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PUISEUX PARAMETRIC EQUATIONS OF ANALYTIC SETS

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ABSTRACT. We prove the existence of local Puiseux-type parameterizations of complex analytic sets via Laurent series convergent on wedges. We describe the wedges in terms of the Newton polyhedron of a function vanishing on the discriminant locus of a projection. The existence of a local parameterization of quasi-ordinary singularities of complex analytic sets of any codimension will come as a consequence of our main result.

1. INTRODUCTION

The existence of local parametric equations at smooth points of an analytic set is a consequence of the implicit function theorem. At singular points the existence of a resolution of singularities [5] implies the existence of a finite system of sets of parametric equations covering a neighborhood of the singularity. However, desingularization theorems do not give much information about this system.

For a plane curve singularity, the Puiseux theorem asserts that we can find local parametric equations of the form $z_1 = t^k$, $z_2 = \phi(t)$, where ϕ is a convergent power series. A generalization of this result to non-planar curves can be found in [7].

The Abhyankar-Jung theorem [1] generalizes Puiseux's theorem to quasi-ordinary hypersurface singularities. (An N-dimensional analytic singularity, $(\mathcal{A}, a)$, is said to be quasi-ordinary if it admits a finite projection $p : (\mathcal{A}, a) \longrightarrow (\mathbb{C}^N, \underline{0})$ whose discriminant locus is contained in a normal crossing divisor.)

H. Hironaka introduced in [11] the notion of ν-quasi-ordinary hypersurface singularity (a notion expressed in terms of the Newton polyhedron of the function defining the hypersurface). A proof of the Abhyankar-Jung theorem using this concept is given in [13]. A different proof can be found in [19].

Let $\mathcal{A} \subset \mathbb{C}^{N+M}$, $\underline{0} \in \mathcal{A}$, be an irreducible analytic set with $N = \dim_{\underline{0}}(\mathcal{A})$. When the discriminant locus of the projection $(z_1, \ldots, z_{N+M}) \mapsto (z_1, \ldots, z_N)$ is not contained in the coordinate hyperplanes (i.e., $\underline{0}$ is not a quasi-ordinary singularity), parametric equations of $\mathcal{A}$ around $\underline{0}$ of the form

$$z_i = {t_i}^k,\ i = 1, \ldots, N, \qquad z_{N+j} = \phi_j(t_1, \ldots, t_N),\ j = 1, \ldots, M, \tag{1.1}$$

where the ϕ_j's are convergent power series do not always exist.

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In order to state a result valid for non-quasi-ordinary singularities, it is then necessary to consider a ring bigger than the ring of ordinary convergent power series. For algebraic hypersurfaces, J. McDonald showed in [14] the existence of parametric equations of the form (1.1), where ϕ is a convergent Laurent series whose set of exponents is contained in a strongly convex polyhedral cone. With an additional hypothesis, P.D. González showed in [10] that the supporting cone can be chosen to be a cone of the Newton polytope of the discriminant of the polynomial defining the hypersurface with respect to z_{N+1}.

For a general analytic set $\mathcal{A} \subset \mathbb{C}^{N+M}$, $\underline{0} \in \mathcal{A}$, with $N = \dim_{\underline{0}}(\mathcal{A})$, we will show the existence of "local parametric equations" of the form (1.1), where the ϕ_j's are convergent Laurent series with exponents contained in a strongly convex polyhedral cone. Due to the strong convexivity of the supporting cone, the common domain of convergence of the ϕ_j's contains a non-empty open set. By analytic continuation, a set of such equations determines uniquely an irreducible branch of $\mathcal{A}$.

Series with exponents in a strongly convex cone appear naturally as Laurent series expansions of rational functions [9, Chapt. 6], as solutions of partial differential equations [4], and as solutions of holonomic systems [16]. Also, A. D. Bruno uses them in [6] to construct a method for solving non-linear differential equations.

We describe the supporting cones of the exponents of the parameterizations in terms of the discriminant locus of a projection. As a corollary we state a generalization of the Abhyankar-Jung theorem to quasi-ordinary complex analytic sets of any codimension.

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2. Notation and basic definitions

Given $A \subset \mathbb{R}$ and $\alpha \in \mathbb{R}$ set $A_{\geq \alpha} := \{a \in A;\, a \geq \alpha\}$. Let $\tau : \mathbb{C}^N \longrightarrow (\mathbb{R}_{\geq 0})^N$, and let $\mathrm{Log} : (\mathbb{R}_{>0})^N \longrightarrow \mathbb{R}^N$ be the maps defined by

$$\tau(z_1, z_2, \ldots, z_N) := (|z_1|, |z_2|, \ldots, |z_N|),$$
$$\mathrm{Log}(x_1, x_2, \ldots, x_N) := (\log x_1, \log x_2, \ldots, \log x_N).$$

A set $\Omega \subset \mathbb{C}^N$ is called a *Reinhardt set* if $\tau^{-1}(\tau(\Omega)) = \Omega$. A Reinhardt set $\Omega \subset (\mathbb{C}^*)^N (:= (\mathbb{C} \setminus \{0\})^N)$ is said to be *logarithmically convex* if the set $\mathrm{Log}(\tau(\Omega))$ is convex.

Let k be a natural number, and let $\xi_k : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ be defined by

$$\xi_k(z_1, z_2, \ldots, z_N) := ({z_1}^k, {z_2}^k, \ldots, {z_N}^k).$$

Given $\Omega \subset \mathbb{C}^N$, set $\sqrt[k]{\Omega} := {\xi_k}^{-1}(\Omega)$ and $|\sqrt[k]{\Omega}| := \tau({\xi_k}^{-1}(\Omega))$.

Given $\mathcal{G} = \{u^{(1)}, u^{(2)}, \ldots, u^{(M)}\} \subset \mathbb{Z}^N$, the *convex polyhedral cone* generated by $\mathcal{G}$ is the convex set $\langle \mathcal{G} \rangle := \{\lambda_1 u^{(1)} + \lambda_2 u^{(2)} + \cdots + \lambda_M u^{(M)};\, \lambda_j \in \mathbb{R}_{\geq 0}\}$. A cone σ is *rational* if there exists $\mathcal{G} \subset \mathbb{Z}^N$ such that $\sigma = \langle \mathcal{G} \rangle$. By *cone* we will mean convex polyhedral cone. A cone is said to be *strongly convex* if it does not contain any proper linear subspace.

For $\vartheta \in \mathbb{R}^N \setminus \{\underline{0}\}$ and $a \in \mathbb{R}$, we will denote by $H^+(\vartheta; a)$ the affine half-space $\{u \in \mathbb{R}^N; u \cdot \vartheta \geq a\}$.

Given $\varrho = (\varrho_1, \ldots, \varrho_N) \in (\mathbb{R}_{>0})^N$, we set

$$\bar{\mathbb{D}}_\varrho := \{z = (z_1, \ldots, z_N) \in \mathbb{C}^N; \quad |z_i| \leq \varrho_i,\ i = 1, \ldots, N\}$$

and $\bar{\mathbb{D}}^*_\varrho := \bar{\mathbb{D}}_\varrho \cap (\mathbb{C}^*)^N$, where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

Given $u = (I_1, I_2, \ldots, I_N) \in \mathbb{Z}^N$ and $z = (z_1, z_2, \ldots, z_N)$ we set

$$z^u := {z_1}^{I_1} {z_2}^{I_2} \cdots {z_N}^{I_N}.$$

Given an M-tuple of vectors $\mathcal{G} = \{u^{(1)}, \ldots, u^{(M)}\} \subset \mathbb{Z}^N$, set $\Phi_\mathcal{G} : (\mathbb{C}^*)^N \longrightarrow (\mathbb{C}^*)^M$ to be the map defined by

$$\Phi_\mathcal{G}(z) := (z^{u^{(1)}}, z^{u^{(2)}}, \ldots, z^{u^{(M)}}). \tag{2.1}$$

Given a Laurent series $s = \sum_{I \in \mathbb{Z}^N} a_I z^I$, we call $\mathcal{E}(s) := \{I \in \mathbb{Z}^N;\ a_I \neq 0\}$ the *set of exponents* of s.

Let $\delta : (\mathbb{C}^N, \underline{0}) \longrightarrow (\mathbb{C}, 0)$ be a germ of an analytic function. The set of exponents of its Taylor series development is contained in $(\mathbb{R}_{\geq 0})^N$. The *Newton polyhedron* of δ, $\mathrm{NP}(\delta)$, is the convex hull of the set $\mathcal{E}(\delta) + (\mathbb{R}_{\geq 0})^N$. See Figure 1.

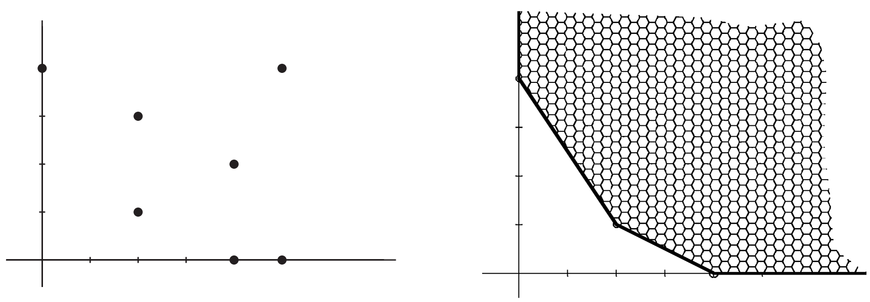

FIGURE 1. The dots on the left are the set of exponents of $f := ({z_1}^4 + {z_1}^5 + {z_2}^4 + {z_1}^2 z_2 + {z_1}^4 {z_2}^2 + {z_1}^2 {z_2}^3 + {z_1}^5 {z_2}^4)$. On the right is the Newton polyhedron $\mathrm{NP}(f)$.

Let A be a vertex of the Newton polyhedron of an analytic function δ. The set of vectors $v \in \mathbb{R}^N$ such that $(A + \lambda v) \in \mathrm{NP}(\delta)$ for some positive real number λ is a cone σ. We will say that σ is the *cone of the Newton polyhedron* of δ, associated to the vertex A. The cone σ is just the cone spanned by the faces of $\mathrm{NP}(\delta)$ that contain A. Cones of the Newton polyhedron of an analytic function are always strongly convex and contain the non-negative orthant $(\mathbb{R}_{\geq 0})^N$. See Figure 2.

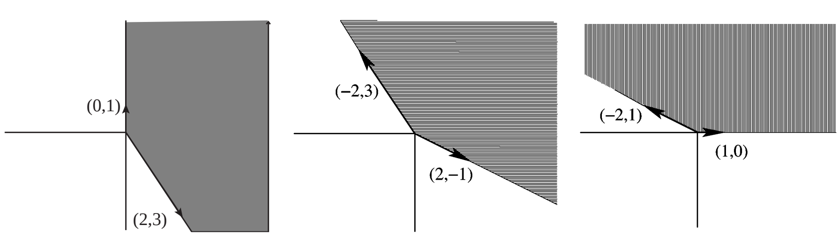

FIGURE 2. Cones associated to the vertices of $\mathrm{NP}(f)$.

3. Wedges and cones

We introduce the notion of σ-wedge, which is the analog for series with exponents in a cone σ, to polydiscs for series with non-negative exponents.

Suppose s is a Laurent series convergent at a point $p \in (\mathbb{C}^*)^N$; then s is also convergent at any point $z \in (\mathbb{C}^*)^N$ with $\tau(z)^I < \tau(p)^I$ for all $I \in \mathcal{E}(s)$.

Definition 3.1. Let $\sigma \subset \mathbb{R}^N$ be a cone. For $\varrho \in (\mathbb{R}_{>0})^N$, the σ-*wedge* of polyradius ϱ is the set

$$\mathrm{W}(\sigma, \varrho) := \left\{ z \in (\mathbb{C}^*)^N ;\, \tau(z)^u \leq \varrho^u,\, \forall\, u \in \sigma \cap \mathbb{Z}^N \right\}.$$

It is clear that σ-wedges are Reinhardt sets.

Example 3.2. Let $\mathcal{C}$ be the canonical basis of $\mathbb{R}^N$. For any $\varrho \in (\mathbb{R}_{>0})^N$, the $\langle \mathcal{C} \rangle$-wedge of polyradius ϱ is $\bar{\mathbb{D}}^*_\varrho$.

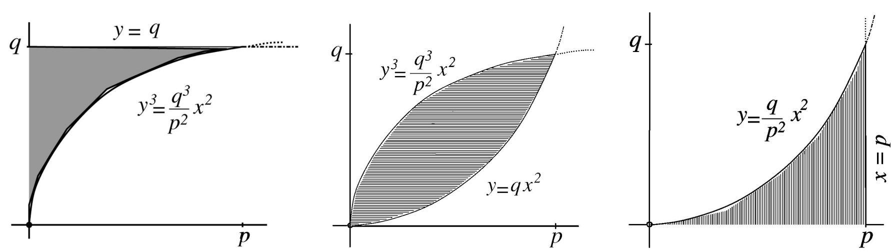

Figure 3. The image under τ of σ-wedges of polyradius (p, q), where the cones σ are those of $\mathrm{NP}(f)$.

Let σ be the cone generated by $\mathcal{G} = \{u^{(1)}, \ldots, u^{(M)}\} \subset \mathbb{Z}^N$. For any $\varrho \in (\mathbb{R}_{>0})^N$ we have:

$$\mathrm{W}(\sigma, \varrho) = \left\{ z \in (\mathbb{C}^*)^N ;\, \tau(z)^{u^{(i)}} \leq \varrho^{u^{(i)}},\, \forall\, i = 1, \ldots, M \right\}; \tag{3.1}$$

$$\mathrm{W}(\sigma, \varrho) = \Phi_{\mathcal{G}}^{-1} \left(\bar{\mathbb{D}}^*_{\Phi_{\mathcal{G}}(\varrho)} \right); \tag{3.2}$$

$$\text{if} \quad \sigma \subset \sigma', \quad \text{then} \quad \mathrm{W}(\sigma', \varrho) \subset \mathrm{W}(\sigma, \varrho); \tag{3.3}$$

$$\sqrt[k]{\mathrm{W}(\sigma, \varrho)} = \mathrm{W}(\sigma, |\sqrt[k]{\varrho}|); \tag{3.4}$$

$$\mathrm{Log}(\tau(\mathrm{W}(\sigma, \varrho))) = \bigcap_{u \in \sigma \cap \mathbb{Z}^N} H^+(-u; -\log \varrho^u). \tag{3.5}$$

Proposition 3.3. *Let σ be a rational cone. A σ-wedge is a connected set with connected interior.*

Proof. For any $\varrho \in (\mathbb{R}_{>0})^N$, $\mathrm{Log}(\tau(\mathrm{W}(\sigma, \varrho)))$ is the intersection of affine half hyperplanes (see (3.5)) and therefore a connected set of connected interior. The result follows from the fact that Log is a homeomorphism and τ is a projection with connected fiber. ☐

Proposition 3.4. *Let σ be a strongly convex rational cone. For any pair $\varrho, \varrho' \in (\mathbb{R}_{>0})^N$, the interior of* $\mathrm{W}(\sigma, \varrho) \cap \mathrm{W}(\sigma, \varrho')$ *is not empty.*

Proof. Suppose σ is strongly convex, and let ϑ be a vector of integer coordinates such that $\sigma \subset H^+(\vartheta;0)$ and $\sigma \cap \vartheta^\perp = \{\underline{0}\}$ [8, 1.2 (13)]. Each $u \in \sigma \cap \mathbb{Z}^N \setminus \{\underline{0}\}$ can be written in a unique way as a sum $u = v_u + \lambda_u \vartheta$ where $v_u \in \vartheta^\perp$ and $\lambda_u \in \mathbb{Q}_{>0}$.

Let $\mathcal{G} \subset \mathbb{Z}^N$ be a finite set of generators of σ. Set $\eta = \dfrac{1}{2} \min\limits_{u \in \mathcal{G}} \{\min\{\varrho^{\frac{1}{\lambda_u} u}, \varrho'^{\frac{1}{\lambda_u} u}\}\}$; since the λ_u's are positive rational numbers, $\eta > 0$.

Let $\mathcal{B} = \{e^{(1)}, \ldots, e^{(N-1)}, \vartheta\} \subset \mathbb{Z}^N$ be an orthogonal basis of $\mathbb{R}^N$. Let $\Phi_\mathcal{B}$ be as in (2.1); since $\mathcal{B}$ is a basis, $\Phi_\mathcal{B}$ is an automorphism of ${\mathbb{C}^*}^N$. Let $\omega = {\Phi_\mathcal{B}}^{-1}(1, \ldots, 1, \eta)$. For each $u \in \sigma$,

$$\tau(\omega)^u = \tau(\omega)^{v_u} \tau(\omega)^{\lambda_u \vartheta} = |\omega^{v_u}| |\omega^\vartheta|^{\lambda_u} = 1\eta^{\lambda_u} \leq \frac{1}{2} \min\{\varrho^u, \varrho'^u\} < \min\{\varrho^u, \varrho'^u\}.$$

Hence ω belongs to the interior of $W(\sigma, \varrho) \cap W(\sigma, \varrho')$. □

Remark 3.5. Suppose that σ is not strongly convex. Then there exists $\vartheta \in \mathbb{Z}^n \setminus \{\underline{0}\}$ such that $\langle \vartheta, -\vartheta \rangle \subset \sigma$. For any $\varrho \in (\mathbb{R}_{>0})^N$,

$$\begin{aligned} \mathrm{W}(\sigma, \varrho) \overset{(3.3)}{\subset} \mathrm{W}(\langle \vartheta, -\vartheta \rangle, \varrho) \overset{(3.1)}{=} & \left\{ z \in (\mathbb{C}^*)^N ;\, \tau(z)^\vartheta \leq \varrho^\vartheta, \tau(z)^{-\vartheta} \leq \varrho^{-\vartheta} \right\} \\ = & \left\{ z \in (\mathbb{C}^*)^N ;\, \tau(z)^\vartheta = \varrho^\vartheta \right\} \end{aligned}$$

(so its interior is empty).

Proposition 3.6. *Let σ be a rational cone. A σ-wedge is a logarithmically convex Reinhardt set. If σ is strongly convex, then its interior is a non-empty logarithmically convex Reinhardt domain.*

Proof. For any $\varrho \in (\mathbb{R}_{>0})^N$, $\mathrm{Log}(\tau(\mathrm{W}(\sigma, \varrho)))$ is the intersection of affine half hyperplanes (see (3.5)) and therefore a convex set. If σ is strongly convex, then its interior is connected (Proposition 3.3) and non-empty (Proposition 3.4). □

4. Laurent series development over a σ-wedge

A holomorphic function φ, defined on a Reinhardt domain Ω, has a (unique) Laurent series expansion that converges uniformly to φ on compact subsets of Ω [2, I.6].

The Taylor development of a function, holomorphic on a disc centered at the origin, is a series with exponents in the non-negative orthant. In this section we will see a similar result for σ-wedges and series with exponents in the cone σ.

Definition 4.1. Let σ be a cone. We will say that a set $\Omega \subset (\mathbb{C}^*)^N$ is σ-*complete* if for any $z \in \Omega$ the σ-wedge of polyradius $\tau(z)$ is contained in Ω.

A σ-wedge is σ-complete. A σ-complete set is a Reinhardt set. If σ is strongly convex, from Propositions 3.3 and 3.4, it follows that a σ-complete set is connected with connected interior.

Remark 4.2. The domain of convergence of a Laurent series with exponents in a cone σ is σ-complete.

Remark 4.3. As a consequence of (3.4) we have that, if Ω is σ-complete and k is a natural number, then $\sqrt[k]{\Omega}$ is σ-complete.

Lemma 4.4. *Let Ω be a Reinhardt domain, let $\varphi : \Omega \longrightarrow \mathbb{C}$ be a bounded holomorphic function, and take $\vartheta \in \mathbb{Z}^N \setminus \{\underline{0}\}$. If Ω is $H^+(\vartheta;0)$-complete, then the set of exponents of the Laurent series expansion of φ is contained in $H^+(\vartheta;0)$.*

Proof. Let $\mathcal{B} = \{e^{(1)}, \dots, e^{(N-1)}, \vartheta\} \subset \mathbb{Z}^N$ be an orthogonal basis of $\mathbb{R}^N$. We have $H^+(\vartheta;0) = \langle e^{(1)}, \dots, e^{(N-1)}, -e^{(1)}, \dots, -e^{(N-1)}, \vartheta\rangle$.

For each $I \in \mathbb{Z}^N$ let $I_{\mathcal{B}}$ be the coordinates of I in the basis $\mathcal{B}$. Since $\mathcal{B} \subset \mathbb{Z}^N$, there exists a natural number d such that $dI_{\mathcal{B}} \in \mathbb{Z}^N$ for any $I \in \mathbb{Z}^N$.

Let $s = \sum_{I\in\mathbb{Z}^N} a_I z^I$ be the Laurent series expansion of φ, and set

$$t := \sum_{I\in\mathbb{Z}^N} a_I y^{dI_{\mathcal{B}}}, \qquad \text{where} \quad y = (y_1, \dots, y_N).$$

For any $z \in \sqrt[d]{\Omega}$, $t(\Phi_{\mathcal{B}}(z)) = s(z^d)$. Hence t is convergent and bounded in $\Phi_{\mathcal{B}}(\sqrt[d]{\Omega})$.

From the way t has been defined:

S) $\mathcal{E}(s) \subset H^+(\vartheta;0)$ if and only if $\mathcal{E}(t) \subset H^+((0,\dots,0,1);0)$.

Let π be the map defined by $\pi(y_1,\dots,y_N) := (y_1,\dots,y_{N-1})$, and let us rewrite the series t in the form

$$t = \sum_{j\in\mathbb{Z}} g_j(\pi(y)) {y_N}^j, \qquad \text{where} \quad g_j = \sum_{(0,\dots,0,1)\cdot dI_{\mathcal{B}}=j} a_I x^{d\pi(I_{\mathcal{B}})}$$

for $x = (x_1, \dots, x_{N-1})$.

Now statement **S)** becomes:

S′) $\mathcal{E}(s) \subset H^+(\vartheta;0)$ if and only if $g_j(x) \equiv 0$ for all $j < 0$.

For any $\varrho \in \tau(\sqrt[d]{\Omega})$, by Remark 4.3, the wedge $\mathrm{W}(H^+(\vartheta,0),\varrho)$ is contained in $\sqrt[d]{\Omega}$. So, $\Phi_{\mathcal{B}}(\mathrm{W}(H^+(\vartheta,0),\varrho)) \subset \Phi_{\mathcal{B}}(\sqrt[d]{\Omega})$. Hence t is convergent and bounded on:

$$\Phi_{\mathcal{B}}(\mathrm{W}(H^+(\vartheta,0),\varrho)) = \left\{ y \in (\mathbb{C}^*)^N; \quad |y_N| \le \varrho^{\vartheta},\, |y_i| = \varrho^{e^{(i)}},\, i = 1,\dots,N-1 \right\}.$$

For any $z \in \sqrt[d]{\Omega}$, the series in one variable,

$$t_z(y_N) := t(\pi(\Phi_{\mathcal{B}}(z)), y_N) = \sum_{j\in\mathbb{Z}} g_j(\pi(\Phi_{\mathcal{B}}(z))) {y_N}^j$$

is convergent and bounded on the punctured disc $\mathbb{D}^*_{\tau(z)^{\vartheta}}$. Then, there exists a unique function, holomorphic on $\mathbb{D}_{\tau(z)^{\vartheta}}$, that extends t_z. Therefore t_z is necessarily the Taylor development of that function and, as a consequence, it cannot have negative exponents.

We have seen that, for $j < 0$, $g_j(x) = 0$ for all $x \in \pi(\Phi_{\mathcal{B}}(\sqrt[d]{\Omega}))$. Since $\pi(\Phi_{\mathcal{B}}(\sqrt[d]{\Omega}))$ is a non-empty open subset of $\mathbb{C}^{N-1}$, $g_j \equiv 0$. □

Proposition 4.5. *Let σ be a cone, and let $\Omega \subset (\mathbb{C}^*)^N$ be a σ-complete Reinhardt domain. If φ is a bounded holomorphic function on Ω, and s is its Laurent series expansion in Ω, then the set of exponents of s is contained in σ.*

Proof. Let $\vartheta \in \mathbb{Z}^N \setminus \{\underline{0}\}$ be such that $\sigma \subset H^+(\vartheta;0)$. By (3.3), Ω is $H^+(\vartheta;0)$-complete. So (Lemma 4.4) $\mathcal{E}(s) \subset H^+(\vartheta;0)$, and the result follows from the fact that

$$\sigma = \bigcap_{\vartheta\in\mathbb{Z}^N,\, \sigma\subset H^+(\vartheta;0)} H^+(\vartheta;0),$$

for any cone σ. □

5. Cones of a polyhedron and wedges not intersecting the zero-locus

Proposition 5.1. *Let $\delta : (\mathbb{C}^N, \underline{0}) \longrightarrow (\mathbb{C}, 0)$ be a germ of an analytic function, and let σ be a cone of the Newton polyhedron of δ. There exists a σ-wedge that does not intersect the zero locus of δ.*

Proof. Let $\mathcal{T}(\delta) = \sum_{I\in\mathcal{E}(\delta)} b_I z^I$ be the Taylor development of δ, and let Ω be its domain of convergence. Let A be the vertex of σ in $\mathrm{NP}(\delta)$. The meromorphic function $f = z^{-A}\delta$ is holomorphic on $\Omega^* = \Omega \cap (\mathbb{C}^*)^N$, and, outside the coordinate hyperplanes, the zero locus of δ coincides with the zero locus of f. We will find a σ-wedge where f does not vanish.

Notice that the Laurent series development of f on Ω^*,

$$s = \frac{1}{z^A}\mathcal{T}(\delta) = \sum_{I-A\in\sigma\cap\mathbb{Z}^N} b_I z^{I-A} = \sum_{I\in\sigma\cap\mathbb{Z}^N} a_I z^I, \qquad a_I = b_{I+A},$$

is a series with exponents in σ.

Let $\mathcal{G} = \{u^{(1)}, u^{(2)}, \ldots, u^{(M)}\}$ be a set of generators of the semigroup $\sigma \cap \mathbb{Z}^N$ (Gordan's lemma [8, 1.2, Prop. 1]). For each $I \in \mathcal{E}(s)$, let $I_\mathcal{G}$ be an M-tuple of natural numbers $I_\mathcal{G} = (I^1_\mathcal{G}, \ldots, I^M_\mathcal{G})$ such that $I = I^1_\mathcal{G} u^{(1)} + I^2_\mathcal{G} u^{(2)} + \cdots + I^M_\mathcal{G} u^{(M)}$.

The series in the variables $y = (y_1, y_2, \ldots, y_M)$ defined by

$$t(y) = \sum_{I\in\mathcal{E}(\delta)} a_I y^{I_\mathcal{G}} \tag{5.1}$$

is a series with positive exponents.

Let $\Phi_\mathcal{G}$ be as in (2.1). For any $z \in (\mathbb{C}^*)^N$ contained in the domain of convergence of s, we have that

$$t(\Phi_\mathcal{G}(z)) = s(z). \tag{5.2}$$

In particular, t is convergent in $\Phi_\mathcal{G}(z)$. Now, since t has positive exponents, t converges in $\bar{\mathbb{D}}_{\Phi_\mathcal{G}(\tau(z))}$ (the closed polydisc of polyradius $\tau(\Phi_\mathcal{G}(z))$).

The series t is convergent, does not have negative exponents, and $t(\underline{0}) = b_{\underline{0}} = a_A \neq 0$. So there exists $\varsigma \in (\mathbb{R}_{>0})^M$ such that $t(y) \neq 0$ for any $y \in \bar{\mathbb{D}}^*_\varsigma$. Take $\varrho \in \tau(\Phi_\mathcal{G}^{-1}(\bar{\mathbb{D}}^*_\varsigma))$. By (3.2), the σ-wedge of polyradius ϱ has the desired property. ☐

6. Coverings of logarithmically convex Reinhardt domains

In this section we extend the well-known fact that coverings of the punctured disc are isomorphic to some covering $t \mapsto t^k$.

Lemma 6.1. *The fundamental group of a logarithmically convex Reinhardt domain $\Omega \subset \mathbb{C}^{*N}$ is isomorphic to $\mathbb{Z}^N$.*

Proof. The projection $\tau : \Omega \longrightarrow \tau(\Omega)$ is a fiber bundle with fiber an N-dimensional torus. $\mathrm{Log}(\tau(\Omega))$ is convex and then contractible. Since Log is a homeomorphism, $\tau(\Omega)$ is also contractible. Then (see, for example, [17, §11.6]) Ω is homeomorphic to $\mathbb{T}^N \times \tau(\Omega)$. Again as a consequence of the contractibility of $\tau(\Omega)$ we have the result. ☐

Proposition 6.2. *Let $\mathcal{A}$ be a complex analytic set. Let $\mathrm{p} : \mathcal{A} \longrightarrow \mathbb{C}^N$ be a finite morphism. Let $\Omega \subset (\mathbb{C}^*)^N$ be a logarithmically convex Reinhardt domain contained in $\mathrm{p}(\mathcal{A})$ such that $\mathrm{p} : \mathcal{A} \cap \mathrm{p}^{-1}(\Omega) \longrightarrow \Omega$ is locally biholomorphic. Then for each* *connected component* C *of* $\mathrm{p}^{-1}(\Omega)$ *there exists a natural number* k *and a morphism* $\varphi : \sqrt[k]{\Omega} \longrightarrow \mathrm{C}$ *such that* $\varphi(\sqrt[k]{\Omega}) = \mathrm{C}$ *and* $\mathrm{p} \circ \varphi = \xi_k$.

Proof. Let C be a connected component of $\mathrm{p}^{-1}(\Omega)$, and let k be the cardinal of the generic fiber of $\mathrm{p}|_{\mathrm{C}}$. Since both $\mathrm{p}|_{\mathrm{C}}$ and $\xi_k|_{\mathrm{C}}$ are locally biholomorphic, the pairs (C, p) and $(\sqrt[k]{\Omega}, \xi_k)$ are, respectively, a k-sheeted and a k^N-sheeted covering of Ω:

$$\begin{tikzcd} \sqrt[k]{\Omega} \arrow[dr, "\xi_k"'] & & \mathrm{C} \arrow[dl, "\mathrm{p}"] \\ & \Omega & \end{tikzcd}$$

Choose a point $z_0 \in \Omega$, a point $z_1 \in {\xi_k}^{-1}(z_0)$, and a point P in $\mathrm{p}^{-1}(z_0) \cap \mathrm{C}$. Consider the induced monomorphisms on the fundamental groups:

$$\begin{tikzcd} \pi_1(\sqrt[k]{\Omega}, z_1) \arrow[dr, "\xi_{k*}"'] & & \pi_1(\mathrm{C}, P) \arrow[dl, "\mathrm{p}_*"] \\ & \pi_1(\Omega, z_0) & \end{tikzcd}$$

An element $\gamma \in \pi_1(\Omega, z_0)$ is in the subgroup $\xi_{k*}\pi_1(\Omega, z_0)$ if and only if $\gamma = \alpha^k$ for some $\alpha \in \pi_1(\Omega, z_0)$.

On the other hand, the index of $\mathrm{p}_*(\pi_1(\mathrm{C}, P))$ in $\pi_1(\Omega, z_0)$ is equal to k (see, for example, [15, V§7]). Since $\pi_1(\Omega, z_0)$ is abelian, the cosets of $\mathrm{p}_*(\pi_1(\mathrm{C}, P))$ in $\pi_1(\Omega, z_0)$ form a group of order k. Therefore, for any $\alpha \in \pi_1(\Omega, z_0)$ the element α^k belongs to $\mathrm{p}_*(\pi_1(\mathrm{C}, P))$. Then $\xi_{k*}(\pi_1(\sqrt[k]{\Omega}, z_1)) \subset \mathrm{p}_*(\pi_1(\mathrm{C}, P))$.

The lifting lemma [15, Theorem V.5.1] ensures the existence of a (unique) map φ, such that $\varphi(z_1) = P$ and the diagram

$$\begin{tikzcd} \sqrt[k]{\Omega} \arrow[rr, "\varphi"] \arrow[dr, "\xi_k"'] & & \mathrm{C} \arrow[dl, "\mathrm{p}"] \\ & \Omega & \end{tikzcd}$$

commutes. □

Remark 6.3. In Proposition 6.2, k can be chosen to be the degree of the covering $\mathrm{p} : \mathrm{C} \longrightarrow \Omega$. Since for each $P \in \mathrm{p}^{-1}(z_0) \cap \mathrm{C}$ there exists a unique φ such that $\varphi(z_1) = P$, there exist k different φ's.

7. The theorem

Let $\mathcal{A}$ be an analytic set, and assume that $\underline{0} \in \mathcal{A} \subset \mathbb{C}^{N+M}$, where $N = \dim_{\underline{0}} \mathcal{A}$ is the dimension of $\mathcal{A}$ at $\underline{0}$. There exists a linear projection $p : \mathbb{C}^{N+M} \longrightarrow \mathbb{C}^N$, and a neighborhood U of $\underline{0}$ such that the restriction of p to $\mathcal{A} \cap U$ is a finite morphism. Moreover, the set of such p is open and everywhere dense in the space of linear projections.

Let $\mathrm{p} : \mathcal{A} \cap U \longrightarrow \mathbb{C}^N$ be a finite morphism. Then $\mathrm{p}(\mathcal{A} \cap U)$ is a neighborhood of $\mathrm{p}(\underline{0})$ and there exists an analytic set $\Delta \subset \mathbb{C}^N$, with $\dim_{\mathrm{p}(\underline{0})} \Delta < N$, such that $\mathrm{p} : \mathcal{A} \cap U \setminus \mathrm{p}^{-1}(\Delta) \longrightarrow \mathrm{p}(\mathcal{A} \cap U) \setminus \Delta$ is locally biholomorphic [7, 3.7]. The smallest set Δ with this property will be called the *discriminant locus* of p.

Theorem 7.1. *Let $\mathcal{A}$ be an analytic subset of $\mathbb{C}^{N+M}$, $\underline{0} \in \mathcal{A}$, $\dim_{\underline{0}}(\mathcal{A}) = N$. Let U be a neighborhood of $\underline{0}$, and let π be the restriction to $\mathcal{A} \cap U$ of the projection $(z_1, z_2, \ldots, z_{N+M}) \mapsto (z_1, z_2, \ldots, z_N)$. Assume π is a finite morphism. Let δ be an analytic function vanishing on the discriminant locus of π.*

For each cone σ of the Newton polyhedron of δ, there exist $k \in \mathbb{N}$ and an M-tuple of convergent Laurent series, $(s_1, \ldots, s_M)$, such that

$$\mathcal{E}(s_i) \subset \sigma, \qquad i = 1, \ldots, M,$$

and

$$f({z_1}^k, \ldots, {z_N}^k, s_1, \ldots, s_M) = 0$$

for any f vanishing on $\mathcal{A}$, and any z in the domain of convergence of the s_i.

Proof. Let us choose ϱ_1 small enough so that $\bar{\mathbb{D}}_{\varrho_1}$ is contained in $\pi(\mathcal{A} \cap U)$ and so that $\pi^{-1}(\mathbb{D}_{\varrho_1})$ is a bounded set of $\mathbb{C}^{N+M}$. Since σ contains the non-negative orthant, by (3.3), $\mathrm{W}(\sigma, \varrho_1) \subset \pi(\mathcal{A} \cap U)$.

By Proposition 5.1, we can find an N-tuple of positive real numbers, ϱ_2, such that $\mathrm{W}(\sigma, \varrho_2)$ does not intersect the zero locus of δ. Since δ vanishes on the discriminant locus of π, $\mathrm{W}(\sigma, \varrho_2)$ does not intersect the discriminant locus, and π restricted to $\mathrm{W}(\sigma, \varrho_2)$ is locally biholomorphic.

Since σ is strongly convex, the intersection $\mathrm{W}(\sigma, \varrho_1) \cap \mathrm{W}(\sigma, \varrho_2)$ is non-empty (Proposition 3.4). Take $\varrho \in \tau(\mathrm{W}(\sigma, \varrho_1) \cap \mathrm{W}(\sigma, \varrho_2))$. Let Ω be the interior of $\mathrm{W}(\sigma, \varrho)$.

Since σ is strongly convex, by Proposition 3.6, Ω is a non-empty logarithmically convex Reinhardt domain.

Let C be a connected component of $\pi^{-1}(\Omega)$. Then, by Proposition 6.2, there exists a natural number k and a morphism $\varphi : \sqrt[k]{\Omega} \longrightarrow \mathrm{C}$, such that $\pi \circ \varphi = \xi_k$. We can express φ as

$$\begin{array}{cccc} \varphi : & \sqrt[k]{\Omega} & \longrightarrow & \mathcal{A} \\ & (x_1, \ldots, x_N) & \mapsto & ({x_1}^k, \ldots, {x_N}^k, \varphi_1, \ldots, \varphi_M), \end{array}$$

where $\varphi_i : \sqrt[k]{\Omega} \longrightarrow \mathbb{C}$ is a holomorphic function for $i \in \{1, \ldots, M\}$. Moreover, since $\varphi(\sqrt[k]{\Omega}) \subset \mathcal{A} \cap U$, each φ_i is bounded.

Since $\sqrt[k]{\Omega}$ is σ-complete, by Proposition 4.5, we have that for each $i \in \{1, \ldots, M\}$, there is a Laurent series s_i that converges uniformly to φ_i in compact subsets of $\sqrt[k]{\Omega}$ and such that $\mathcal{E}(s_i) \subset \sigma$.

Now, for any z in the domain of convergence of the s_i's, the point of $\mathbb{C}^{N+M}$, $P = ({z_1}^k, \ldots, {z_N}^k, s_1(z), \ldots, s_M(z))$ is in $\mathcal{A}$ (or in an analytic extension of $\mathcal{A}$) and, therefore, any function f vanishing on $\mathcal{A}$ also vanishes on P. $\square$

Remark 7.2. Taking as C the different connected components of $\pi^{-1}(\Omega)$, we may in fact find $k_1, \ldots, k_r$ natural numbers with $\sum_{i=1}^{r} k_i$ equal to the degree of π and, for each $i \in \{1, \ldots, r\}$, k_i different M-tuples of series. In other words, we have seen that if d is the degree of π, there are d M-tuples of Puiseux series such that $f(z_1, \ldots, z_N, s_1, \ldots, s_M) = 0$ for any $f \in I(\mathcal{A})$.

Corollary 7.3 (Abhyankar-Jung)**.** *Let $\underline{0}$ be a quasi-ordinary singularity of a complex analytic set $\mathcal{A} \subset \mathbb{C}^{N+M}$, $\dim_{\underline{0}}(\mathcal{A}) = N$. Then there exists a natural number* *k, and M convergent power series $\phi_j, j = 1, \dots, M$, such that*

$$z_i = {t_i}^k,\ i = 1, \dots, N, \qquad z_{N+j} = \phi_j(t_1, \dots, t_N),\ j = 1, \dots, M, \tag{7.1}$$

are parametric equations of $\mathcal{A}$ about $\underline{0}$.

Proof. By definition of a quasi-ordinary singularity, the discriminant locus of the projection π is contained in the coordinate hyperplanes, the function $\delta(z) = z_1 \cdots z_N$ vanishes there and its Newton polyhedron has just one cone which is the non-negative orthant. The result is then a consequence of the theorem. □

Remark 7.4. Let $\mathcal{A} \subset \mathbb{C}^{N+M}$ and π be as in Theorem 7.1, and suppose that $\mathcal{A}$ is a surface (i.e., $N = 2$). The discriminant locus of π is contained in an analytic curve $\delta \subset \mathbb{C}^2$. Performing the quadratic transformations needed to solve δ to both δ and $\mathcal{A}$ we arrive at a new surface $\tilde{\mathcal{A}}$ and a projection $\tilde{\pi}$ whose discriminant locus $\tilde{\delta}$ has only normal crossings. This fact is used (see [12, 18]) to get a resolution of $\mathcal{A}$. In [3, Chapt. 2] it is shown that the transform of a σ-wedge is a tubular neighborhood of a connected piece of the exceptional divisor of $\tilde{\delta}$.

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