 = 0$. By Riemann’s characterization of zero set of theta (Theorem VI.3.1 in [6]), this last condition is equivalent to $\underline{x} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \Delta_0 = \int_{p_0}^p d\vec{w} \pmod{\mathbb{Z}^2 + \tau\mathbb{Z}^2} + \Delta_0$, for some p . Cancelling out Δ_0 from both sides, the last condition is equivalent to $\int_{p_0}^p d\vec{w} \in \mathbb{R}^2/(\mathbb{Z}^2 + \tau\mathbb{Z}^2)$, for some path from p_0 to p . Using the fact that $d\vec{w} = \frac{1}{2}(d\vec{\omega} + i * d\vec{\omega})$, we see that this last condition is equivalent to $[\int_{p_0}^p *d\vec{\omega}] = [0]$, for some path from p_0 to p . If it is so, then $\underline{x} = \text{Re}(\int_{p_0}^p d\vec{w} + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}) \pmod{\mathbb{Z}^2}$. If the path of integration is taken along the loop \mathbb{T}_0 , Figure 1(b), then $\int_{p_0}^p d\vec{w}$ is a real vector (and hence $\int_{p_0}^p *d\vec{\omega} = \underline{0}$). When p traces the loop \mathbb{T}_0 for the first time,

$$\int_{p_0}^p d\vec{w} = \frac{1}{2} \begin{bmatrix} \omega_1(p) - \omega_1(p_0) \\ \omega_2(p) - \omega_2(p_0) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \omega_1(p) \\ \omega_2(p) \end{bmatrix} \in \mathbb{R}^2,$$

and when p traces \mathbb{T}_0 for the k th time, $k \in \mathbb{Z}^*$ (here, $k < 0$ denotes $-k$ revolutions in the negative direction),

$$\int_{p_0}^p d\vec{w} = \frac{1}{2} \begin{bmatrix} \omega_1(p) \\ \omega_2(p) \end{bmatrix} + \begin{bmatrix} k \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\omega_1(p) + k \\ \frac{1}{2}\omega_2(p) \end{bmatrix}.$$

Since $\vec{\omega}(p)$ is a continuous map, the image of p under the map $\int_{p_0}^p d\vec{w}$ is a curve with the parameterization $\begin{bmatrix} \frac{1}{2}\omega_1(p) + k \\ \frac{1}{2}\omega_2(p) \end{bmatrix}$, $k \in \mathbb{Z}^*$. Therefore, the translation of the image of $\int_{p_0}^p d\vec{w}$ by $\frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has the parameterization

$$\begin{bmatrix} \frac{1}{2}\omega_1(p) + k + \frac{1}{2} \\ \frac{1}{2}\omega_2(p) + \frac{1}{2} \end{bmatrix}, k \in \mathbb{Z}.$$

Note that the second component, $\frac{1}{2}\omega_2(p) + \frac{1}{2}$ has the same behavior in each revolution. Furthermore, $0 \leq \frac{1}{2}\omega_2(p) + \frac{1}{2} \leq 1$ for all p . Thus, this curve lies in a horizontal strip of one unit width. By plotting the points $\begin{bmatrix} \frac{1}{2}\omega_1(p_0) + \frac{1}{2} \\ \frac{1}{2}\omega_2(p_0) + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$,

$\begin{bmatrix} \frac{1}{2}\omega_1(p_1) + \frac{1}{2} \\ \frac{1}{2}\omega_2(p_1) + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$, and $\begin{bmatrix} \frac{1}{2}\omega_1(p_0) + 1 + \frac{1}{2} \\ \frac{1}{2}\omega_2(p_0) + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$, using the above parameterization and \mathbb{Z}^2 periodicity of F , we get a portion of the zero divisor of F in \mathbb{R}^2 as in Figure 2(a). We claim that this locus is the complete locus of all zeros of F . To establish this, it is sufficient to show that $\int_{p_0}^p *d\vec{\omega} \neq \underline{0}$, whenever p is off the loop \mathbb{T}_0 . The function $\tilde{\omega}_1(p) = \int_{p_0}^p *d\omega_1$ is a single-valued continuous function on \overline{D} and harmonic in $\mathcal{G} = D \setminus \{I_1 \cup I_2\}$ if the path of integration does not go around any boundary components. Moreover, it is zero on I_0 . For the remainder of the proof we use the notation $\gamma_0, \gamma_1, \gamma_2$ as in Figure 1(a). Under the above condition on the path of integration, since the restriction to the boundary ∂D of the differential $*d\omega_1 = \frac{\partial\omega_1}{\partial\eta} ds$, and $\frac{\partial\omega_1}{\partial\eta} > 0$ on $\gamma_1, \frac{\partial\omega_1}{\partial\eta} < 0$ on γ_0, γ_2 , one can see that $\tilde{\omega}_1(p) \geq 0$ on $I_0 \cup \gamma_1 \cup I_1 \cup \gamma_2 \cup I_2 \cup \gamma_0$. Note that the orientation of the boundary is consistent with the orientation for the arc length. Thus,

$$\int_{p_0}^{p_5} *d\omega_1 = - \int_{\gamma_0} *d\omega_1 = - \int_{\gamma_0} \frac{\partial\omega_1}{\partial\eta} ds > 0.$$

Therefore, using the maximum principle, $\tilde{\omega}_1$ never vanishes on \mathcal{G} intersected with the upper half-plane. Using the symmetry of the region and the anti-conformal involution map J , one can show that $\tilde{\omega}_1$ will never vanish off the loop \mathbb{T}_0 . This proves that the above zero locus is the complete zero divisor of $F(\underline{x})$. \square

In a similar manner one sees that the complete zero sets of G and H are the union of the arcs with parameterizations

$$\begin{bmatrix} \frac{1}{2}\omega_1(p) + \frac{1}{2} \\ \frac{1}{2}\omega_2(p) + \frac{1}{2} + k \end{bmatrix}, k \in \mathbb{Z}, \quad \text{and} \quad \begin{bmatrix} \frac{1}{2}\omega_1(p) - k + \frac{1}{2} \\ \frac{1}{2}\omega_2(p) + k + \frac{1}{2} \end{bmatrix}, k \in \mathbb{Z},$$

respectively, where p varies over \mathbb{T}_2 and \mathbb{T}_1 , respectively. The details of these cases are given in [2].

Remark. As can be seen from the Figures 2(a), 2(b), 2(c), the zero sets of F, G, H have the following geometric property. A line in \mathbb{R}^2 with positive slope will intersect these zero sets infinitely often.

4. THEOREM 2

Let the genus $g = 2$. In the sequel we assume that $\phi \in L^\infty(dm_a), \phi = \overline{\phi}$, and we put $m_i = \text{essinf}_{t \in b_i} \phi(t), M_i = \text{esssup}_{t \in b_i} \phi(t), i = 0, 1, 2$. Also, we assume that the intervals $[m_0, M_0], [m_1, M_1],$ and $[m_2, M_2]$ are separated by gaps. This means they are mutually disjoint. We are going to establish a sufficient condition for the existence of sequences of eigenvalues of T_ϕ in the gaps in the range of ϕ .

When t is in one of the gaps, the function $f(t)$, in Section 2, is in the form $f(t) = \theta(\underline{x}(t) + \frac{1}{2}\tau\underline{\lambda})$, where $\underline{\lambda} \in \mathbb{Z}^2/2\mathbb{Z}^2$, and $\underline{x}(t) = t_a + \frac{1}{2\pi} \int_{\partial D} \log|\phi - t|d\vec{\nu}$, $t \in (\text{gap})$. Therefore, using the theorem of Riemann, one needs to investigate the intersections of zero loci of the theta function θ in the real horizontal planes $P_\underline{\lambda} = \{z = \underline{x} + \frac{1}{2}\tau\underline{\lambda} : \underline{x} \in \mathbb{R}^2\}$ with the path $\underline{x}(t) = t_a + \frac{1}{2\pi} \int_{\partial D} \log|\phi - t|d\vec{\nu}$, $t \in (\text{gap})$.

Using the notation introduced in the preceding section, one needs only to study the intersections of the zero loci of the theta functions $F, G,$ and H , from Section 3, in \mathbb{R}^2 with the path $\underline{x}(t)$.

TABLE 1.

order of intervals	gap	$\theta(t_a - \frac{1}{2\pi} \int_{\partial D} \text{Log}(\phi - t) d\vec{w})$	existence of eigenvalues
$M_2 < m_1 \leq M_1 < m_0$	$M_2 < m_1$	$\theta(\underline{x}(t) - \frac{1}{2}\tau \begin{bmatrix} 0 \\ 1 \end{bmatrix})$	two infinite sequences
	$M_1 < m_0$	$\theta(\underline{x}(t) - \frac{1}{2}\tau \begin{bmatrix} 1 \\ 1 \end{bmatrix})$	at least one infinite sequence
$M_1 < m_2 \leq M_2 < m_0$	$M_1 < m_2$	$\theta(\underline{x}(t) - \frac{1}{2}\tau \begin{bmatrix} 1 \\ 0 \end{bmatrix})$	two infinite sequences
	$M_2 < m_0$	$\theta(\underline{x}(t) - \frac{1}{2}\tau \begin{bmatrix} 1 \\ 1 \end{bmatrix})$	at least one infinite sequence
$M_2 < m_0 \leq M_0 < m_1$	$M_2 < m_0$	$\theta(\underline{x}(t) - \frac{1}{2}\tau \begin{bmatrix} 0 \\ 1 \end{bmatrix})$	two infinite sequences
	$M_0 < m_1$	$\theta(\underline{x}(t) - \frac{1}{2}\tau \begin{bmatrix} -1 \\ 0 \end{bmatrix})$	two infinite sequences
$M_1 < m_0 \leq M_0 < m_2$	$M_1 < m_0$	$\theta(\underline{x}(t) - \frac{1}{2}\tau \begin{bmatrix} 1 \\ 0 \end{bmatrix})$	two infinite sequences
	$M_0 < m_2$	$\theta(\underline{x}(t) - \frac{1}{2}\tau \begin{bmatrix} 0 \\ -1 \end{bmatrix})$	two infinite sequences
$M_0 < m_2 \leq M_2 < m_1$	$M_0 < m_2$	$\theta(\underline{x}(t) - \frac{1}{2}\tau \begin{bmatrix} -1 \\ -1 \end{bmatrix})$	at least one infinite sequence
	$M_2 < m_1$	$\theta(\underline{x}(t) - \frac{1}{2}\tau \begin{bmatrix} -1 \\ 0 \end{bmatrix})$	two infinite sequences
$M_0 < m_1 \leq M_1 < m_2$	$M_0 < m_1$	$\theta(\underline{x}(t) - \frac{1}{2}\tau \begin{bmatrix} -1 \\ -1 \end{bmatrix})$	at least one infinite sequence
	$M_1 < m_2$	$\theta(\underline{x}(t) - \frac{1}{2}\tau \begin{bmatrix} 0 \\ -1 \end{bmatrix})$	two infinite sequences

The following theorem covers the case where $M_2 < m_1 \leq M_1 < m_0$.

Theorem 2. *Suppose D is a conjugate symmetric region with two holes. Let $\phi \in L^\infty(dm_a)$, $\phi = \bar{\phi}$. If $M_2 < m_1 \leq M_1 < m_0$, then*

(i) *There exist two infinite sequences $\{\lambda_n\}_{n=1}^\infty$ of eigenvalues in the gap (M_2, m_1) , one converging to M_2 if $\int_{b_2} \log|\phi - M_2| dm_a = -\infty$ and the other converging to m_1 if $\int_{b_1} \log|\phi - m_1| dm_a = -\infty$.*

(ii) *There is an infinite sequence $\{\lambda_n\}_{n=1}^\infty$ of eigenvalues in the gap (M_1, m_0) , converging to m_0 if $\int_{b_0} \log|\phi - m_0| dm_a = -\infty$.*

Below we will prove Theorem 2. For the convenience of the reader we collect the analogues of this theorem in Table 1. This table demonstrates all possible forms of the function $f(t)$, and the final column indicates the existence of an infinite sequence of eigenvalues in gaps and under the hypotheses that $\int_{b_i} \log|\phi - m_i| dm_a$ is infinity or $\int_{b_i} \log|\phi - M_i| dm_a$ is infinity.

Proof. (i): When t is in the gap (M_2, m_1) of the spectrum of T_ϕ , the function $f(t) = \theta(t_a - \frac{1}{2\pi i} \int_{\partial D} \text{Log}(\phi - t) d\vec{w})$ associated with the function in the denominator of the resolvent formula is in the form $F(\underline{x}(t)) = \theta(\underline{x}(t) + \frac{1}{2}\tau \begin{bmatrix} 0 \\ 1 \end{bmatrix})$, where

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = t_a + \frac{1}{2\pi} \int_{\partial D} \log|\phi - t| d\vec{v} \in \mathbb{R}^2.$$

In Section 3, Figure 2(a), we exhibited the zero locus of the function in F in \mathbb{R}^2 . In order to see the intersection of the zero locus of the function F with the path $\underline{x}(t)$

we consider that if $\int_{b_2} \log|\phi - M_2|dm_a = -\infty$, then $\int_{b_2} \log|\phi - M_2|d\vec{\nu} = \begin{bmatrix} +\infty \\ -\infty \end{bmatrix}$.

In fact, for $j = 1, 2$, we can write, for example,

$$\int_{b_2} \log|\phi - M_2|d\nu_j = \int_{b_2} \log|\phi - M_2|(2\pi i \frac{\nu_j}{\Omega_{a-Ja}})dm_a,$$

where Ω_{a-Ja} is the normalized differential of the third kind with poles at a, Ja on the double X . The function $2\pi i \frac{\nu_j}{\Omega_{a-Ja}}$ is a symmetric multiple-valued meromorphic function having no poles on ∂D . The symmetric meromorphic differential $\frac{-1}{2\pi i} \Omega_{a-Ja}$ is positive on ∂D . Moreover, ν_j is positive on b_j and negative on $b_i, i \neq j$. Hence, it is easy to see that

$$\int_{b_2} \log|\phi - M_2|dm_a = -\infty \iff \int_{b_2} \log|\phi - M_2|d\vec{\nu} = \begin{bmatrix} +\infty \\ -\infty \end{bmatrix}.$$

Therefore, it is easy to see that if the above hypotheses hold, then $\int_{b_2} \log|\phi - t|d\vec{\nu} \rightarrow \begin{bmatrix} \text{large positive} \\ \text{large negative} \end{bmatrix}$ as t approaches M_2 from the right. Note that whenever $d\nu_j, j = 1, 2$, is considered only on one of the boundary components, it will be a positive or a negative measure. This property allows us to use the Lebesgue Dominated Convergence Theorem. Thus, if the above hypotheses hold, then the path $\underline{x}(t) = t_a - \frac{1}{2\pi i} \int_{\partial D} \log|\phi - t|d\vec{w}$ intersects the above zero locus infinitely often as t approaches M_2 from the right. Similarly, one can see that

$$\int_{b_1} \log|\phi - m_1|dm_a = -\infty \iff \int_{b_1} \log|\phi - M_2|d\vec{\nu} = \begin{bmatrix} -\infty \\ +\infty \end{bmatrix}.$$

Therefore, if the above hypotheses hold, then $\int_{b_1} \log|\phi - t|d\vec{\nu} \rightarrow \begin{bmatrix} \text{large negative} \\ \text{large positive} \end{bmatrix}$ as t approaches m_1 from the left. Thus, if $\int_{b_1} \log|\phi - m_1|dm_a = -\infty$, then $\int_{b_1} \log|\phi - m_1|d\vec{\nu} = \begin{bmatrix} -\infty \\ +\infty \end{bmatrix}$ and hence the path $\underline{x}(t) = t_a - \frac{1}{2\pi i} \int_{\partial D} \log|\phi - t|d\vec{w}$ intersects the above zero locus infinitely often as t approaches m_1 from the left. See Figure 3.

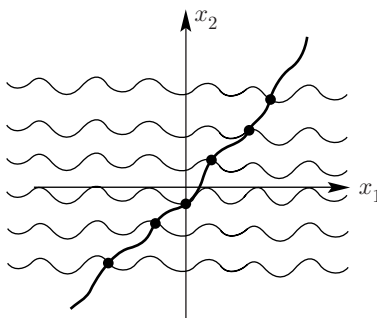


FIGURE 3.

For each one of these zeros one can choose an appropriate ω_z , by moving z if necessary, so that the zero becomes an isolated singularity of the resolvent form. Note

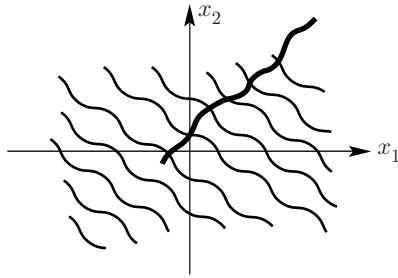


FIGURE 4.

that once we make a zero an isolated singularity this zero would be an eigenvalue of T_ϕ .

(ii): When t is in the gap (M_1, m_0) , the function

$$f(t) = \theta(t_a - \frac{1}{2\pi i} \int_{\partial D} \text{Log}(\phi - t) d\vec{w})$$

is in the form $f(t) = \theta(\underline{x}(t) + \frac{1}{2}\tau\lambda)$, where $\lambda = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^2$. One can see that if $\int_{b_0} \log|\phi - m_0| dm_a = -\infty$, then $\int_{b_0} \log|\phi - m_0| d\vec{\nu} = \begin{bmatrix} +\infty \\ +\infty \end{bmatrix}$. Thus,

$$\int_{b_0} \log|\phi - t| dw_j = \int_{b_0} \log|\phi - t| d\nu_j, \quad j = 1, 2,$$

gets large positive when t approaches m_0 from the left. Therefore, referring to the picture of the zero locus of the theta function G in P_λ , Section 3, Figure 2(b), one can see that the path $\underline{x}(t) = t_a - \frac{1}{2\pi i} \int_{\partial D} \log|\phi - t| d\vec{w}$ intersects the above zero locus infinitely often as t approaches m_0 from the left, as in Figure 4. This ends the proof of Theorem 2. □

Corollary 1. *If ϕ is a simple function in $L^\infty(dm_a)$, $\phi = \bar{\phi}$, and $M_2 < m_1 \leq M_1 < m_0$, then there always exist infinitely many eigenvalues of T_ϕ in the gaps as indicated in Table 1.*

We close with an example that illustrates the effectiveness of the criteria that have been developed for detecting eigenvalues.

Example 1. Let D be a region obtained from the open unit disc $\{z : |z| < 1\}$ by removing 2 disjoint closed discs \bar{D}_1, \bar{D}_2 of radius $0 < r_0 < \frac{1}{2}$, centered at the points $-\frac{1}{2}$ and $\frac{1}{2}$, respectively, in the real axis, as in Figure 5. Let m_a be the harmonic measure based at a fixed point a in D , and let ϕ be an element in $L^\infty(dm_a)$, $\bar{\phi} = \phi$, with $\phi_i = \phi|_{b_i} \equiv c_i$, $i = 0, 1, 2$ and $c_0 \neq c_1 \neq c_2$. To show the existence of eigenvalues of the related self-adjoint Toeplitz operator T_ϕ on $H^2(dm_a)$, without loss of generality, one can assume that $c_2 = -1, c_1 = 1$, and $c_0 = c$. Let us assume that $1 < c$. If τ is the B-period matrix of the marked double X of D , then we have the following.

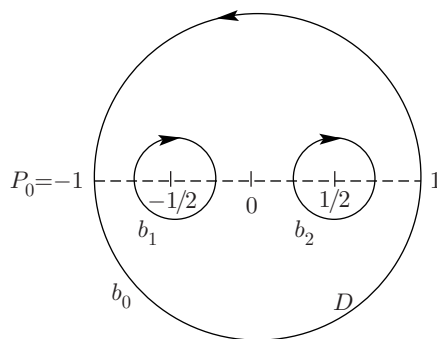


FIGURE 5.

(i) For any t in $(-1, 1)$,

$$\begin{aligned} \theta(t_a + \frac{1}{2\pi} \int_{\partial D} \text{Log}(\phi - t)d\vec{w}) &= \theta(t_a + \frac{1}{2\pi} \int_{\partial D} \log|\phi - t|d\vec{v} + \frac{1}{2}\tau \begin{bmatrix} 0 \\ 1 \end{bmatrix}) \\ &= F(t_a + \frac{1}{2\pi} \int_{\partial D} \log|\phi - t|d\vec{v}). \end{aligned}$$

Since

$$\begin{aligned} \underline{x}(t) &= t_a + \frac{1}{2\pi} \int_{\partial D} \log|\phi - t|d\vec{v} \\ &= t_a + \frac{1}{2\pi} \int_{b_0} \log|c - t|d\vec{v} + \frac{1}{2\pi} \int_{b_1} \log|1 - t|d\vec{v} + \frac{1}{2\pi} \int_{b_2} \log|1 + t|d\vec{v} \\ &= t_a + \frac{1}{2\pi} \log|c - t| \int_{b_0} d\vec{v} + \frac{1}{2\pi} \log|1 - t| \int_{b_1} d\vec{v} + \frac{1}{2\pi} \log|1 + t| \int_{b_2} d\vec{v}, \end{aligned}$$

for t near -1 , in the gap $(-1, 1)$, the path $\underline{x}(t)$ can be written as

$$\underline{x}(t) = \underline{r}(t) + \frac{1}{2\pi} \log|1 + t| \int_{b_2} d\vec{v} = \underline{r}(t) + \frac{1}{2} \log|1 + t| \begin{bmatrix} P_{12} \\ P_{22} \end{bmatrix},$$

where $\underline{r}(t) = (r_1(t), r_2(t))$ and the $r_i(t)$ ($i = 1, 2$) are bounded. Since $\log|1 + t| \rightarrow -\infty$, as $t \rightarrow (-1)^+$,

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \rightarrow \begin{bmatrix} \pm\infty \\ -\infty \end{bmatrix}, \text{ as } t \rightarrow (-1)^+.$$

Similarly,

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \rightarrow \begin{bmatrix} -\infty \\ \pm\infty \end{bmatrix}, \text{ as } t \rightarrow 1^-.$$

Therefore, there exist two infinite sequences of eigenvalues in the gap $(-1, 1)$.

(ii) For any t in $(1, c)$,

$$\begin{aligned} \theta(t_a + \frac{1}{2\pi} \int_{\partial D} \text{Log}(\phi - t)d\vec{w}) \\ = \theta(t_a + \frac{1}{2\pi} \int_{\partial D} \log|\phi - t|d\vec{v} + \frac{1}{2}\tau \begin{bmatrix} 1 \\ 1 \end{bmatrix}) &= H(t_a + \frac{1}{2\pi} \int_{\partial D} \log|\phi - t|d\vec{v}). \end{aligned}$$

Since

$$\begin{aligned}\underline{x}(t) &= t_a + \frac{1}{2\pi} \int_{\partial D} \log|\phi - t| d\vec{\nu} \\ &= t_a + \frac{1}{2\pi} \int_{b_0} \log|c - t| d\vec{\nu} + \frac{1}{2\pi} \int_{b_1} \log|1 - t| d\vec{\nu} + \frac{1}{2\pi} \int_{b_2} \log|1 + t| d\vec{\nu} \\ &= t_a + \frac{1}{2\pi} \log|c - t| \int_{b_0} d\vec{\nu} + \frac{1}{2\pi} \log|1 - t| \int_{b_1} d\vec{\nu} + \frac{1}{2\pi} \log|1 + t| \int_{b_2} d\vec{\nu},\end{aligned}$$

for t near c , in the gap $(1, c)$, the path $\underline{x}(t)$ can be written as

$$\underline{x}(t) = \underline{r}(t) + \frac{1}{2\pi} \log|c - t| \int_{b_0} d\vec{\nu} = \underline{r}(t) - \frac{1}{2} \log|c - t| \begin{bmatrix} P_{11} + P_{12} \\ P_{12} + P_{22} \end{bmatrix},$$

where $\underline{r}(t) = (r_1(t), r_2(t))$ and the $r_i(t)$ ($i = 1, 2$) are bounded. Since $\log|c - t| \rightarrow -\infty$, as $t \rightarrow c^-$,

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \rightarrow \begin{bmatrix} +\infty \\ +\infty \end{bmatrix}, \text{ as } t \rightarrow c^-.$$

Therefore, there exists an infinite sequence of eigenvalues in the gap $(1, c)$.

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