PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 10, Pages 3083–3088
S 0002-9939(04)07366-6
Article electronically published on May 12, 2004

ON THE ELLIPTIC EQUATION $\Delta u + K(x)e^{2u} = 0$ ON B^2

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(Communicated by Richard A. Wentworth)

ABSTRACT. In this paper we consider the existence problem for the elliptic equation $\Delta u + K(x)e^{2u} = 0$ on $B^2 = \{x \in R^2 \mid |x| < 1\}$, which arises in the study of conformal deformation of the hyperbolic disc. We prove an existence result for the above equation.

1. INTRODUCTION

Let $B^2 = \{x \in R^2 \mid |x| < 1\}$. We study the existence problem for the elliptic equation

$$\Delta u + K(x)e^{2u} = 0 \quad \text{on } B^2, \tag{1}$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the standard Laplacian and $K(x)$ is a locally Hölder continuous function.

Equation (1) arises in the study of conformal deformation of the hyperbolic disc H^2.

In general, given a two-dimensional Riemannian manifold (M, g), a function K on M is the Gaussian curvature of a conformal metric $\tilde{g} = e^{2u}g$ if and only if u is a solution of the following elliptic equation:

$$\Delta_g u - k_g + Ke^{2u} = 0, \tag{2}$$

where k_g and Δ_g are the Gaussian curvature function and the Laplace-Beltrami operator on M with respect to the metric g. This problem has been studied by many authors (cf. [1], [3], [9]).

In the special case where M is the hyperbolic disc H^2, equation (2) becomes

$$\Delta_h u + 1 + K(x)e^{2u} = 0, \tag{3}$$

where Δ_h is the hyperbolic Laplacian. Since the hyperbolic metric

$$g_h = \frac{4}{(1 - |x|^2)^2}\eta \tag{4}$$

Received by the editors January 6, 2003 and, in revised form, May 3, 2003.

2000 *Mathematics Subject Classification.* Primary 53C21; Secondary 35J60.

Key words and phrases. Semilinear elliptic PDE, Gaussian curvature, conformal Riemannian metric.

The first author was supported in part by the China National Education Committee Science Research Foundation.

is conformal to the Euclidean metric η on B^2, a solution of (1) provides a conformal metric of g_h:

$$e^{2u}\eta = e^{2u+2\ln(1-|x|^2)-\ln 4} \cdot g_h. \tag{5}$$

Now we can state our main result as follows:

Theorem 1. *Suppose $K(x)$ is a locally Hölder continuous function on B^2 that is positive somewhere and for some positive constants C and σ, $0 < \sigma < 1$, the inequality*

$$|K(x)| \leq \frac{C}{|1-|x||^{\sigma}} \tag{6}$$

holds for $|x| < 1$. Then equation (1) admits a C^2 solution.

Remark. J. Bland and M. Kalka [1] and A. Ratto, M. Rigoli, and L. Véron [9] use the technique of supsolution and subsolution to study the equation (1), and get good results for the case that $K(x)$ is negative near the boundary of B^2. Our Theorem 1 allows $K(x)$ to go to positive infinity near the boundary of B^2.

2. PRELIMINARIES

For $-\frac{1}{2} < \alpha < 0$, consider the conformal metric

$$g_\alpha = (1-r^2)^{2\alpha}\eta, \tag{7}$$

where $r = |x|$. Since the Gaussian curvature of g_α is $\frac{4\alpha}{(1-r^2)^{2+2\alpha}}$, the Gaussian curvature $K(x)$ of the conformal metric $e^{2u}g_\alpha$ satisfies the following equation:

$$\Delta_\alpha u - \frac{4\alpha}{(1-r^2)^{2+2\alpha}} + K(x)e^{2u} = 0, \tag{8}$$

where Δ_α is the Laplace-Beltrami operator on (B^2, g_α).

It is easy to verify that

$$\Delta_\alpha(-\alpha\ln(1-r^2)) = \frac{4\alpha}{(1-r^2)^{2+2\alpha}}; \tag{9}$$

then make the substitution $u = w - \alpha\ln(1-r^2) + lr^2$ for a constant $l < 0$. We have (still denote by u)

$$\Delta_\alpha u + \frac{4l}{(1-r^2)^{2\alpha}} + K_l(x)e^{2u} = 0, \tag{10}$$

where $K_l(x) = K(x)(1-r^2)^{-2\alpha}e^{2lr^2}$ and

$$\int_{B^2} \frac{4l}{(1-r^2)^{2\alpha}} dA = 4\pi l, \tag{11}$$

where dA is the area element of (B^2, g_α).

Let H_k denote the Hilbert space of L^2_{loc}-functions for which

$$\|u\|_k = [\sum_{j=0}^{k} \int_{B^2} |D^j u|^2 dA]^{\frac{1}{2}} < +\infty, \tag{12}$$

where $|D^j u|$ is the pointwise norm (with respect to g_α) of the j^{th} covariant derivative of u. In particular,

$$\|u\|_1 = [\int_{B^2} u^2 dA + \int_{B^2} |\nabla u|^2 dA]^{\frac{1}{2}}, \tag{13}$$

where ∇u is the gradient of u with respect to g_α.

The following is a type of Trudinger inequality (cf. [7], which we also follow in the proof).

Proposition 2. *There exist positive constants β, γ such that*

$$\int_{B^2} e^{\beta u^2} dA \leq \gamma \tag{14}$$

for all $u \in H_1$ with $\int_{B^2} u\, dA = 0$ and $\|\nabla u\|_{L^2} \leq 1$.

Proof. We apply symmetrization which is based on the isoperimetric inequality that holds on (B^2, g_α) (cf. [6], [7]); to be specific, with $u(x)$ we associate a nonincreasing radial function $u^*(r)$ by the requirement

$$\mu\{x \mid u^* > \rho\} = \mu\{x \mid u > \rho\}$$

for every ρ, where μ denotes the measure on (B^2, g_α).

Since the Dirichlet norm is a conformal invariant and symmetrization decreases the Dirichlet norm,

$$\|\nabla u^*\|_{L^2} \leq \|\nabla u\|_{L^2}.$$

Thus we may assume that $u = u(r) = u^*(r)$. Now introduce $\omega(t) = \sqrt{4\pi}u(r)$, where $r^2 = e^t$; then

$$\|\dot\omega\|_{L^2(dt)} = \|\nabla u\|_{L^2},$$

where $\dot\omega = \frac{d\omega}{dt}$, and we must show that

$$\int_{-\infty}^0 e^{\frac{\beta}{4\pi}\omega^2} \cdot (1 - e^t)^{2\alpha} e^t dt \leq \frac{\gamma}{\pi}. \tag{15}$$

Using the Schwarz inequality, we find that

$$(\omega(t) - \omega(s))^2 \leq |t - s| \int_s^t \dot\omega^2 dt \leq |t - s|,$$

or

$$-|t - s|^{\frac{1}{2}} \leq \omega(t) - \omega(s) \leq |t - s|^{\frac{1}{2}}. \tag{16}$$

Let $\rho(t) = C(1 - e^t)^{2\alpha} e^t$ be such that (in this paper we use C to denote different positive constants)

$$\int_{-\infty}^0 \rho(t) dt = 1$$

and

$$\int_{-\infty}^0 \omega(t)\rho(t) dt = 0.$$

Multiplying (16) by $\rho(s)$ and integrating ds, we find that

$$|\omega(t)|^2 \leq (\int_{-\infty}^0 |t - s|^{\frac{1}{2}} \rho(s) ds)^2 \leq |t| + C. \tag{17}$$

Thus we obtain (15) provided $\beta < 4\pi$. □

Corollary 3. *If* $\beta < 4\pi$, *then*

$$\int_{B^2} e^{\delta|v|} dA \le C e^{(\delta^2/(4\beta))\|\nabla v\|^2_{L^2}} \tag{18}$$

for all $v \in H_1$ *with* $\int_{B^2} v\, dA = 0$.

Proof. Write $v = \|\nabla v\| \cdot u$, so that $\|\nabla u\| \le 1$. Apply Proposition 2 to u using $\delta|v| \le \beta u^2 + \delta^2\|\nabla v\|^2/(4\beta)$. □

The next result is a type of Poincaré inequality, which we also need in the proof of Theorem 1.

Proposition 4. *There exists a positive constant* C_1 *such that*

$$\|v\|^2_{L^2} \le C_1 \|\nabla v\|^2_{L^2} \tag{19}$$

for all $v \in H_1$ *with* $\int_{B^2} v\, dA = 0$.

Proof. Let $v = \|\nabla v\| \cdot u$, so that $\|\nabla u\| \le 1$. It suffices to show that

$$\|u\|^2_{L^2} \le C.$$

We may use symmetrization to assume that $u = u(r)$ and introduce $\omega(t)$ as in the proof of Proposition 2. Using (17), we find that

$$\|u\|^2_{L^2} \le C \int_{-\infty}^{0} |\omega|^2 (1 - e^t)^{2\alpha} e^t dt \le C.$$

□

3. Proof of Theorem 1

Now we can give the following.

Proof of Theorem 1. Define E by

$$E = \left\{ u \in H_1(B^2) \mid \int_{B^2} K_l e^{2u} dA = -4\pi l \right\}. \tag{20}$$

Since $K(x)$ is positive somewhere and $l < 0$, it is easy to see that E is not empty.

We shall minimize the functional

$$J(u) = \int_{B^2} \left(\frac{1}{2} |\nabla u|^2 - \frac{4l}{(1 - r^2)^{2\alpha}} \cdot u \right) dA \tag{21}$$

for $u \in E$. Writing $u = v + b$, so that $\int_{B^2} v\, dA = 0$, and solving for b in the constraint (20), one sees that J can be expressed as

$$J(u) = \frac{1}{2} \int_{B^2} |\nabla v|^2 dA + 2\pi l \ln \int_{B^2} K_l e^{2v} dA - 2\pi l \ln(-4\pi l) - \int_{B^2} \frac{4l}{(1 - r^2)^{2\alpha}} \cdot v\, dA. \tag{22}$$

By assumption (6) (take $\alpha = -\frac{1}{2}\sigma$), K_l is bounded. Then use (18) and choose $-l$ small such that $\left|\frac{2\pi l}{\beta}\right| < \frac{1}{8}$. Also, we note that $\frac{4l}{(1-r^2)^{2\alpha}}$ is bounded on B^2. Then using (19) we have

$$\left| \int_{B^2} \frac{4l}{(1 - r^2)^{2\alpha}} \cdot v\, dA \right| \le \epsilon \int_{B^2} v^2 dA + \frac{1}{4\epsilon} \int_{B^2} \frac{4l}{(1 - r^2)^{2\alpha}} dA \le \epsilon C_1 \|\nabla v\|^2_{L^2} + C.$$

Taking $\epsilon = \frac{1}{8C_1}$, we find that

$$J(u) \geq \frac{1}{4}\|\nabla v\|^2_{L^2} - C. \tag{23}$$

Now let $u_j = v_j + c_j \in E$ be a minimal sequence, where $\int_{B^2} v_j dA = 0$. Since $J(u_j)$ is bounded, we have $\|\nabla v_j\|_{L^2} \leq C$ by (23). Then it follows from (19) that $\|v_j\|_1$ is bounded. Because the unit ball of any Hilbert space is weakly compact, we can extract a subsequence (which we again denote by v_j) converging weakly in H_1 to v, and this implies that $\int_{B^2} v\, dA = 0$.

The fact that $\|\nabla v_j\|_{L^2} \leq C$ and (19) hold for all v_j implies that $\{v_j\}$ is precompact in L^2 (cf. [7], [10]). Then $v_j \to v$ strongly in L^2. Using the inequality $|e^z - 1| \leq |z|e^{|z|}$ and (18), we find that $\int_{B^2} K_l e^{2v_j} dA \to \int_{B^2} K_l e^{2v} dA$:

$$|\int_{B^2} K_l(e^{2v_j} - e^{2v})dA| \leq C\int_{B^2} e^{2|v|}|v_j - v|e^{2|v_j - v|}dA \leq C\|v_j - v\|_{L^2}.$$

Since $e^{2c_j}\int_{B^2} K_l e^{2v_j} dA = -4\pi l$ for each j, we find that

$$c_j \to c = \frac{1}{2}\ln[-4\pi l/\int_{B^2} K_l e^{2v}\, dA],$$

and $u = v + c \in E$ gives the minimum for J on E. Then by standard Lagrange multiplier theory we find that there is a constant λ such that

$$\int_{B^2}[\nabla u \cdot \nabla\phi - \frac{4l}{(1-r^2)^{2\alpha}} \cdot \phi]dA = \lambda\int_{B^2} K_l e^{2u}\phi\, dA \tag{24}$$

for all $\phi \in H_1$. Taking $\phi \equiv 1$ we find that $\lambda = 1$. So u is a weak solution of (10). Local regularity implies that $u \in C^2(B^2)$. This completes the proof of Theorem 1. □

Acknowledgments

The authors thank the referee for the detailed and valuable suggestions for improving the paper.

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