PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 10, Pages 2875–2883
S 0002-9939(04)07424-6
Article electronically published on May 12, 2004

# DISTINGUISHED REPRESENTATIONS AND POLES OF TWISTED TENSOR $L$-FUNCTIONS

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(Communicated by Wen-Ching Winnie Li)

ABSTRACT. Let $E/F$ be a quadratic extension of $p$-adic fields. If $\pi$ is an admissible representation of $GL_n(E)$ that is parabolically induced from discrete series representations, then we prove that the space of $P_n(F)$-invariant linear functionals on $\pi$ has dimension one, where $P_n(F)$ is the mirabolic subgroup. As a corollary, it is deduced that if $\pi$ is distinguished by $GL_n(F)$, then the twisted tensor $L$-function associated to $\pi$ has a pole at $s = 0$. It follows that if $\pi$ is a discrete series representation, then at most one of the representations $\pi$ and $\pi \otimes \chi$ is distinguished, where $\chi$ is an extension of the local class field theory character associated to $E/F$. This is in agreement with a conjecture of Flicker and Rallis that relates the set of distinguished representations with the image of base change from a suitable unitary group.

## 1. INTRODUCTION

Let $F$ be a non-archimedean local field of characteristic zero and $E$ a quadratic extension of $F$. Let $\pi$ be an irreducible admissible representation of $GL_n(E)$. It is known that $\mathrm{Hom}_{GL_n(F)}(\pi, 1)$, the space of $GL_n(F)$-invariant linear forms on the space of $\pi$, has dimension at most one (Proposition 11, [6]). We say that $\pi$ is distinguished with respect to $GL_n(F)$, or simply distinguished, when $\mathrm{Hom}_{GL_n(F)}(\pi, 1)$ is not zero. More generally, if $\mu$ is a character of $F^*$, $\pi$ is said to be $\mu$-distinguished if $\mathrm{Hom}_{GL_n(F)}(\pi, \mu)$ is not zero. There has been a lot of interest in the study of these representations and their global counterparts ever since the work of Harder, Langlands and Rapoport [10]. For various aspects of the local theory, we refer to [1], [6], [9], [11], [12], [13], and the references therein.

Assume that $\pi$ is generic, i.e., that $\pi$ admits a non-zero Whittaker functional. Fix a nontrivial additive character $\psi$ of $E$ which restricts trivially to $F$. Let $\mathcal{W}(\pi, \psi)$ denote the $\psi$-Whittaker model of $\pi$. Let $P_n(F)$ denote the mirabolic subgroup of $GL_n(F)$. That is, elements of $P_n(F)$ are elements of $GL_n(F)$ with last row $(0, 0, ..., 0, 1)$. Let $N_n(F)$ denote the group of upper triangular unipotent matrices. Consider the linear form $\ell$ defined on the Whittaker model $\mathcal{W}(\pi, \psi)$ of $\pi$ by

$$\ell(W) = \int_{N_n(F)\backslash P_n(F)} W(p)\, d_r p, \tag{1}$$

Received by the editors September 11, 2002 and, in revised form, June 3, 2003.
2000 *Mathematics Subject Classification.* Primary 11F70, 11F85.
*Key words and phrases.* Distinguished representations, local twisted tensor $L$-function, Asai $L$-function, Bernstein-Zelevinsky derivatives.

where $d_r p$ is a right Haar measure on $N_n(F)\backslash P_n(F)$. The integral in the definition of the linear form displayed above converges if $\pi$ is also unitary (see Lemma, p. 306, [5]). Moreover, $\ell$ is non-zero (p. 309, [5]). Thus if $\pi$ is unitary and generic, then $\ell$ gives rise to a non-zero $P_n(F)$-invariant linear form on $\pi$. Our first result says that, for a large class of representations $\pi$ of $GL_n(E)$, $\ell$ is the only $P_n(F)$-invariant linear form on $\pi$ up to a scalar.

**Theorem 1.1.** *Let $\pi$ be a representation of $GL_n(E)$ that is parabolically induced from discrete series representations. Then,*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{P_n(F)}(\pi, 1) = 1.$$

Note that we assume that the central character of a discrete series representation is unitary. In particular, a representation parabolically induced from discrete series is necessarily irreducible and unitary. These are precisely the tempered representations of $GL_n(F)$. Theorem 1.1 is proved using a result of Flicker (Lemma 2.4, see also Main Lemma, [7]) and the Bernstein-Zelevinsky filtration of the restriction of $\pi$ to $P_n(E)$ ([2], [3], [4]).

**Corollary 1.2.** *Let $\pi$ be as in Theorem 1.1. Suppose $\pi$ is distinguished with respect to $GL_n(F)$. Then $\ell$ gives a non-trivial $GL_n(F)$-invariant functional on $\mathcal{W}(\pi,\psi)$.*

Indeed, any $GL_n(F)$-invariant functional is $P_n(F)$-invariant; hence, from Theorem 1.1, it must be a multiple of $\ell$.

We also consider the linear form $\ell'$ defined on $\mathcal{W}(\pi,\psi)$ by

$$\ell'(W) = \int_{N_n(F)\backslash P_n(F)} \widetilde{W}(p)\, d_r p, \tag{2}$$

where $\widetilde{W}(g) = W(w{}^t g^{-1})$ and $w$ is the longest Weyl element. Let $\omega_\pi$ denote the central character of $\pi$.

**Theorem 1.3.** *Let $\pi$ be as in Theorem 1.1 and assume that $\omega_\pi|_{F^*} = 1$. Then $\pi$ is distinguished if and only if $\ell$ and $\ell'$ are equal up to a non-zero scalar multiple.*

Now let $L(s, r(\pi))$ denote the local twisted tensor $L$-function attached to $\pi$. This $L$-function was introduced by Flicker in [7]. Then $\pi$ being distinguished is reflected in $L(s, r(\pi))$.

**Theorem 1.4.** *Let $\pi$ be as in Theorem 1.1. Suppose $\pi$ is distinguished with respect to $GL_n(F)$. Then $L(s, r(\pi))$ has a pole at $s = 0$.*

The proof of Theorem 1.4 uses Flicker's theory of zeta integrals and the results stated above.

**Corollary 1.5.** *Let $\pi$ be a discrete series representation of $GL_n(E)$. Then $\pi$ is distinguished if and only if $L(s, r(\pi))$ has a pole at $s = 0$.*

One direction in this corollary is Theorem 1.4. The other direction follows from Theorem 4 of [11].

Before stating the next corollary let us introduce $\chi_{E/F}$, the character associated to $E/F$ by local class field theory.

**Corollary 1.6.** *Suppose $\pi$ is a discrete series representation. Then $\pi$ is not both distinguished and $\chi_{E/F}$-distinguished.*

This can be proved by combining Theorem 1.4 with the identity

$$L(s, \pi \times \pi^\sigma) = L(s, r(\pi))L(s, r(\pi) \otimes \chi_{E/F})$$

established in Theorem 6 of [11]. One has only to note that the Rankin-Selberg $L$-function on the left can have at most a simple pole at $s = 0$ when $\pi$ is in the discrete series.

We would like to remark on a conjectural context for Corollary 1.6. According to a conjecture of Flicker and Rallis [6], distinguished representations of $GL_n(E)$ should be those that are in the image of stable (respectively, unstable) base change from the quasisplit unitary group in $n$ variables when $n$ is odd (respectively, even). Also, if $\pi$ is a stable base change lift, then $\pi \otimes \chi_{E/F}$ is an unstable base change lift. It is expected that the images of the two base change maps are disjoint on square integrable representations. Corollary 1.6 is in accord with these expectations.

Finally, a word or two about the origin of this paper. The first and third named authors initially had a preprint that contained results of the flavour of Theorems 1.1 and 1.4. Then they came to know about a paper of the second named author in which a study of the poles of the twisted tensor $L$-function is carried out, resulting in a proof of the square integrable case of Jacquet's conjecture about distinguished representations [11]. Around that time, the second named author was also thinking about a statement like Corollary 1.6, which refines Jacquet's conjecture in this case. The three of us thought it best to write a joint paper on the subject, reflecting our independent progress on these questions.

## 2. Preliminaries

Let $E/F$ be a quadratic extension of local fields. Let $q = q_F$ denote the cardinality of the residue field of $F$. Let $\psi$ be a non-trivial character of $E$ that is trivial on $F$. Note that such a character takes the form $\psi(x) = \psi'(\mathrm{trace}_{E/F}(\Delta x))$, where $\psi'$ is a nontrivial character of $F$ and $\Delta$ is a trace zero element of $E$. Then $\psi$ gives rise to a character of the unipotent upper triangular group $N_n(E)$ of $GL_n(E)$ which is trivial on $N_n(F)$. We again denote it by $\psi$. Thus, $\psi(u) = \psi(\sum_{i=1}^{n-1} u_{i,i+1})$ for $u = (u_{i,j}) \in N_n(E)$. Let $w \in GL_n(F)$ be the matrix whose antidiagonal entries are 1 and whose other entries are 0; $w$ represents the longest element of the Weyl group of $GL_n(F)$ with respect to the standard choices of maximal split torus and positive system. Let $\mathcal{S}(F^n)$ denote the space of Schwartz functions on $F^n$. We introduce the Fourier transform

$$\hat{\Phi}(x) = \int_{F^n} \Phi(y)\psi'(\sum_i x_i y_i)\, dy$$

with respect to the self-dual Haar measure $dy$ for $\psi'$.

Suppose that $\pi$ is an irreducible admissible unitary generic representation of $GL_n(E)$. Denote by $\mathcal{W}(\pi, \psi)$ its $\psi$-Whittaker model. For $W \in \mathcal{W}(\pi, \psi)$ and a Schwartz-Bruhat function $\Phi \in \mathcal{S}(F^n)$, consider the integral

$$\Psi(s, W, \Phi) = \int_{N_n(F)\backslash GL_n(F)} W(g)\Phi(\eta g)|\det g|_F^s\, dg,$$

where $\eta = (0, 0, ..., 0, 1)$ and $dg$ is the right $GL_n(F)$-invariant measure on the quotient space.

With this notation, we have (main theorem, [7])

**Theorem 2.1.** *(i) For each $W \in \mathcal{W}(\pi, \psi)$ and $\Phi \in \mathcal{S}(F^n)$, the integral $\Psi(s, W, \Phi)$ is absolutely convergent, when* $\mathrm{Re}(s)$ *is sufficiently large, to a rational function of $X = q^{-s}$.*

*(ii) There exists a polynomial $P(X) \in \mathbb{C}[X]$ with $P(0) = 1$ such that the integrals $\Psi(s, W, \Phi)$ span the fractional ideal $L(s, r(\pi))\mathbb{C}[X, X^{-1}]$ of the ring $\mathbb{C}[X, X^{-1}]$, where $L(s, r(\pi)) = P(X)^{-1}$.*

*(iii) There exist an integer $m(\pi, \psi)$ and a non-zero complex number $c(\pi, \psi)$ such that*

$$\frac{\Psi(1-s, \widetilde{W}, \hat{\Phi})}{L(1-s, r(\tilde{\pi}))} = \omega_\pi(-1)^{n-1}\epsilon(s, r(\pi), \psi)\frac{\Psi(s, W, \Phi)}{L(s, r(\pi))}$$

*for all $W \in \mathcal{W}(\pi, \psi)$, $\Phi \in \mathcal{S}(F^n)$. Here $\omega_\pi$ is the central character of $\pi$, $\tilde{\pi}$ is the contragredient of $\pi$, $\epsilon(s, r(\pi), \psi) = c(\pi, \psi)X^{m(\pi,\psi)}$, and $\widetilde{W}(g) = W(w\,{}^tg^{-1})$.*

*Remark.* In fact, there exists some positive constant $\epsilon$ such that the integral $\Psi(s, W, \Phi)$ converges absolutely, uniformly in compact subsets, for $\mathrm{Re}(s) > 1 - \epsilon$ (Proposition, [8]).

One key ingredient in the proof of $(iii)$ in Theorem 2.1 is the following lemma (Main Lemma, [7]). Note that $\nu = \nu_E$ stands for the normalized absolute value $|\ |_E$ of $E$.

**Lemma 2.2.** *With the exception of finitely many values of $X = q^{-s}$, the dimension of* $\mathrm{Hom}_{P_n(F)}(\pi \otimes \nu^{\frac{s-1}{2}}, 1)$ *is at most one.*

Now we summarise certain facts which we will be using, from the work of Bernstein and Zelevinsky (section 3, [3]; see also [2] and [4]). Note that we always use normalized induction unless otherwise specified. This is consistent with [3] and [4], but not with [2].

In $GL_n(E)$ we consider the mirabolic subgroup $P_n(E)$ and its unipotent radical $U_n(E)$. A non-trivial additive character $\psi$ of $E$ gives a character of $U_n(E)$ by $\psi((u_{i,j})) = \psi(u_{n-1,n})$. Let Alg $G$ denote the category of smooth representations of an $l$-group $G$. There are four functors $\Psi^-$, $\Psi^+$, $\Phi^-$ and $\Phi^+$. The functors $\Psi^-$ and $\Phi^-$ map from $\mathrm{Alg}\, P_n(E)$ to $\mathrm{Alg}\, GL_{n-1}(E)$ and $\mathrm{Alg}\, P_{n-1}(E)$, respectively; $\Psi^-$ is the normalized Jacquet functor and $\Phi^-$ the normalized $\psi$-twisted Jacquet functor. If $(\tau, V)$ is a smooth representation of $P_n(E)$, let $J(V) = \{\tau(u)v - v \mid v \in V, u \in U_n(E)\}$. Then $\Psi^-\tau$ is realized on the space $V/J(V)$ with the action $\Psi^-(\tau)(g)(v + J(V)) = |\det g|_E^{-1/2}(\tau(g)v + J(V))$. Similarly, let $J_\psi(V) = \{\tau(u)v - \psi(u)v \mid v \in V, u \in U_n(E)\}$. Then $\Phi^-(\tau)$ is realized on the space $V/J_\psi(V)$ with the action $\Phi^-(\tau)(p)(v + J_\psi(V)) = |\det p|_E^{-1/2}(\tau(p)v + J_\psi(V))$.

The functors $\Psi^+$ and $\Phi^+$ map, from $\mathrm{Alg}\, GL_{n-1}(E)$ and $\mathrm{Alg}\, P_{n-1}(E)$ respectively, to $\mathrm{Alg}\, P_n(E)$. The functor $\Psi^+$ is normalized inflation; thus, if $\sigma \in \mathrm{Alg}\, GL_{n-1}(E)$ then $\Psi^+(\sigma)$ is realized on the space of $\sigma$ with the action

$$\Psi^+(\sigma)(gu)v = |\det g|_E^{1/2}\sigma(g)v$$

for $u \in U_n(E)$. The functor $\Phi^+$ is normalized compactly supported induction. Thus, if $\sigma$ is a smooth representation of $P_{n-1}(E)$, then

$$\Phi^+(\sigma) = \operatorname{ind}_{P_{n-1}(E)U_n(E)}^{P_n(E)}\big((|\det|_E^{1/2}\sigma) \otimes \psi\big),$$

where ind is non-normalized using smooth functions of compact support modulo $P_{n-1}(E)U_n(E)$.

These functors are used to define the $k$-th derivative $\tau^{(k)}$ of a smooth representation $\tau$ of $P_n(E)$. Let $\tau^{(k)} \in \operatorname{Alg} GL_{n-k}(E)$ be $\tau^{(k)} = \Psi^-(\Phi^-)^{k-1}(\tau)$ for $1 \leq k \leq n$. There exists a natural filtration of $\tau$ by $P_n(E)$-submodules $0 \subset \tau_n \subset \tau_{n-1} \subset \cdots \subset \tau_1 = \tau$ such that $\tau_k/\tau_{k+1} = (\Phi^+)^{k-1}\Psi^+(\tau^{(k)})$. Here $\tau_k = (\Phi^+)^{k-1}(\Phi^-)^{k-1}(\tau)$. It follows that if $\tau$ is also irreducible, then it is equivalent to a representation of the form $(\Phi^+)^{k-1}\Psi^+(\rho)$, where $1 \leq k \leq n$ and $\rho$ is an irreducible representation of $GL_{n-k}(E)$. The index $k$ and the representation $\rho$ are uniquely determined by $\tau$.

If $\pi$ is a smooth representation of $GL_n(E)$, set $\tau = \pi|_{P_n(E)}$. Also set $\pi^{(0)} = \pi$ and $\pi^{(k)} = \tau^{(k)}$ for $k = 1, 2, ..., n$. The $\pi^{(k)}$ are called the derivatives of $\pi$. We have the following result (Theorem 4.4, [3]; Lemma 4.5, [3]; p. 35, [4]).

**Proposition 2.3.** *(i) Let $\pi$ be a supercuspidal representation of $GL_r(E)$. Then $\pi^{(k)} = 0$ for $0 < k < r$ and $\pi^{(r)} = 1$.*

*(ii) Let $\pi = [\rho, \rho\nu, \ldots, \rho\nu^{l-1}]$ be the unique irreducible quotient of the representation parabolically induced from the indicated segment, where $\rho$ is a supercuspidal representation of $GL_r(E)$. Then $\pi^{(k)} = 0$ if $k$ is not a multiple of $r$, $\pi^{(kr)} = [\rho\nu^k, \rho\nu^{k+1}, \ldots, \rho\nu^{l-1}]$ for $1 \leq k \leq l-1$, and $\pi^{(lr)} = 1$.*

*(iii) Let $\pi = \operatorname{Ind}(\Delta_1 \otimes \ldots \otimes \Delta_t)$, where $\Delta_i$ is an irreducible representation of $GL_{n_i}(E)$, so that $n = n_1 + \ldots + n_t$. Then $\pi^{(k)}$ has a filtration whose subquotients are the representations $\operatorname{Ind}(\Delta_1^{(k_1)} \otimes \ldots \otimes \Delta_t^{(k_t)})$ with $k = k_1 + \ldots + k_t$.*

The following is extracted from the proof of the Main Lemma in [7], taking into account the change from the unnormalized to the normalized versions of the functors.

**Lemma 2.4.** *If $\rho$ is a representation of $GL_k(E)$, then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{P_n(F)}((\Phi^+)^{n-k-1}\Psi^+(\rho), 1) = \dim_{\mathbb{C}} \operatorname{Hom}_{GL_k(F)}(\rho \otimes \nu^{1/2}, 1).$$

Finally, we recall the following facts (cf. appendix, pp. 474-477, [14]). Let $G$ be a locally compact group and $H$ a closed subgroup. Let $\rho$ be a strictly positive Borel function on $G$, bounded above and below on compact subsets, and such that $\rho(hg) = (\Delta_G(h)/\Delta_H(h))\rho(g)$ for all $h \in H$ and $g \in G$. Here $\Delta_G$ and $\Delta_H$ are the modular functions of $G$ and $H$, respectively. A function with these properties is called a rho-function. If we fix a rho-function $\rho$, then associated to it we have a quasi-invariant measure $\mu_\rho$ such that

$$\int_G f(x)\rho(x)d_r x = \int_{H\backslash G}\int_H f(hx)\, d_r h\; d\mu_\rho(\dot{x})$$

for $f$ continuous with compact support.

Suppose $H_1 \subset H_2$ are closed subgroups of $G$. We have the following lemma (Lemma A 1.2, [14]).

**Lemma 2.5.** *Suppose $H_1\backslash H_2$ admits a positive $H_2$ right invariant measure $\nu_2$. Let $\rho_2$ be a rho-function on $G$ for $H_2$ (and hence also for $H_1$ since $\Delta_{H_2}|_{H_1} = \Delta_{H_1}$). Let $\mu_1$ and $\mu_2$ be the associated quasi-invariant measures on $H_1\backslash G$ and $H_2\backslash G$. Then, for a suitable normalization of Haar measures, we have*

$$\int_{H_1\backslash G} f(x)d\mu_1(x) = \int_{H_2\backslash G} d\mu_2(x) \int_{H_1\backslash H_2} f(hx)d\nu_2(h)$$

*for $f$ continuous with compact support modulo $H_1$.*

## 3. Proof of Theorem 1.1 and Theorem 1.3

As mentioned in the introduction, the proof of Theorem 1.1 falls out of Lemma 2.4. Let $\pi = \mathrm{Ind}(\Delta_1 \otimes \ldots \otimes \Delta_t)$, where the $\Delta_i$ are discrete series representations. Let $\Delta_i = [\rho_i, \rho_i\nu, \ldots, \rho_i\nu^{l_i-1}]$; that is, $\Delta_i$ is the unique irreducible quotient of the representation parabolically induced from the indicated segment, where $\rho_i$ is a supercuspidal representation of $GL_{r_i}(E)$. Let $\tau = \pi|_{P_n(E)}$. Then $\tau$ has the Bernstein-Zelevinsky filtration $0 \subset \tau_n \subset \cdots \subset \tau_1 = \tau$ by $P_n(E)$-submodules, where $\tau_k/\tau_{k+1} \cong (\Phi^+)^{k-1}\Psi^+(\tau^{(k)})$. By Proposition 2.3 (iii), $\tau^{(k)}$ has a filtration whose successive quotients are isomorphic to $\mathrm{Ind}(\Delta_1^{(k_1)} \otimes \ldots \otimes \Delta_t^{(k_t)})$ with $k = k_1 + \ldots + k_t$. We claim that

$$\mathrm{Hom}_{GL_{n-k}(F)}(\mathrm{Ind}(\Delta_1^{(k_1)} \otimes \ldots \otimes \Delta_t^{(k_t)}) \otimes \nu^{1/2}, 1) = 0$$

except when $k_i = n_i$ for $i = 1, \ldots, t$, in which case the space is clearly one-dimensional, since the representation in the claim is just the trivial representation of the trivial group. By Proposition 2.3 (ii), the claim is clear if at least one $k_i$ is not a multiple of $r_i$. So suppose that $k_i = a_i r_i$, $k_i \neq n_i$ (i.e., $a_i \neq l_i$), for all $i = 1, \ldots, t$. Then $\Delta_i^{(k_i)} = [\rho_i\nu^{a_i}, \ldots, \rho_i\nu^{l_i-1}]$. Since the $\Delta_i$ are discrete series representations, we know that $\rho_i\nu^{\frac{l_i-1}{2}}$ is unitary. We conclude that the central character of $\mathrm{Ind}(\Delta_1^{(k_1)} \otimes \ldots \otimes \Delta_t^{(k_t)}) \otimes \nu^{1/2}$ is the product of a unitary character and $\nu^m$, where

$$m = \frac{(n-k)}{2} + \sum_{i=1}^{t} \frac{a_i(n_i - k_i)}{2}.$$

Note that $m = 0$ if and only if $k_i = n_i$ for all $i = 1, \ldots, t$. Since $k_i \neq n_i$ by our assumption, we conclude that $m$ is a non-zero real number, and hence the representation $\mathrm{Ind}(\Delta_1^{(k_1)} \otimes \ldots \otimes \Delta_t^{(k_t)}) \otimes \nu^{1/2}$ is not unitary. The claim follows since the central character of a distinguished representation is unitary.

By Lemma 2.4, we deduce that there is exactly one non-trivial $P_n(F)$-invariant linear form on the semisimplification of $\tau$. Hence $\mathrm{Hom}_{P_n(F)}(\pi, 1)$ has dimension at most one. But then the dimension is equal to one, since the linear form $\ell$ defined on the Whittaker model $\mathcal{W}(\pi, \psi)$ of $\pi$ by (1) belongs to $\mathrm{Hom}_{P_n(F)}(\pi, 1)$. This concludes the proof of Theorem 1.1. Corollary 1.2 follows immediately, since a $GL_n(F)$-invariant functional is, in particular, $P_n(F)$-invariant.

Now we prove Theorem 1.3. Let $\ell$ and $\ell'$ be as in (1) and (2). Suppose $\ell(W) = c \cdot \ell'(W)$ for all $W$, where $c$ is a complex number. Then, for any $x \in P_n(F)$,

$$\begin{aligned} \ell(\pi({}^t x^{-1})W) &= c \cdot \ell'(\pi({}^t x^{-1})W) \\ &= c \cdot \int_{N_n(F)\backslash P_n(F)} \widetilde{\pi({}^t x^{-1})W}(p)\, d_r p \end{aligned}$$

$$
\begin{aligned}
&= c \cdot \int_{N_n(F)\backslash P_n(F)} \widetilde{W}(px)\, d_r p \\
&= c \cdot \int_{N_n(F)\backslash P_n(F)} \widetilde{W}(p)\, d_r p \\
&= c \cdot \ell'(W) \\
&= \ell(W).
\end{aligned}
$$

Thus $\ell$ is invariant under the transpose of $P_n(F)$ as well. Since $\omega_\pi|_{F^*} = 1$, we conclude that $\ell$ is invariant under the standard maximal parabolic and its transpose. These subgroups together generate $GL_n(F)$, and so $\ell$ is a $GL_n(F)$-invariant functional. Hence $\pi$ is distinguished.

Conversely, suppose $\pi$ is distinguished. Then $\tilde{\pi}$ is also distinguished. By Corollary 1.2, $\ell$ is a $GL_n(F)$-invariant functional on $\mathcal{W}(\pi, \psi)$ and the integral in the definition of $\ell'$ is a $GL_n(F)$-invariant functional on $\mathcal{W}(\tilde{\pi}, \psi^{-1})$. Since $\tilde{\pi} \cong \pi'$ where $\pi'(g) = \pi({}^t g^{-1})$, it follows that $\ell'$ gives a $GL_n(F)$-invariant functional on $\pi$ as well. Since the space of $GL_n(F)$-invariant linear forms on $\pi$ has multiplicity one, $\ell$ and $\ell'$ differ by a scalar.

## 4. Proof of Theorem 1.4

Let $G = GL_n(F)$, $H_2 = P_n(F)$, and $H_1 = N_n(F)$. We will apply Lemma 2.5 to this triplet. So let $\nu_2 = d_r p$ be the $P_n(F)$-right invariant measure on $N_n(F)\backslash P_n(F)$. Let $\rho_2(g) = |\det g|_F$. Then $\rho_2$ is a rho-function for $P_n(F)$, since $\Delta_{GL_n(F)} = 1$ and $\Delta_{P_n(F)}(p) = |\det p|_F^{-1}$. As in Lemma 2.5, let $\mu_1$ and $\mu_2$ be the associated quasi-invariant measures on $N_n(F)\backslash GL_n(F)$ and $P_n(F)\backslash GL_n(F)$. It can be verified that if $dg$ is the quotient measure on $N_n(F)\backslash GL_n(F)$, then $d\mu_1(g) = |\det g|_F\, dg$.

Let $\pi$ be as in Theorem 1.4. We need to prove that $s = 0$ is a pole for $L(s, r(\pi))$. Consider

$$
\Psi(1, \widetilde{W}, \hat{\Phi}) = \int_{N_n(F)\backslash GL_n(F)} \widetilde{W}(g)\hat{\Phi}(\eta g)|\det g|_F\, dg,
$$

where $W \in \mathcal{W}(\pi, \psi)$, $\Phi \in \mathcal{S}(F^n)$ and $\eta = (0, 0, ..., 1)$. By Lemma 2.5,

$$
\begin{aligned}
\Psi(1, \widetilde{W}, \hat{\Phi}) &= \int_{P_n(F)\backslash GL_n(F)} \int_{N_n(F)\backslash P_n(F)} \widetilde{W}(pg)\hat{\Phi}(\eta pg)\, d_r p\, d\mu_2(g) \\
&= \int_{P_n(F)\backslash GL_n(F)} \hat{\Phi}(\eta g) \left( \int_{N_n(F)\backslash P_n(F)} \widetilde{W}(pg)\, d_r p \right) d\mu_2(g) \\
&= \int_{P_n(F)\backslash GL_n(F)} \hat{\Phi}(\eta g) \left( \int_{N_n(F)\backslash P_n(F)} (\tilde{\pi}(g)\widetilde{W})(p)\, d_r p \right) d\mu_2(g) \\
&= \int_{P_n(F)\backslash GL_n(F)} \hat{\Phi}(\eta g) d\mu_2(g) \times \left( \int_{N_n(F)\backslash P_n(F)} \widetilde{W}(p)\, d_r p \right) \\
&= c \cdot \Phi(0) \cdot \ell'(W),
\end{aligned}
$$

where $c$ is a non-zero constant. Note that the fourth equality is obtained because we know that $\int_{N_n(F)\backslash P_n(F)} \widetilde{W}(p) d_r p$ gives a $GL_n(F)$-invariant linear form on $\tilde{\pi}$. The integral over $P_n(F)\backslash GL_n(F)$ equals $\Phi(0)$ by Fourier inversion, since $P_n(F)\backslash GL_n(F)$ is $F^n - 0$ and $d\mu_2$ is a Haar measure on $F^n$.

Next, as in Proposition 4, [5], we choose $W$ in $\mathcal{W}(\pi,\psi)$ and $\Phi$ in $\mathcal{S}(F^n)$ such that $\Psi(s,W,\Phi)$ is identically one. This is done as follows (see p. 309, [5]). Fix a congruence subgroup $K'$ of $K$ $(= GL_n(\mathcal{O}_E))$. By Iwahori factorization, we get a $\phi$ which is supported on $N_n(E)(K'\cap P_n(E))$ and right invariant under $K'\cap P_n(E)\cap {}^tB_n(E)$, where $B_n(E)$ is the group of upper triangular matrices. Fix $W$ in $\mathcal{W}(\pi,\psi)$ with $W|_{P_n(E)}=\phi$. Then

$$\int_{N_n(F)\backslash P_n(F)} W(p)|\det p|_F^{s-1} d_r p = \int_{N_n(F)\backslash P_n(F)} \phi(p)|\det p|_F^{s-1} d_r p$$

is a non-zero constant. Let $K_m$ denote the group of $k$ in $K'$ with $\eta k=(\varpi^m x_1,\varpi^m x_2, \ldots,\varpi^m x_{n-1},1+\varpi^m x_n)$, where the $x_i$ are all in the ring $\mathcal{O}_E$. Choose $m$ so that $W$ is right invariant under $K_m\cap {}^tU_n(F)$, where $U_n(F)$ is the unipotent radical of the parabolic subgroup of type $(n-1,1)$. Let $\Phi$ be the characteristic function of the last row of $K_m\cap GL_n(F)$. Then $\Phi$ lies in $\mathcal{S}(F^n)$, and

$$\Psi(s,W,\Phi)=\int_{K(F)} dk \int_{N_n(F)\backslash P_n(F)} W(pk)|\det p|_F^{s-1} d_r p \int_{F^*}\Phi(\eta a k)|a|_F^{ns} d^\times a$$

is a non-zero constant.

Now let us write down the functional equation at $s=0$:

$$\Psi(1,\widetilde{W},\hat{\Phi})/L(1,r(\tilde{\pi}))=\epsilon(0,r(\pi),\psi)\Psi(0,W,\Phi)/L(0,r(\pi)).$$

Choose $W$ and $\Phi$ as above. Then since $\Phi(0)=0$, the left hand side of the functional equation vanishes. Since $W$ and $\Phi$ are such that $\Psi(0,W,\Phi)\neq 0$, it follows that $s=0$ is a pole for $L(s,r(\pi))$.

## Acknowledgements

The first named author would like to thank Professors Dipendra Prasad, T.N. Venkataramana and Yuval Flicker for suggestions related to this paper. The second named author would like to thank his co-authors for inviting him to write a joint paper, and they in turn thank him for accepting the invitation. The authors also thank the referee for many helpful suggestions.

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