ON THE ELLIPTIC EQUATION $\Delta u + K(x)e^{2u} = 0$ ON $B^2$

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Abstract. In this paper we consider the existence problem for the elliptic equation $\Delta u + K(x)e^{2u} = 0$ on $B^2 = \{ x \in \mathbb{R}^2 \mid |x| < 1 \}$, which arises in the study of conformal deformation of the hyperbolic disc. We prove an existence result for the above equation.

1. Introduction

Let $B^2 = \{ x \in \mathbb{R}^2 \mid |x| < 1 \}$. We study the existence problem for the elliptic equation

$$\Delta u + K(x)e^{2u} = 0 \quad \text{on } B^2,$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the standard Laplacian and $K(x)$ is a locally Hölder continuous function.

Equation (1) arises in the study of conformal deformation of the hyperbolic disc $H^2$.

In general, given a two-dimensional Riemannian manifold $(M,g)$, a function $K$ on $M$ is the Gaussian curvature of a conformal metric $\tilde{g} = e^{2u}g$ if and only if $u$ is a solution of the following elliptic equation:

$$\Delta_g u - k_g + Ke^{2u} = 0,$$

where $k_g$ and $\Delta_g$ are the Gaussian curvature function and the Laplace-Beltrami operator on $M$ with respect to the metric $g$. This problem has been studied by many authors (cf. [1], [3], [9]).

In the special case where $M$ is the hyperbolic disc $H^2$, equation (2) becomes

$$\Delta_h u + 1 + K(x)e^{2u} = 0,$$

where $\Delta_h$ is the hyperbolic Laplacian. Since the hyperbolic metric

$$g_h = \frac{4}{(1 - |x|^2)^2} g$$

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is conformal to the Euclidean metric $\eta$ on $B^2$, a solution of (1) provides a conformal metric of $g_h$:

$$e^{2u} \eta = e^{2u + 2 \ln(1 - |x|^2) - \ln 4} \cdot g_h.$$  

(5)

Now we can state our main result as follows:

**Theorem 1.** Suppose $K(x)$ is a locally Hölder continuous function on $B^2$ that is positive somewhere and for some positive constants $C$ and $\sigma$, $0 < \sigma < 1$, the inequality

$$|K(x)| \leq \frac{C}{1 - |x|^\sigma}$$

holds for $|x| < 1$. Then equation (1) admits a $C^2$ solution.

**Remark.** J. Bland and M. Kalka [1] and A. Ratto, M. Rigoli, and L. Véron [9] use the technique of supsolution and subsolution to study the equation (1), and get good results for the case that $K(x)$ is negative near the boundary of $B^2$. Our Theorem 1 allows $K(x)$ to go to positive infinity near the boundary of $B^2$.

2. Preliminaries

For $-\frac{1}{2} < \alpha < 0$, consider the conformal metric

$$g_\alpha = (1 - r^2)^{2\alpha} \eta,$$

where $r = |x|$. Since the Gaussian curvature of $g_\alpha$ is $\frac{4\alpha}{(1 - r^2)^{2 + 2\alpha}}$, the Gaussian curvature $K(x)$ of the conformal metric $e^{2u}g_\alpha$ satisfies the following equation:

$$\Delta_\alpha u - \frac{4\alpha}{(1 - r^2)^{2 + 2\alpha}} + K(x)e^{2u} = 0,$$

where $\Delta_\alpha$ is the Laplace-Beltrami operator on $(B^2, g_\alpha)$.

It is easy to verify that

$$\Delta_\alpha (-\alpha \ln(1 - r^2)) = \frac{4\alpha}{(1 - r^2)^{2 + 2\alpha}},$$

then make the substitution $u = w - \alpha \ln(1 - r^2) + lr^2$ for a constant $l < 0$. We have (still denote by $u$)

$$\Delta_\alpha u + \frac{4l}{(1 - r^2)^{2\alpha}} + K(x)e^{2u} = 0,$$

(10)

where $K(x) = K(x)(1 - r^2)^{-2\alpha} e^{2lr^2}$ and

$$\int_{B^2} \frac{4l}{(1 - r^2)^{2\alpha}} dA = 4\pi l,$$

(11)

where $dA$ is the area element of $(B^2, g_\alpha)$.

Let $H_k$ denote the Hilbert space of $L^2_{loc}$-functions for which

$$||u||_k = \left[ \sum_{j=0}^{k} \int_{B^2} |D^j u|^2 dA \right]^{\frac{1}{2}} < +\infty,$$

(12)
where $|D^j u|$ is the pointwise norm (with respect to $g_a$) of the $j^{th}$ covariant derivative of $u$. In particular,

$$\|u\|_1 = \left[ \int_{B^2} u^2 dA + \int_{B^2} |\nabla u|^2 dA \right]^{\frac{1}{2}},$$

where $\nabla u$ is the gradient of $u$ with respect to $g_a$.

The following is a type of Trudinger inequality (cf. [7], which we also follow in the proof).

**Proposition 2.** There exist positive constants $\beta, \gamma$ such that

$$\int_{B^2} e^{\beta u^2} dA \leq \gamma$$

for all $u \in H^1$ with $\int_{B^2} u dA = 0$ and $\|\nabla u\|_{L^2} \leq 1$.

**Proof.** We apply symmetrization which is based on the isoperimetric inequality that holds on $(B^2, g_a)$ (cf. [6], [7]); to be specific, with $u(x)$ we associate a nonincreasing radial function $u(r)$ by the requirement

$$\mu\{x \mid u^* > \rho\} = \mu\{x \mid u > \rho\}$$

for every $\rho$, where $\mu$ denotes the measure on $(B^2, g_a)$.

Since the Dirichlet norm is a conformal invariant and symmetrization decreases the Dirichlet norm,

$$\|\nabla u^*\|_{L^2} \leq \|\nabla u\|_{L^2}.$$  

Thus we may assume that $u = u(r) = u^*(r)$. Now introduce $\omega(t) = \sqrt{4\pi} u(r)$, where $r^2 = e^t$; then

$$\|\dot{\omega}\|_{L^2(dt)} = \|\nabla u\|_{L^2},$$

where $\dot{\omega} = \frac{d\omega}{dt}$, and we must show that

$$\int_{-\infty}^0 e^{\frac{4}{\pi} \omega^2} \cdot (1 - e^t)^{2\alpha} e^t dt \leq \frac{\gamma}{\pi}.  \tag{15}$$

Using the Schwarz inequality, we find that

$$(\omega(t) - \omega(s))^2 \leq |t - s| \int_s^t \dot{\omega}^2 dt \leq |t - s|,$$

or

$$-|t - s|^\frac{1}{2} \leq \omega(t) - \omega(s) \leq |t - s|^\frac{1}{2}.  \tag{16}$$

Let $\rho(t) = C(1 - e^t)^{2\alpha} e^t$ be such that (in this paper we use $C$ to denote different positive constants)

$$\int_{-\infty}^0 \rho(t) dt = 1$$

and

$$\int_{-\infty}^0 \omega(t) \rho(t) dt = 0.$$  

Multiplying (16) by $\rho(s)$ and integrating $ds$, we find that

$$|\omega(t)|^2 \leq (\int_{-\infty}^0 |t - s|^\frac{1}{2} \rho(s) ds)^2 \leq |t| + C.  \tag{17}$$

Thus we obtain (15) provided $\beta < 4\pi$. \qed
Corollary 3. If $\beta < 4\pi$, then

$$
\int_{B^2} e^{\delta |v|} dA \leq C e^{(\delta^2/(4\beta)) \|
\nabla v\|_{L^2}^2}
$$

for all $v \in H_1$ with $\int_{B^2} v dA = 0$.

Proof. Write $v = \|\nabla v\| \cdot u$, so that $\|\nabla u\| \leq 1$. Apply Proposition 2 to $u$ using $\delta |v| \leq \beta u^2 + \delta^2 \|
\nabla v\|_{L^2}^2/(4\beta)$. □

The next result is a type of Poincaré inequality, which we also need in the proof of Theorem 1.

Proposition 4. There exists a positive constant $C_1$ such that

$$
\|v\|_{L^2}^2 \leq C_1 \|
\nabla v\|_{L^2}^2
$$

for all $v \in H_1$ with $\int_{B^2} v dA = 0$.

Proof. Let $v = \|\nabla v\| \cdot u$, so that $\|\nabla u\| \leq 1$. It suffices to show that

$$
\|u\|_{L^2}^2 \leq C.
$$

We may use symmetrization to assume that $u = u(r)$ and introduce $\omega(t)$ as in the proof of Proposition 2. Using (17), we find that

$$
\|u\|_{L^2}^2 \leq C \int_{-\infty}^{0} |\omega|^2 (1 - e^t)^{2\alpha} e^t dt \leq C.
$$

□

3. Proof of Theorem 1

Now we can give the following.

Proof of Theorem 1. Define $E$ by

$$
E = \left\{ u \in H_1(B^2) \mid \int_{B^2} K_1 e^{2\nu} dA = -4\pi l \right\}.
$$

Since $K(x)$ is positive somewhere and $l < 0$, it is easy to see that $E$ is not empty.

We shall minimize the functional

$$
J(u) = \int_{B^2} \left( \frac{1}{2} |\nabla u|^2 - \frac{4l}{(1 - r^2)^{2\alpha}} \cdot u \right) dA
$$

for $u \in E$. Writing $u = v + b$, so that $\int_{B^2} v dA = 0$, and solving for $b$ in the constraint (20), one sees that $J$ can be expressed as

$$
J(u) = \frac{1}{2} \int_{B^2} |\nabla v|^2 dA + 2\pi l \ln \int_{B^2} K_1 e^{2\nu} dA - 2\pi l \ln(-4\pi l) - \int_{B^2} \frac{4l}{(1 - r^2)^{2\alpha}} \cdot v dA.
$$

By assumption (6) (take $\alpha = -\frac{1}{2}$), $K_1$ is bounded. Then use (18) and choose $-l$ small such that $\mid \frac{2\pi l}{1 - r^2} \mid < \frac{1}{8}$. Also, we note that $\frac{4l}{(1 - r^2)^{2\alpha}}$ is bounded on $B^2$. Then using (19) we have

$$
\left| \int_{B^2} \frac{4l}{(1 - r^2)^{2\alpha}} \cdot v dA \right| \leq \epsilon \int_{B^2} v^2 dA + \frac{1}{4\epsilon} \int_{B^2} \frac{4l}{(1 - r^2)^{2\alpha}} dA \leq \epsilon C_1 \|
\nabla v\|_{L^2}^2 + C.
$$
Taking $\epsilon = \frac{1}{8c_1}$, we find that

$$J(u) \geq \frac{1}{4} \|\nabla v\|^2_{L^2} - C. \tag{23}$$

Now let $u_j = v_j + c_j \in E$ be a minimal sequence, where $\int_{B^2} v_j dA = 0$. Since $J(u_j)$ is bounded, we have $\|\nabla v_j\|_{L^2} \leq C$ by (23). Then it follows from (19) that $\|v_j\|_1$ is bounded. Because the unit ball of any Hilbert space is weakly compact, we can extract a subsequence (which we again denote by $v_j$) converging weakly in $H_1$ to $v$, and this implies that $\int_{B^2} v dA = 0$.

The fact that $\|\nabla v_j\|_{L^2} \leq C$ and (19) hold for all $v_j$ implies that $\{v_j\}$ is pre-compact in $L^2$ (cf. [7], [10]). Then $v_j \to v$ strongly in $L^2$. Using the inequality $|e^z - 1| \leq |z| e^{|z|}$ and (18), we find that $\int_{B^2} K_i e^{2v_j} dA \to \int_{B^2} K_i e^{2v} dA$:

$$\left| \int_{B^2} K_i (e^{2v_j} - e^{2v}) dA \right| \leq C \int_{B^2} e^{2|v_j|} |v_j - v| e^{2|v_j - v|} dA \leq C \|v_j - v\|_{L^2}.$$

Since $e^{2v_j} \int_{B^2} K_i e^{2v} dA = -4\pi l$ for each $j$, we find that

$$c_j \to c = \frac{1}{2} \ln(-4\pi l/\int_{B^2} K_i e^{2v} dA),$$

and $u = v + c \in E$ gives the minimum for $J$ on $E$. Then by standard Lagrange multiplier theory we find that there is a constant $\lambda$ such that

$$\int_{B^2} \nabla u \cdot \nabla \phi - \frac{4l}{(1 - r^2)^2} \phi dA = \lambda \int_{B^2} K_i e^{2u} \phi dA \tag{24}$$

for all $\phi \in H_1$. Taking $\phi \equiv 1$ we find that $\lambda = 1$. So $u$ is a weak solution of (10). Local regularity implies that $u \in C^2(B^2)$. This completes the proof of Theorem 1. \[\square\]

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