Abstract. We prove the existence of local Puiseux-type parameterizations of complex analytic sets via Laurent series convergent on wedges. We describe the wedges in terms of the Newton polyhedron of a function vanishing on the discriminant locus of a projection. The existence of a local parameterization of quasi-ordinary singularities of complex analytic sets of any codimension will come as a consequence of our main result.

1. Introduction

The existence of local parametric equations at smooth points of an analytic set is a consequence of the implicit function theorem. At singular points the existence of a resolution of singularities implies the existence of a finite system of sets of parametric equations covering a neighborhood of the singularity. However, desingularization theorems do not give much information about this system.

For a plane curve singularity, the Puiseux theorem asserts that we can find local parametric equations of the form $z_1 = t^k$, $z_2 = \phi(t)$, where $\phi$ is a convergent power series. A generalization of this result to non-planar curves can be found in [7].

The Abhyankar-Jung theorem generalizes Puiseux’s theorem to quasi-ordinary hypersurface singularities. (An $N$-dimensional analytic singularity, $(\mathcal{A}, a)$, is said to be quasi-ordinary if it admits a finite projection $p : (\mathcal{A}, a) \rightarrow (\mathbb{C}^N, 0)$ whose discriminant locus is contained in a normal crossing divisor.)

H. Hironaka introduced in [11] the notion of $\nu$-quasi-ordinary hypersurface singularity (a notion expressed in terms of the Newton polyhedron of the function defining the hypersurface). A proof of the Abhyankar-Jung theorem using this concept is given in [13]. A different proof can be found in [19].

Let $\mathcal{A} \subset \mathbb{C}^{N+M}$, $0 \in \mathcal{A}$, be an irreducible analytic set with $N = \dim_0(\mathcal{A})$. When the discriminant locus of the projection $(z_1, \ldots, z_N+M) \mapsto (z_1, \ldots, z_N)$ is not contained in the coordinate hyperplanes (i.e., $0$ is not a quasi-ordinary singularity), parametric equations of $\mathcal{A}$ around $0$ of the form

$$z_i = t_i^k, \quad i = 1, \ldots, N,$$

$$z_{N+j} = \phi_j(t_1, \ldots, t_N), \quad j = 1, \ldots, M,$$

where the $\phi_j$’s are convergent power series do not always exist.
In order to state a result valid for non-quasi-ordinary singularities, it is then
necessary to consider a ring bigger than the ring of ordinary convergent power
series. For algebraic hypersurfaces, J. McDonald showed in [14] the existence of
parametric equations of the form (1.1), where \( \phi \) is a convergent Laurent series
whose set of exponents is contained in a strongly convex polyhedral cone. With an
additional hypothesis, P.D. González showed in [10] that the supporting cone can
be chosen to be a cone of the Newton polytope of the discriminant of the polynomial
defining the hypersurface with respect to \( z_{N+1} \).

For a general analytic set \( A \subset \mathbb{C}^{N+M} \), \( \emptyset \in A \), with \( N = \dim_{\mathbb{C}}(A) \), we will show
the existence of "local parametric equations" of the form (1.1), where the \( \phi_j \)'s
are convergent Laurent series with exponents contained in a strongly convex polyhedral
cone. Due to the strong convexity of the supporting cone, the common domain of
convergence of the \( \phi_j \)'s contains a non-empty open set. By analytic continuation,
a set of such equations determines uniquely an irreducible branch of \( A \).

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Series with exponents in a strongly convex cone appear naturally as Laurent se-
ries expansions of rational functions [9, Chapt. 6], as solutions of partial differential
equations [4], and as solutions of holonomic systems [16]. Also, A. D. Bruno uses

We describe the supporting cones of the exponents of the parameterizations in
terms of the discriminant locus of a projection. As a corollary we state a general-
ization of the Abhyankar-Jung theorem to quasi-ordinary complex analytic sets of
any codimension.

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2. Notation and basic definitions

Given \( A \subset \mathbb{R} \) and \( \alpha \in \mathbb{R} \) set \( A_{\geq \alpha} := \{ a \in A; a \geq \alpha \} \). Let \( \tau : \mathbb{C}^N \to (\mathbb{R}_{\geq 0})^N \),
and let \( \Log : (\mathbb{R}_{\geq 0})^N \to \mathbb{R}^N \) be the maps defined by
\[
\tau(z_1, z_2, \ldots, z_N) := (|z_1|, |z_2|, \ldots, |z_N|),
\Log(x_1, x_2, \ldots, x_N) := (\log x_1, \log x_2, \ldots, \log x_N).
\]

A set \( \Omega \subset \mathbb{C}^N \) is called a Reinhardt set if \( \tau^{-1}(\tau(\Omega)) = \Omega \). A Reinhardt set
\( \Omega \subset (\mathbb{C}^*)^N := (\mathbb{C} \setminus \{0\})^N \) is said to be logarithmically convex if the set \( \Log(\tau(\Omega)) \)
is convex.

Let \( k \) be a natural number, and let \( \xi_k : \mathbb{C}^N \to \mathbb{C}^N \) be defined by
\[
\xi_k(z_1, z_2, \ldots, z_N) := (z_1^k, z_2^k, \ldots, z_N^k).
\]

Given \( \Omega \subset \mathbb{C}^N \), set \( \sqrt[k]{\Omega} := \xi_k^{-1}(\Omega) \) and \( |\sqrt[k]{\Omega}| := \tau(\xi_k^{-1}(\Omega)) \).

Given \( G = \{u^{(1)}, u^{(2)}, \ldots, u^{(M)}\} \subset \mathbb{Z}^N \), the convex polyhedral cone generated by
\( G \) is the convex set \( \langle G \rangle := \{ \lambda_1 u^{(1)} + \lambda_2 u^{(2)} + \cdots + \lambda_M u^{(M)}; \lambda_i \in \mathbb{R}_{\geq 0} \} \). A cone \( \sigma \)
is rational if there exists \( G \subset \mathbb{Z}^N \) such that \( \sigma = \langle G \rangle \). By cone we will mean convex
polyhedral cone. A cone is said to be strongly convex if it does not contain any
proper linear subspace.

For \( \vartheta \in \mathbb{R}^N \setminus \{0\} \) and \( a \in \mathbb{R} \), we will denote by \( H^+(\vartheta; a) \) the affine half-space
\( \{ u \in \mathbb{R}^N; u \cdot \vartheta \geq a \} \).
Given \( g = (g_1, \ldots, g_N) \in (\mathbb{R}_{>0})^N \), we set

\[ \mathcal{D}_g := \{ z = (z_1, \ldots, z_N) \in \mathbb{C}^N; \quad |z_i| \leq g_i, \ i = 1, \ldots, N \} \]

and \( \mathcal{D}_g^* := \mathcal{D}_g \cap (\mathbb{C}^*)^N \), where \( \mathbb{C}^* := \mathbb{C} \setminus \{0\} \).

Given \( u = (I_1, I_2, \ldots, I_N) \in \mathbb{Z}^N \) and \( z = (z_1, z_2, \ldots, z_N) \), we set

\[ z^u := z_1^{I_1}z_2^{I_2} \cdots z_N^{I_N}. \]

Given an \( M \)-tuple of vectors \( \mathcal{G} = \{ u^{(1)}, \ldots, u^{(M)} \} \subset \mathbb{Z}^N \), set \( \Phi_\mathcal{G} : (\mathbb{C}^*)^N \to (\mathbb{C}^*)^M \) to be the map defined by

\[
(2.1) \quad \Phi_\mathcal{G}(z) := (z^{u^{(1)}}, z^{u^{(2)}}, \ldots, z^{u^{(M)}}).
\]

Given a Laurent series \( s = \sum_{I \in \mathbb{Z}^N} a_I z^I \), we call \( \mathcal{E}(s) := \{ I \in \mathbb{Z}^N; a_I \neq 0 \} \) the set of exponents of \( s \).

Let \( \delta : (\mathbb{C}^N, 0) \to (\mathbb{C}, 0) \) be a germ of an analytic function. The set of exponents of its Taylor series development is contained in \((\mathbb{R}_{>0})^N\). The Newton polyhedron of \( \delta \), \( \text{NP}(\delta) \), is the convex hull of the set \( \mathcal{E}(\delta) + (\mathbb{R}_{>0})^N \). See Figure 1.

![Figure 1](image1.png)

**Figure 1.** The dots on the left are the set of exponents of \( f := (z_1^4 + z_1^5 + z_2^4 + z_1^2 z_2 + z_1^4 z_2^2 + z_1^2 z_2^3 + z_1^5 z_2^4) \). On the right is the Newton polyhedron \( \text{NP}(f) \).

Let \( A \) be a vertex of the Newton polyhedron of an analytic function \( \delta \). The set of vectors \( v \in \mathbb{R}^N \) such that \((A + \lambda v) \in \text{NP}(\delta)\) for some positive real number \( \lambda \) is a cone \( \sigma \). We will say that \( \sigma \) is the cone of the Newton polyhedron of \( \delta \), associated to the vertex \( A \). The cone \( \sigma \) is just the cone spanned by the faces of \( \text{NP}(\delta) \) that contain \( A \). Cones of the Newton polyhedron of an analytic function are always strongly convex and contain the non-negative orthant \((\mathbb{R}_{\geq 0})^N\). See Figure 2.

![Figure 2](image2.png)

**Figure 2.** Cones associated to the vertices of \( \text{NP}(f) \).
3. Wedges and Cones

We introduce the notion of \( \sigma \)-wedge, which is the analog for series with exponents in a cone \( \sigma \), to polydiscs for series with non-negative exponents.

Suppose \( s \) is a Laurent series convergent at a point \( p \in \mathbb{C}^N \); then \( s \) is also convergent at any point \( z \in \mathbb{C}^N \) with \( \tau(z)^I < \tau(p)^I \) for all \( I \in \mathcal{E}(s) \).

**Definition 3.1.** Let \( \sigma \subset \mathbb{R}^N \) be a cone. For \( \varrho \in (\mathbb{R}_{>0})^N \), the \( \sigma \)-wedge of polyradius \( \varrho \) is the set

\[
W(\sigma, \varrho) := \left\{ z \in (\mathbb{C}^*)^N; \tau(z)^u \leq \varrho^u, \forall u \in \sigma \cap \mathbb{Z}^N \right\}.
\]

It is clear that \( \sigma \)-wedges are Reinhardt sets.

**Example 3.2.** Let \( \mathcal{C} \) be the canonical basis of \( \mathbb{R}^N \). For any \( \varrho \in (\mathbb{R}_{>0})^N \), the \( \langle \mathcal{C} \rangle \)-wedge of polyradius \( \varrho \) is \( \mathbb{D}^*_\varrho \).

\[
\begin{align*}
\text{Figure 3.} & \quad \text{The image under } \tau \text{ of } \sigma \text{-wedges of polyradius } (p, q) \text{, where the cones } \sigma \text{ are those of NP}(f). \\
\end{align*}
\]

Let \( \sigma \) be the cone generated by \( \mathcal{G} = \{u^{(1)}, \ldots, u^{(M)}\} \subset \mathbb{Z}^N \). For any \( \varrho \in (\mathbb{R}_{>0})^N \) we have:

\[
\begin{align*}
(3.1) \quad & \quad W(\sigma, \varrho) = \left\{ z \in (\mathbb{C}^*)^N; \tau(z)^{u^{(i)}} \leq \varrho^{u^{(i)}}, \forall i = 1, \ldots, M \right\}; \\
(3.2) \quad & \quad W(\sigma, \varrho) = \varrho^{-1} \left( \mathbb{D}^{*\varrho}(\varrho) \right); \\
(3.3) \quad & \quad \text{if } \sigma \subset \sigma', \text{ then } W(\sigma', \varrho) \subset W(\sigma, \varrho); \\
(3.4) \quad & \quad \overline{\bar{W}(\sigma, \varrho)} = W(\sigma, \sqrt[\varrho]{\varrho}); \\
(3.5) \quad & \quad \text{Log}(\tau(W(\sigma, \varrho))) = \bigcap_{u \in \sigma \cap \mathbb{Z}^N} H^+(-u; -\log \varrho^u).
\end{align*}
\]

**Proposition 3.3.** Let \( \sigma \) be a rational cone. A \( \sigma \)-wedge is a connected set with connected interior.

**Proof.** For any \( \varrho \in (\mathbb{R}_{>0})^N \), \( \text{Log}(\tau(W(\sigma, \varrho))) \) is the intersection of affine half hyperplanes (see (3.5)) and therefore a connected set of connected interior. The result follows from the fact that \( \text{Log} \) is a homeomorphism and \( \tau \) is a projection with connected fiber.

**Proposition 3.4.** Let \( \sigma \) be a strongly convex rational cone. For any pair \( \varrho, \varrho' \in (\mathbb{R}_{>0})^N \), the interior of \( W(\sigma, \varrho) \cap W(\sigma, \varrho') \) is not empty.
Proposition 3.6. Let \( \sigma \) be a rational cone. A \( \sigma \)-wedge is a logarithmically convex Reinhardt set. If \( \sigma \) is strongly convex, then its interior is a non-empty logarithmically convex Reinhardt domain.

Proof. For any \( \varrho \in (\mathbb{R}_{>0})^N \), \( \log(\sigma(\varrho)) \) is the intersection of affine half hyperplanes (see (3.3)) and therefore a convex set. If \( \sigma \) is strongly convex, then its interior is connected (Proposition 3.3) and non-empty (Proposition 3.4). \( \square \)

4. Laurent Series Development over a \( \sigma \)-Wedge

A holomorphic function \( \varphi \), defined on a Reinhardt domain \( \Omega \), has a (unique) Laurent series expansion that converges uniformly to \( \varphi \) on compact subsets of \( \Omega \) [2, I.6].

The Taylor development of a function, holomorphic on a disc centered at the origin, is a series with exponents in the non-negative orthant. In this section we will see a similar result for \( \sigma \)-wedges and series with exponents in the cone \( \sigma \).

Definition 4.1. Let \( \sigma \) be a cone. We will say that a set \( \Omega \subset (\mathbb{C}^*)^N \) is \( \sigma \)-complete if for any \( z \in \Omega \) the \( \sigma \)-wedge of polyradius \( \tau(z) \) is contained in \( \Omega \).

A \( \sigma \)-wedge is \( \sigma \)-complete. A \( \sigma \)-complete set is a Reinhardt set. If \( \sigma \) is strongly convex, from Propositions 3.3 and 3.4 it follows that a \( \sigma \)-complete set is connected with connected interior.

Remark 4.2. The domain of convergence of a Laurent series with exponents in a cone \( \sigma \) is \( \sigma \)-complete.

Remark 4.3. As a consequence of (3.4) we have that, if \( \Omega \) is \( \sigma \)-complete and \( k \) is a natural number, then \( \sqrt[\kappa]{\Omega} \) is \( \sigma \)-complete.
Lemma 4.4. Let $\Omega$ be a Reinhardt domain, let $\varphi : \Omega \to \mathbb{C}$ be a bounded holomorphic function, and take $\vartheta \in \mathbb{Z}^N \setminus \{0\}$. If $\Omega$ is $H^+(\vartheta; 0)$-complete, then the set of exponents of the Laurent series expansion of $\varphi$ is contained in $H^+(\vartheta; 0)$.

Proof. Let $\mathcal{B} = \{e^{(1)}, \ldots, e^{(N-1)}, \vartheta\} \subset \mathbb{Z}^N$ be an orthogonal basis of $\mathbb{R}^N$. We have $H^+(\vartheta; 0) = \langle e^{(1)}, \ldots, e^{(N-1)}, -e^{(1)}, \ldots, -e^{(N-1)}, \vartheta \rangle$.

For each $I \in \mathbb{Z}^N$ let $I_B$ be the coordinates of $I$ in the basis $\mathcal{B}$. Since $\mathcal{B} \subset \mathbb{Z}^N$, there exists a natural number $d$ such that $dI_B \in \mathbb{Z}^N$ for any $I \in \mathbb{Z}^N$.

Let $s = \sum_{I \in \mathbb{Z}^N} a_I z^I$ be the Laurent series expansion of $\varphi$, and set

$$t := \sum_{I \in \mathbb{Z}^N} a_I y^d I_B,$$

where $y = (y_1, \ldots, y_N)$.

For any $z \in \sqrt[\vartheta]{\Omega}$, $t(\Phi_B(z)) = s(z^d)$. Hence $t$ is convergent and bounded in $\Phi_B(\sqrt[\vartheta]{\Omega})$.

From the way $t$ has been defined:

S) $E(s) \subset H^+(\vartheta; 0)$ if and only if $E(t) \subset H^+((0, \ldots, 0, 1); 0)$.

Let $\pi$ be the map defined by $\pi(y_1, \ldots, y_N) := (y_1, \ldots, y_{N-1})$, and let us rewrite the series $t$ in the form

$$t = \sum_{j \in \mathbb{Z}} g_j(\pi(y)) y_N^j,$$

where $g_j = \sum_{(0, \ldots, 0, 1) \cdot dI_B = j} a_I x^d \pi(I_B)$

for $x = (x_1, \ldots, x_{N-1})$.

Now statement S) becomes:

S') $E(s) \subset H^+(\vartheta; 0)$ if and only if $g_j(x) \equiv 0$ for all $j < 0$.

For any $\varphi \in \tau(\sqrt[\vartheta]{\Omega})$, by Remark 4.3 the wedge $W(H^+(\vartheta, 0), \varphi)$ is contained in $\sqrt[\vartheta]{\Omega}$. So, $\Phi_B(W(H^+(\vartheta, 0), \varphi)) \subset \Phi_B(\sqrt[\vartheta]{\Omega})$. Hence $t$ is convergent and bounded on:

$$\Phi_B(W(H^+(\vartheta, 0), \varphi)) = \left\{ y \in (\mathbb{C}^*)^N : \left| y_N \right| \leq \varphi^0, \left| y_i \right| = \varphi^{e(i)}, i = 1, \ldots, N-1 \right\}.$$

For any $z \in \sqrt[\vartheta]{\Omega}$, the series in one variable,

$$t_z(y_N) := t(\pi(\Phi_B(z)), y_N) = \sum_{j \in \mathbb{Z}} g_j(\pi(\Phi_B(z))) y_N^j$$

is convergent and bounded on the punctured disc $D^*_{\tau(z)^0}$. Then, there exists a unique function, holomorphic on $D^*_{\tau(z)^0}$, that extends $t_z$. Therefore $t_z$ is necessarily the Taylor development of that function and, as a consequence, it cannot have negative exponents.

We have seen that, for $j < 0$, $g_j(x) = 0$ for all $x \in \pi(\Phi_B(\sqrt[\vartheta]{\Omega}))$. Since $\pi(\Phi_B(\sqrt[\vartheta]{\Omega}))$ is a non-empty open subset of $\mathbb{C}^{N-1}$, $g_j \equiv 0$. \hfill $\square$

Proposition 4.5. Let $\sigma$ be a cone, and let $\Omega \subset (\mathbb{C}^*)^N$ be a $\sigma$-complete Reinhardt domain. If $\varphi$ is a bounded holomorphic function on $\Omega$, and $s$ is its Laurent series expansion in $\Omega$, then the set of exponents of $s$ is contained in $\sigma$.

Proof. Let $\vartheta \in \mathbb{Z}^N \setminus \{0\}$ be such that $\sigma \subset H^+(\vartheta; 0)$. By Lemma 4.4, $\Omega$ is $H^+(\vartheta; 0)$-complete. So (Lemma 4.4) $E(s) \subset H^+(\vartheta; 0)$, and the result follows from the fact that

$$\sigma = \bigcap_{\vartheta \in \mathbb{Z}^N, \sigma \subset H^+(\vartheta; 0)} H^+(\vartheta; 0),$$

for any cone $\sigma$. \hfill $\square$
5. Cones of a polyhedron and wedges not intersecting the zero-locus

Proposition 5.1. Let \( \delta : (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}, 0) \) be a germ of an analytic function, and let \( \sigma \) be a cone of the Newton polyhedron of \( \delta \). There exists a \( \sigma \)-wedge that does not intersect the zero locus of \( \delta \).

Proof. Let \( T(\delta) = \sum_{I \in \mathcal{E}(\delta)} b_I z^I \) be the Taylor development of \( \delta \), and let \( \Omega \) be its domain of convergence. Let \( A \) be the vertex of \( \sigma \) in \( \text{NP}(\delta) \). The meromorphic function \( f = z^{-A} \delta \) is holomorphic on \( \Omega^* = \Omega \cap (\mathbb{C}^*)^N \), and, outside the coordinate hyperplanes, the zero locus of \( \delta \) coincides with the zero locus of \( f \). We will find a \( \sigma \)-wedge where \( f \) does not vanish.

Notice that the Laurent series development of \( f \) on \( \Omega^* \),

\[
s = \frac{1}{z^A} T(\delta) = \sum_{I - A \in \mathbb{N}^N} b_I z^{I-A} = \sum_{I \in \mathbb{N}^N} a_I z^I, \quad a_I = b_{I+A},
\]

is a series with exponents in \( \sigma \).

Let \( \mathcal{G} = \{u^{(1)}, u^{(2)}, \ldots, u^{(M)}\} \) be a set of generators of the semigroup \( \sigma \cap \mathbb{Z}^N \) (Gordan’s lemma [8, 1.2, Prop. 1]). For each \( I \in \mathcal{E}(s) \), let \( I_G \) be an \( M \)-tuple of natural numbers \( I_G = (I_G^{(1)}, \ldots, I_G^{(M)}) \) such that \( I = I_G^{(1)} + I_G^{(2)} + \cdots + I_G^{(M)} \).

The series in the variables \( y = (y_1, y_2, \ldots, y_M) \) defined by

\[
\sum_{I \in \mathcal{E}(\delta)} a_I y^{I_G}
\]

is a series with positive exponents.

Let \( \Phi_G \) be as in \((2.1)\). For any \( z \in (\mathbb{C}^*)^N \) contained in the domain of convergence of \( s \), we have that

\[
t = \Phi_G(z) \implies t(\Phi_G(z)) = s(z).
\]

In particular, \( t \) is convergent in \( \Phi_G(z) \). Now, since \( t \) has positive exponents, \( t \) converges in \( \mathbb{D}_{\Phi_G(\tau(z))} \) (the closed polydisc of polyradius \( \tau(\Phi_G(z)) \)).

The series \( t \) converges, does not have negative exponents, and \( t(0) = b_0 = a_A \neq 0 \). So there exists \( z \in (\mathbb{R}_{>0})^M \) such that \( t(y) \neq 0 \) for any \( y \in \mathbb{D}^*_z \). Take \( g = \tau(\Phi_G(\mathbb{D}^*_z)) \). By \((3.2)\), the \( \sigma \)-wedge of polyradius \( g \) has the desired property. \( \square \)

6. Coverings of logarithmically convex Reinhardt domains

In this section we extend the well-known fact that coverings of the punctured disc are isomorphic to some covering \( t \mapsto t^k \).

Lemma 6.1. The fundamental group of a logarithmically convex Reinhardt domain \( \Omega \subset \mathbb{C}^N \) is isomorphic to \( \mathbb{Z}^N \).

Proof. The projection \( \tau : \Omega \rightarrow \tau(\Omega) \) is a fiber bundle with fiber an \( N \)-dimensional torus. \( \text{Log}(\tau(\Omega)) \) is convex and then contractible. Since \( \text{Log} \) is a homeomorphism, \( \tau(\Omega) \) is also contractible. Then (see, for example, [17, §11.6]) \( \Omega \) is homeomorphic to \( \mathbb{T}^N \times \tau(\Omega) \). Again as a consequence of the contractibility of \( \tau(\Omega) \) we have the result. \( \square \)

Proposition 6.2. Let \( \mathcal{A} \) be a complex analytic set. Let \( p : \mathcal{A} \rightarrow \mathbb{C}^N \) be a finite morphism. Let \( \Omega \subset (\mathbb{C}^*)^N \) be a logarithmically convex Reinhardt domain contained in \( p(\mathcal{A}) \) such that \( p : \mathcal{A} \cap p^{-1}(\Omega) \rightarrow \Omega \) is locally biholomorphic. Then for each
connected component \( C \) of \( p^{-1}(\Omega) \) there exists a natural number \( k \) and a morphism \( \varphi : \sqrt[\xi_k]{\Omega} \to C \) such that \( \varphi(\sqrt[\xi_k]{\Omega}) = C \) and \( p \circ \varphi = \xi_k \).

**Proof.** Let \( C \) be a connected component of \( p^{-1}(\Omega) \), and let \( k \) be the cardinal of the generic fiber of \( p|_C \). Since both \( p|_C \) and \( \xi_k|_C \) are locally biholomorphic, the pairs \((C, p)\) and \((\sqrt[\xi_k]{\Omega}, \xi_k)\) are, respectively, a \( k \)-sheeted and a \( k^N \)-sheeted covering of \( \Omega \):

\[
\begin{array}{ccc}
\sqrt[\xi_k]{\Omega} & \xrightarrow{\varphi} & C \\
\xi_k \downarrow & & \downarrow p \\
\Omega & & \\
\end{array}
\]

Choose a point \( z_0 \in \Omega \), a point \( z_1 \in \xi_k^{-1}(z_0) \), and a point \( P \) in \( p^{-1}(z_0) \cap C \). Consider the induced monomorphisms on the fundamental groups:

\[
\begin{array}{ccc}
\pi_1(\sqrt[\xi_k]{\Omega}, z_1) & \xrightarrow{\xi_k} & \pi_1(C, P) \\
\pi_1(\Omega, z_0) \downarrow & & \downarrow P_* \\
\end{array}
\]

An element \( \gamma \in \pi_1(\Omega, z_0) \) is in the subgroup \( \pi_1(\Omega, z_0) \) if and only if \( \gamma = \alpha^k \) for some \( \alpha \in \pi_1(\Omega, z_0) \).

On the other hand, the index of \( p_\ast(\pi_1(C, P)) \) in \( \pi_1(\Omega, z_0) \) is equal to \( k \) (see, for example, [15, V\$7]). Since \( \pi_1(\Omega, z_0) \) is abelian, the cosets of \( p_\ast(\pi_1(C, P)) \) in \( \pi_1(\Omega, z_0) \) form a group of order \( k \). Therefore, for any \( \alpha \in \pi_1(\Omega, z_0) \) the element \( \alpha^k \) belongs to \( p_\ast(\pi_1(C, P)) \). Then \( \xi_k^\ast(\pi_1(\sqrt[\xi_k]{\Omega}, z_1)) \subset p_\ast(\pi_1(C, P)) \).

The lifting lemma [15, Theorem V.5.1] ensures the existence of a (unique) map \( \varphi \), such that \( \varphi(z_1) = P \) and the diagram

\[
\begin{array}{ccc}
\sqrt[\xi_k]{\Omega} & \xrightarrow{\varphi} & C \\
\xi_k \downarrow & & \downarrow p \\
\Omega & & \\
\end{array}
\]

commutes. \( \square \)

**Remark 6.3.** In Proposition 6.2, \( k \) can be chosen to be the degree of the covering \( p : C \to \Omega \). Since for each \( P \in p^{-1}(z_0) \cap C \) there exists a unique \( \varphi \) such that \( \varphi(z_1) = P \), there exist \( k \) different \( \varphi \)'s.

7. The theorem

Let \( \mathcal{A} \) be an analytic set, and assume that \( \emptyset \in \mathcal{A} \subset \mathbb{C}^{N+M} \), where \( N = \dim_0 \mathcal{A} \) is the dimension of \( \mathcal{A} \) at \( \emptyset \). There exists a linear projection \( p : \mathbb{C}^{N+M} \to \mathbb{C}^N \), and a neighborhood \( U \) of \( \emptyset \) such that the restriction of \( p \) to \( \mathcal{A} \cap U \) is a finite morphism. Moreover, the set of such \( p \) is open and everywhere dense in the space of linear projections.

Let \( p : \mathcal{A} \cap U \to \mathbb{C}^N \) be a finite morphism. Then \( p(\mathcal{A} \cap U) \) is a neighborhood of \( p(\emptyset) \) and there exists an analytic set \( \Delta \subset \mathbb{C}^N \), with \( \dim_{p(\emptyset)} \Delta < N \), such that \( p : \mathcal{A} \cap U \setminus p^{-1}(\Delta) \to p(\mathcal{A} \cap U) \setminus \Delta \) is locally biholomorphic [15, 3.7]. The smallest set \( \Delta \) with this property will be called the **discriminant locus** of \( p \).
Theorem 7.1. Let $A$ be an analytic subset of $\mathbb{C}^{N+M}$, $0 \in A$, $\dim_{0}(A) = N$. Let $U$ be a neighborhood of $0$, and let $\pi$ be the restriction to $A \cap U$ of the projection $(z_1, z_2, \ldots, z_{N+M}) \mapsto (z_1, z_2, \ldots, z_N)$. Assume $\pi$ is a finite morphism. Let $\delta$ be an analytic function vanishing on the discriminant locus of $\pi$.

For each cone $\sigma$ of the Newton polyhedron of $\delta$, there exist $k \in \mathbb{N}$ and an $M$-tuple of convergent Laurent series, $(s_1, \ldots, s_M)$, such that

$$E(s_i) \subset \sigma, \quad i = 1, \ldots, M,$$

and

$$f(z_1^k, \ldots, z_N^k, s_1, \ldots, s_M) = 0$$

for any $f$ vanishing on $A$, and any $z$ in the domain of convergence of the $s_i$.

Proof. Let us choose $\varrho_1$ small enough so that $D_{\varrho_1}$ is contained in $\pi(A \cap U)$ and so that $\pi^{-1}(D_{\varrho_1})$ is a bounded set of $\mathbb{C}^{N+M}$. Since $\sigma$ contains the non-negative orthant, by Proposition 3.4, $W(\sigma, \varrho_1) \subset \pi(A \cap U)$.

By Proposition 3.1 we can find an $N$-tuple of positive real numbers, $\varrho_2$, such that $W(\sigma, \varrho_2)$ does not intersect the zero locus of $\delta$. Since $\delta$ vanishes on the discriminant locus of $\pi$, $W(\sigma, \varrho_2)$ does not intersect the discriminant locus, and $\pi$ restricted to $W(\sigma, \varrho_2)$ is locally biholomorphic.

Since $\sigma$ is strongly convex, the intersection $W(\sigma, \varrho_1) \cap W(\sigma, \varrho_2)$ is non-empty (Proposition 5.3). Take $\varrho \in \tau(W(\sigma, \varrho_1) \cap W(\sigma, \varrho_2))$. Let $\Omega$ be the interior of $W(\sigma, \varrho)$.

Since $\sigma$ is strongly convex, by Proposition 5.6 $\Omega$ is a non-empty logarithmically convex Reinhardt domain.

Let $C$ be a connected component of $\pi^{-1}(\Omega)$. Then, by Proposition 6.2 there exists a natural number $k$ and a morphism $\varphi : \sqrt[\bar{\Omega}]{A} \longrightarrow C$, such that $\pi \circ \varphi = \xi_k$.

We can express $\varphi$ as

$$\varphi : \sqrt[\bar{\Omega}]{A} \longrightarrow A,$$

$$(x_1, \ldots, x_N) \mapsto (x_1^k, \ldots, x_N^k, \varphi_1, \ldots, \varphi_M),$$

where $\varphi_i : \sqrt[\bar{\Omega}]{A} \longrightarrow \mathbb{C}$ is a holomorphic function for $i \in \{1, \ldots, M\}$. Moreover, since $\varphi(\sqrt[\bar{\Omega}]{A}) \subset A \cap U$, each $\varphi_i$ is bounded.

Since $\sqrt[\bar{\Omega}]{A}$ is $\sigma$-complete, by Proposition 4.5 we have that for each $i \in \{1, \ldots, M\}$, there is a Laurent series $s_i$ that converges uniformly to $\varphi_i$ in compact subsets of $\sqrt[\bar{\Omega}]A$ and such that $E(s_i) \subset \sigma$.

Now, for any $z$ in the domain of convergence of the $s_i$’s, the point of $\mathbb{C}^{N+M}$, $P = (z_1^k, \ldots, z_N^k, s_1(z), \ldots, s_M(z))$ is in $A$ (or in an analytic extension of $A$) and, therefore, any function $f$ vanishing on $A$ also vanishes on $P$. \hfill $\square$

Remark 7.2. Taking as $C$ the different connected components of $\pi^{-1}(\Omega)$, we may in fact find $k_1, \ldots, k_r$ natural numbers with $\sum_{i=1}^{r} k_i$ equal to the degree of $\pi$ and, for each $i \in \{1, \ldots, r\}$, $k_i$ different $M$-tuples of series. In other words, we have seen that if $d$ is the degree of $\pi$, there are $d$ $M$-tuples of Puiseux series such that $f(z_1, \ldots, z_N, s_1, \ldots, s_M) = 0$ for any $f \in I(A)$.

Corollary 7.3 (Abhyankar-Jung). Let $0$ be a quasi-ordinary singularity of a complex analytic set $A \subset \mathbb{C}^{N+M}$, $\dim_{0}(A) = N$. Then there exists a natural number
\[ k, \text{ and } M \text{ convergent power series } \phi_j, j = 1, \ldots, M, \text{ such that} \]
\[ z_i = t_i^k, i = 1, \ldots, N, \quad z_{N+j} = \phi_j(t_1, \ldots, t_N), j = 1, \ldots, M, \]
are parametric equations of \( \mathcal{A} \) about \( \mathbf{0} \).

**Proof.** By definition of a quasi-ordinary singularity, the discriminant locus of the projection \( \pi \) is contained in the coordinate hyperplanes, the function \( \delta(z) = z_1 \cdots z_N \) vanishes there and its Newton polyhedron has just one cone which is the non-negative orthant. The result is then a consequence of the theorem. \( \square \)

**Remark 7.4.** Let \( \mathcal{A} \subset \mathbb{C}^{N+M} \) and \( \pi \) be as in Theorem 7.1 and suppose that \( \mathcal{A} \) is a surface (i.e., \( N = 2 \)). The discriminant locus of \( \pi \) is contained in an analytic curve \( \delta \subset \mathbb{C}^2 \). Performing the quadratic transformations needed to solve \( \delta \) to both \( \delta \) and \( \mathcal{A} \) we arrive at a new surface \( \tilde{\mathcal{A}} \) and a projection \( \tilde{\pi} \) whose discriminant locus \( \tilde{\delta} \) has only normal crossings. This fact is used (see [12, 18]) to get a resolution of \( \mathcal{A} \). In [3, Chapt. 2] it is shown that the transform of a \( \sigma \)-wedge is a tubular neighborhood of a connected piece of the exceptional divisor of the resolution.

**References**


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