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POLYNOMIALS GENERATED BY LINEAR OPERATORS

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Abstract. We study the class of Banach algebra-valued n-homogeneous polynomials generated by the n^{th} powers of linear operators. We compare it with the finite type polynomials. We introduce a topology w_{EF} on E, similar to the weak topology, to clarify the features of these polynomials.

The finite type scalar-valued polynomials play an important role when dealing with algebras of holomorphic functions. For instance, the spectrum of the Fréchet algebra of entire functions of bounded type on a complex Banach space E with the approximation property coincides with E'' if, and only if, the finite type polynomials are dense in the algebra [2]. When dealing with holomorphic mappings with values in a complex Banach algebra F, a new class of "finite type" polynomials arises: In [12] the subspace $\mathbb{P}_f(^nE, F)$ of all n-homogeneous polynomials generated by n^{th} powers of linear operators was introduced. In this paper we study the relationship of this subspace with other commonly used classes of polynomials. Unless otherwise stated, the space of n-homogeneous polynomials $P(^nE, F)$ will be endowed with the norm topology. For uniform Banach algebras F and Banach spaces E with the approximation property, we prove that the closure of the n-homogeneous finite type polynomials $P_f(^nE, F)$ and the closure of $\mathbb{P}_f(^nE, F)$ coincide if, and only if, all linear operators from E into F are compact. We also show that it may happen that the closure of $\mathbb{P}_f(^nE, F)$ lies strictly between the closure of $P_f(^nE, F)$ and the whole space of polynomials $P(^nE, F)$.

We introduce a topology w_{EF} on E, like the weak topology, which is the coarsest that makes continuous all elements in $\mathbb{P}_f(^nE, F)$ $\forall n \in \mathbb{N}$. When $F = \mathbb{C}$, this topology coincides with the weak topology w in E. It is known that the finite type scalar polynomials on E are the weakly continuous ones. However, it may happen that a polynomial is w_{EF}-continuous but not in $\mathbb{P}_f(^nE, F)$ ([12]; see also section 2). It is shown that the Aron-Berner extension of w_{EF}-continuous polynomials is $w_{E''F''}$-continuous and also that it extends polynomials in $\mathbb{P}_f(^nE, F)$ to polynomials in $\mathbb{P}_f(^nE'', F'')$ where F'' carries the Arens product.

For background on polynomials on Banach spaces we refer to [6].

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1. Polynomials generated by linear operators

Let E be a complex Banach space, and let $n \in \mathbb{N}$. If F is a complex Banach algebra with identity, let $\mathbb{P}_f({}^nE, F)$ denote the space generated by $\{T^n : T \in L(E,F)\}$, where $T^n(x) = (T(x))^n$ for each $x \in E$. This space was introduced in [12] where it is shown that $P_f({}^nE, F) \subset \mathbb{P}_f({}^nE, F)$. Recall that $P_f({}^nE, F) = P_f({}^nE) \otimes F$, where $P_f({}^nE)$ is the space generated by the n^{th} powers of elements in E'. This section begins with the study of the relationship between the norm closures of $\mathbb{P}_f({}^nE, F)$ and the finite type polynomials. Later we will check that the Aron-Berner extension of a polynomial in $\mathbb{P}_f({}^nE, F)$ is the n^{th} power of the bitranspose of the generating linear operators.

Recall that given an arbitrary Banach space $(F, || \; ||)$ one can define in F a product $\odot$ and a norm $||| \; |||$ such that $(F, ||| \; |||)$ is a commutative Banach algebra with identity. Indeed, let $e \in F$ with $||e|| = 1$ and consider $\varphi \in F'$ such that $\varphi(e) = 1$. Let $F_1 = \ker(\varphi)$. Define $u \odot v := abe + (bx + ay)$ for $u = ae + x$ and $v = be + y$ where $a, b \in \mathbb{C}$ and $x, y \in F_1$. Moreover, define $|||u||| = |a| + ||x||$ for all $u = ae + x$ with $a \in \mathbb{C}$ and $x \in F_1$. It is easy to verify that $||| \; |||$ is equivalent to $|| \; ||$ and that $(F, ||| \; |||)$ endowed with the product $\odot$ is a commutative Banach algebra with identity. It is also clear that such a φ turns out to be a multiplicative form.

If E is a Banach space and F is a Banach algebra, it is easy to check that $\varphi \circ P \in P_f({}^nE)$ whenever $P \in \mathbb{P}_f({}^nE, F)$ and $\varphi \in F'$ is multiplicative.

Observe that $\overline{P_f({}^nE)} = P({}^nE)$ whenever $\overline{\mathbb{P}_f({}^nE, F)} = P({}^nE, F)$ for some Banach algebra F. Indeed, let $P \in P({}^nE)$. Take a homomorphism $\varphi : F \longrightarrow \mathbb{C}$ and $e \in F$ such that $\varphi(e) = 1$ and define $Q : E \longrightarrow F$ by $Q(x) := P(x) \cdot e \; \forall x \in E$. Clearly $P = \varphi \circ Q$ and $Q \in P({}^nE, F)$. Since $\overline{\mathbb{P}_f({}^nE, F)} = P({}^nE, F)$ by hypothesis, there exists $(Q_k) \subset \mathbb{P}_f({}^nE, F)$ such that $Q_k \xrightarrow{|| \; ||} Q$ and so $||\varphi \circ Q_k - \varphi \circ Q|| \leq ||\varphi|| \cdot ||Q_k - Q|| \longrightarrow 0$. Since $(\varphi \circ Q_k) \subset P_f({}^nE)$ and $\varphi \circ Q = P$, we have $P \in \overline{P_f({}^nE)}$.

It is known that $\overline{P_f({}^nE)} = P({}^nE)$ does not imply $\overline{P_f({}^nE, F)} = P({}^nE, F)$ for every Banach space F. The same situation occurs for $\mathbb{P}_f({}^nE, F)$, that is, it may happen that $\overline{\mathbb{P}_f({}^nE, F)} \neq P({}^nE, F)$ despite $\overline{P_f({}^nE)} = P({}^nE)$. We give some explanation for the reader's convenience. Indeed, take $E = \ell_3$ and $F = \ell_2$ endowed with the product $\odot$ introduced above. All $\mathbb{C}$-valued 2-homogeneous polynomials on ℓ_3 are weakly uniformly continuous on bounded sets by Pitt's theorem, hence $\overline{P_f({}^2\ell_3)} = P({}^2\ell_3)$. On the other hand, since $L(\ell_3, \ell_2) = K(\ell_3, \ell_2)$, it follows from Proposition 2.5 in [12] that $\overline{\mathbb{P}_f({}^2\ell_3, \ell_2)} = \overline{P_f({}^2\ell_3, \ell_2)}$. This shows that $\mathbb{P}_f({}^2\ell_3, \ell_2)$ is not dense, since the polynomial $(x_i) \in \ell_3 \mapsto (x_i^2) \in \ell_2$ is not a compact one.

Now we are going to establish conditions under which $\overline{\mathbb{P}_f({}^nE, F)} = \overline{P_f({}^nE, F)}$. For this we require the following result.

Theorem 1.1. *Let E be a Banach space, and let F be a uniform Banach algebra. If $T \in L(E, F)$ is such that $T^n, n \in \mathbb{N}$, is a (weakly) compact polynomial, then T is a (resp. weakly) compact operator.*

Proof. Since F is a closed subspace of $\mathcal{C}(K)$ for some compact set K, it suffices to prove that T is a (weakly) compact operator from E into $\mathcal{C}(K)$.

Assume first that T^n is compact. By the Ascoli Theorem it suffices to show that $T(B_E)$ is an equicontinuous subset of $\mathcal{C}(K)$. Let $u \in K$. If $T(x)(u) = 0$ for all $x \in B_E$, given $\epsilon > 0$ there exists a neighbourhood U of u in K such that $|T^n(x)(t)| <$

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ϵ^n for all $t \in U$ and for all $x \in B_E$ since $T^n(B_E)$ is clearly equicontinuous. So $|T(x)(t) - T(x)(u)| \leq \epsilon \ \forall t \in U$ and $\forall x \in B_E$.

Suppose now that there exists $a \in E$ such that $T(a)(u) \neq 0$. Without loss of generality, we may suppose that $T(a)(u) = 1$. Given $\epsilon > 0$ there exists a neighbourhood W of u such that $|T^{n-1}(a)(t) - 1| < \frac{\epsilon}{2||T||}$ for all $t \in W$ since $T^{n-1}(a) \in \mathcal{C}(K)$. Moreover, $T^{n-1}(a)T(B_E)$ is equicontinuous since $T^{n-1}(a) \cdot T$ is a compact operator from E into $\mathcal{C}(K)$ by Proposition 3.4 of [3]. Consequently, there exists a neighbourhood V of u in K such that $V \subset W$ and

$$|[T^{n-1}(a) \cdot T(x)](t) - T(x)(u)| < \frac{\epsilon}{2}$$

for all $x \in B_E$ and for all $t \in V$. Therefore,

$$\begin{aligned}
&|T(x)(t) - T(x)(u)| \\
&\quad \leq |T(x)(t) - [T^{n-1}(a) \cdot T(x)](t)| + |[T^{n-1}(a) \cdot T(x)](t) - T(x)(u)| \\
&\quad = |T(x)(t)| \cdot |1 - T^{n-1}(a)(t)| + |[T^{n-1}(a) \cdot T(x)](t) - T(x)(u)| \\
&\quad \leq ||T|| \frac{\epsilon}{2||T||} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

for all $x \in B_E$ and for all $t \in V$.

The argument for the weakly compact case is similar. According to [7], Theorem 14, p. 269, it suffices to show that $T(B_E)$ is quasi-equicontinuous, that is, to show that given $u \in K$ and a net $(u_\alpha)_{\alpha \in A} \subset K$ converging to u, for $\epsilon > 0$ and $\alpha_0 \in A$ there are a finite number of indices, $\alpha_1, \ldots, \alpha_m \in A$ with $\alpha_i \geq \alpha_0$ such that

$$\min_{1 \leq i \leq m} |T(x)(u_{\alpha_i}) - T(x)(u)| \leq \epsilon \quad \forall x \in B_E.$$

Since $T^n(B_E)$ is weakly relatively compact, it is quasi-equicontinuous; hence in case $T(x)(u) = 0$ for all $x \in B_E$, we get, for $\epsilon > 0$ and $\alpha_0 \in A$, the existence of $\alpha_1, \ldots, \alpha_m \in A$ with $\alpha_i \geq \alpha_0$ such that

$$\min_{1 \leq i \leq m} |T^n(x)(u_{\alpha_i})| = \min_{1 \leq i \leq m} |T^n(x)(u_{\alpha_i}) - T^n(x)(u)| \leq \epsilon^n \quad \forall x \in B_E,$$

which is enough.

Suppose now that there exists $a \in E$ such that $T(a)(u) = 1$. Given $\epsilon > 0$ and $\alpha_0 \in A$ the continuity of T^{n-1} leads to α_0' such that

$$|1 - T^{n-1}(a)(u_\alpha)| = |T^{n-1}(a)(u) - T^{n-1}(a)(u_\alpha)| \leq \frac{\epsilon}{2||T||} \ \ \forall \alpha \geq \alpha_0'.$$

By Theorem 3.2 in [14] the derivative mapping $nT^{n-1}(a)T : E \to \mathcal{C}(K)$ is a weakly compact mapping. In particular, $[T^{n-1}(a)T](B_E)$ is a weakly relatively compact set and therefore $\alpha_1, \ldots, \alpha_m \in A$ with $\alpha_i \geq \alpha_0'$ may be found such that

$$\min_{1 \leq i \leq m} |T^{n-1}(a)(u_{\alpha_i}) \cdot T(x)(u_{\alpha_i}) - T^{n-1}(a)(u) \cdot T(x)(u)| \leq \frac{\epsilon}{2} \quad \forall x \in B_E.$$

Then

$$\begin{aligned}
&|T(x)(u_{\alpha_i}) - T(x)(u)| \\
&\leq |T(x)(u_{\alpha_i}) - [T^{n-1}(a) \cdot T(x)](u_{\alpha_i})| + |[T^{n-1}(a) \cdot T(x)](u_{\alpha_i}) - T(x)(u)| \\
&= |T(x)(u_{\alpha_i})| \cdot |1 - T^{n-1}(a)(u_{\alpha_i})| + |[T^{n-1}(a) \cdot T(x)](u_{\alpha_i}) - T^{n-1}(a)(u)T(x)(u)|.
\end{aligned}$$

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Thus for all $x \in B_E$,

$$\begin{aligned}&\min_{1\le i\le m} |T(x)(u_{\alpha_i}) - T(x)(u)|\\ &\qquad \le \frac{\epsilon}{2} + \min_{1\le i\le m} |T^{n-1}(a)(u_{\alpha_i}) \cdot T(x)(u_{\alpha_i}) - T^{n-1}(a)(u)\cdot T(x)(u)| \le \epsilon.\end{aligned}$$

□

Corollary 1.2. *Let E be a Banach space such that E' has the approximation property, and let F be a uniform Banach algebra. The following statements are equivalent:*

(1) $L(E,F) = K(E,F)$.
(2) $\overline{P_f(^nE,F)} = \overline{I\!P_f(^nE,F)}$ $\forall n \ge 1$.
(3) $\overline{P_f(^nE,F)} = \overline{I\!P_f(^nE,F)}$ *for some* $n \ge 1$.

Proof. (1) ⇒ (2) Apply Proposition 2.5 of [12].

(3) ⇒ (1) If $\overline{P_f(^nE,F)} = \overline{I\!P_f(^nE,F)}$, then every element of $\overline{I\!P_f(^nE,F)}$ is compact. Now, given $T \in L(E,F)$ we have that T^n is a compact polynomial and so, by Theorem 1.1, T is compact. □

Under the assumptions of the above corollary the equivalent conditions mean also that $\overline{I\!P_f(^nE,F)}$ coincides with the space of all compact polynomials. Theorem 1.1 shows that all elements in $L(E,F)$ are weakly compact if, and only if, all elements in $I\!P_f(^nE,F)$ are weakly compact. It is tempting to ask whether in this case $\overline{I\!P_f(^nE,F)}$ coincides with the space of all weakly compact polynomials; however, the example quoted before Theorem 1.1 prevents such a coincidence.

Remark 1.3. Theorem 1.1 is not true if the Banach algebra F is not uniform as the following example shows:

Consider $T : c_0 \to L(\ell_\infty, \ell_\infty)$ defined by

$$T(x)(y) = (0, x_1\cdot y_1, 0, x_3\cdot y_3, 0, \dots, 0, x_{2n+1}y_{2n+1}, 0, \dots)$$

for all $x = (x_i) \in c_0$ and $y = (y_i) \in \ell_\infty$. It is clear that T is a linear operator such that T^2 is compact since $T^2 \equiv 0$. Nevertheless T is not compact since the sequence $(T(e_{2n+1}))$ does not have convergent subsequences. Neither T is weakly compact; otherwise the transpose mapping T^t would be a weakly compact operator into the Schur space ℓ_1, and hence T^t would be compact.

Recall that a set $A \subset E$ is said to be a limited set if weak* null sequences in E^* are uniformly convergent on A. The Josefson-Nissenzweig Theorem guarantees that the unit ball of any infinite-dimensional Banach space is not a limited set (see [6], p. 234).

Theorem 1.4. *Let E be a Grothendieck space having a quotient isomorphic to ℓ_p. Then*

$$\overline{P_f(^nE,c)} \subsetneq \overline{I\!P_f(^nE,c)} \subsetneq P(^nE,c)\ \ \forall n \ge p,\ \ n \in \mathbb{N}.$$

Proof. Since E is a Grothendieck space, all elements in $L(E,c)$ are weakly compact. Recall that in a uniform algebra the product of two weakly relatively compact sets is a weakly relatively compact set. Then we have that every element in $I\!P_f(^nE,c)$ maps bounded sets into weakly relatively compact sets, and the same holds for all the elements in $\overline{I\!P_f(^nE,c)}$.

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Fix $n \geq p$. In order to prove that $\overline{\mathbb{P}_f(^nE,c)} \subsetneq P(^nE,c)$ we define a continuous n-homogeneous polynomial $Q : \ell_p \longrightarrow c$ by

$$Q(x) = (-\sum_{i=1}^{\infty} x_i^n, (-1)^2 \sum_{i=2}^{\infty} x_i^n, \ldots, (-1)^k \sum_{i=k}^{\infty} x_i^n, \ldots)$$

for every $x = (x_i)_{i=1}^{\infty} \subset \ell_p$. We remark that Q is not weakly compact since the sequence $(Q(e_m))_{m=1}^{\infty} \subset Q(\overline{B_{\ell_p}})$ does not have a weakly convergent subsequence. Now, if $q : E \to \ell_p$ is the quotient mapping, there exists $\lambda > 0$ such that $\lambda Q(B_{\ell_p}) \subset (Q \circ q)(B_E)$. Therefore, $Q \circ q$ is not weakly compact and $(Q \circ q) \in P(^nE,c) \setminus \overline{\mathbb{P}_f(^nE,c)}$.

Since c is a Banach algebra with identity, $\overline{P_f(^nE,c) \subset \mathbb{P}_f(^nE,c)}$ ([12], Proposition 2.2). Furthermore, $\overline{P_f(^nE,c)} \subsetneq \overline{\mathbb{P}_f(^nE,c)}$ because otherwise $\overline{\mathbb{P}_f(^nE,c)} = \overline{P_f(^nE,c)}$ and then Corollary 1.2 would yield that $L(E,c) = K(E,c)$, which is not possible since there are noncompact operators from E into c because the unit ball of E is not limited. □

Corollary 1.5. *Let E be a Grothendieck space. If $\overline{\mathbb{P}_f(^nE,c)} = P(^nE,c)$ for some $n > 1$, then E is reflexive.*

Proof. Suppose that E is nonreflexive. By Proposition 1 of [8], E contains a copy of ℓ_1, and hence it has a quotient isomorphic to ℓ_2. Therefore by Theorem 1.4 we have that $\overline{\mathbb{P}_f(^nE,c)} \subsetneq P(^nE,c)$ for all $n \geq 2$, a contradiction. □

The elements in $\mathbb{P}_f(^nE,F)$ behave in the most suitable way regarding the Aron-Berner extension to the bidual. We refer the reader to [5] for a survey of properties of the Arens extension to F'' of the product in F.

Proposition 1.6. *The Aron-Berner extension of every $P \in \mathbb{P}_f(^nE,F)$ belongs to $\mathbb{P}_f(^nE'',F'')$ when F'' is considered as carrying the left Arens product.*

Proof. We have to show that for any $L \in L(E,F)$, L''^n, the nth power of the double transpose $L'' : E'' \to F''$, coincides with the Aron-Berner extension of the polynomial L^n.

Recall that for a multilinear mapping $A \in L(^nE,F)$, its Aron-Berner extension $\tilde{A}$ and fixed $a_1,\ldots,a_{j-1} \in E$ and $a''_{j+1},\ldots,a''_n \in E''$, the mapping

$$x'' \in E'' \mapsto \tilde{A}(a_1,\ldots,a_{j-1},x'',a''_{j+1},\ldots,a''_n)$$

is weak*-weak* continuous (see [6], 6.2).

Now, for $T_1,\ldots,T_n \in L(E,F)$ the n-linear mapping $A := T_1 \cdot \ldots \cdot T_n : E \times \ldots \times E \to F$ given by $A(x_1,\ldots,x_n) = T_1(x_1) \cdot \ldots \cdot T_n(x_n)$ may be extended according to $(x''_1,\ldots,x''_n) \in E'' \times \ldots \times E'' \mapsto T''_1(x_1) \cdot \ldots \cdot T''_n(x_n) \in F''$. Since the transpose mappings are always weak*-weak* continuous and for the left Arens product the mappings $f'' \in F'' \mapsto f'' \cdot g'' \in F''$ and $f'' \in F'' \mapsto h \cdot f'' \in F''$ are weak*-weak* continuous for fixed $g'' \in F''$ and $h \in F$ respectively (see [5], p. 311), it follows that the mapping

$$x'' \in E'' \mapsto T_1(a_1) \cdot \ldots \cdot T_{j-1}(a_{j-1}) \cdot T''_j(x'') \cdot T_{j+1}(a''_{j+1}) \cdot \ldots \cdot T''_n(a''_n)$$

is also weak*-weak* continuous. Therefore by the weak* density of E in E'', we obtain that $\tilde{A} = T''_1 \cdot \ldots \cdot T''_n$. □

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2. The w_{EF} topology and the w_{EF}-continuous polynomials

Let E and $F \neq \{0\}$ be complex normed spaces. For each $T \in L(E,F)$ we set $p_T(x) := ||T(x)||$ for all $x \in E$. Then p_T is a seminorm on E, and the family $\{p_T : T \in L(E,F)\}$ defines a locally convex topology w_{EF} on E. This topology is the weakest topology on E that makes continuous all $T \in L(E,F)$ and hence, all polynomials in $\mathcal{P}_f({}^nE,F)$ $\forall n \in \mathbb{N}$. We denote by $L_{wsc}(E,F)$ the subspace of all weakly sequentially continuous elements of $L(E,F)$.

It is easy to verify that the weak topology is coarser than the w_{EF} topology, which in turn is also coarser than the norm topology and consequently, w_{EF} is compatible with the dual pair (E,E'). Note that $w_{EF} = w$ in case $F = \mathbb{C}$, and w_{EF} is the norm topology in case E is a subspace of F. Later, Propositions 2.3, 2.4 and 2.5 show that these topologies can be different.

Remark 2.1. The relationship between different w_{EF} topologies when varying the reference spaces may be summarized as follows: For fixed E, whenever Y is a subspace of Z, then $w_{EY} \preceq w_{EZ}$, and for fixed Y, whenever E is a subspace of F, then $w_{FY_{|_E}} \preceq w_{EY}$.

Proposition 2.2. *Let E and F be Banach spaces, and let T be a linear mapping from E into F. Then T is continuous if and only if $T : (E, w_{EF}) \longrightarrow (F,w)$ is continuous.*

Proof. If $T : (E, w_{EF}) \longrightarrow (F,w)$ is continuous, $T(x_n) \xrightarrow{w} T(x)$ whenever $x_n \xrightarrow{w_{EF}} x$. Now, if $(x_n, T(x_n)) \longrightarrow (x,y)$ in $E \times F$ we have $T(x_n) \xrightarrow{w} T(x)$ since w_{EF} is weaker than the norm topology and clearly $T(x_n) \xrightarrow{w} y$. So, $T(x) = y$ and we have that $T : E \longrightarrow F$ is continuous by the Closed Graph Theorem.

The converse is clearly true since $w \preceq w_{EF}$. □

Proposition 2.3. *Let E and F be Banach spaces. The following statements are equivalent:*

1. *$L(E,F) = L_{wsc}(E,F)$.*
2. *Given any sequence $(x_n)_{n\in\mathbb{N}}$ in E we have that $(x_n)_{n\in\mathbb{N}}$ converges to x in (E,w) if and only if $(x_n)_{n\in\mathbb{N}}$ converges to x in (E, w_{EF}).*
3. *The weak topology w and the w_{EF} topology coincide on the weakly compact subsets of E.*

Proof. The equivalence between (1) and (2) is clear. It is also clear that (3) implies (2).

(2) ⇒ (3) Let K be a weakly compact subset of E. It is enough to show that $\overline{A}^{w} = \overline{A}^{w_{EF}}$ for all $A \subset K$. Since $A \subset K$ is weakly relatively compact, given any $x \in \overline{A}^{w}$, there exists, by the Eberlein-Smulian Theorem, $(x_n)_{n\in\mathbb{N}} \subset A$ such that $x_n \xrightarrow{w} x$. By using (2) we get $\overline{A}^{w} \subset \overline{A}^{w_{EF}}$, and the proof is complete since clearly $\overline{A}^{w_{EF}} \subset \overline{A}^{w}$. □

To emphasize the different behaviour of the w_{EF} topology and the norm or the weak topology, we point out, for instance, that the canonical embedding $\iota : E \to E''$ may not be $(w_{EF} - w_{E''F''})$-continuous: just pick $E = \ell_\infty$ and $F = c_0$, and remark that $L(E,F) = L_{wsc}(E,F)$ since $E = \ell_\infty$ is a Grothendieck space and has the Dunford-Pettis property. Since any weakly convergent sequence $(x_n) \subset \ell_\infty$ is w_{EF}-convergent by Proposition 2.3, if $\iota : E \to E''$ were $(w_{EF} - w_{E''F''})$-continuous,

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$(\iota(x_n))$ would be $w_{E''F''}$-convergent. In particular, for the extension to E'' of the mapping $\iota : E \to \ell_\infty$, that is, the projection $\pi : (\ell_\infty)'' \to \ell_\infty$, since $\pi \in L(E'', F'')$, we would obtain that $(x_n) = (\pi(\iota(x_n)))$ norm converges in $F'' = \ell_\infty$. Consequently, ℓ_∞ would be a Schur space.

Proposition 2.4. *Let E and F be Banach spaces. The following statements are equivalent:*

(1) $L(E,F) = K(E,F)$.
(2) *Given any bounded net $(x_\beta)_{\beta\in I}$ in E we have that $(x_\beta)_{\beta\in I}$ converges to x in (E,w) if, and only if, $(x_\beta)_{\beta\in I}$ converges to x in (E,w_{EF}).*
(3) *The weak topology w and the w_{EF} topology coincide on the bounded subsets of E.*

Proof. The equivalence between (2) and (3) is clear.

(1) $\Rightarrow$ (2) Let $(x_\beta)_{\beta\in I}$ be a bounded net in E such that $x_\beta \xrightarrow{w} x$. Since $L(E,F) = K(E,F)$, every $T \in L(E,F)$ is weakly continuous on the bounded set $\{x_\beta : \beta \in I\} \cup \{x\}$ and consequently $T(x_\beta) \longrightarrow T(x)$ for all $T \in L(E,F)$. This means that $x_\beta \xrightarrow{w_{EF}} x$. From this the result follows.

(2) $\Rightarrow$ (1) If (2) is true, by Proposition 2.3 we have that $L(E,F) = L_{wsc}(E,F)$. So, if $L(E,F) \neq K(E,F)$ there exists $T \in L_{wsc}(E,F)$ such that $T \notin K(E,F)$. Consequently, there exists a bounded subset X of E such that $T|_X$ is not weakly continuous. So, there exists $x \in X$ and $(x_\beta)_{\beta\in I} \subset X$ such that $x_\beta \xrightarrow{w} x$ but $T(x_\beta) \overset{\|\,\|}{\not\longrightarrow} T(x)$. This means that $x_\beta \overset{w_{EF}}{\not\longrightarrow} x$ despite $x_\beta \xrightarrow{w} x$. □

Proposition 2.5.

(1) *If E is a reflexive infinite-dimensional Banach space, then there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in E such that $x_k \xrightarrow{w_{E\ell_1}} x$ and $x_k \overset{\|\,\|}{\not\longrightarrow} x$.*
(2) *If $L(E,F) = L_{wsc}(E,F)$ and E is not a Schur space, then there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in E such that $x_k \xrightarrow{w_{EF}} x$ and $x_k \overset{\|\,\|}{\not\longrightarrow} x$.*

Proof. (1) Since E is reflexive, there exists $(x_k)_{k\in\mathbb{N}} \subset E$ such that $x_k \xrightarrow{w} x$ and $x_k \overset{\|\,\|}{\not\longrightarrow} x$. On the other hand, $x_k \xrightarrow{w_{E\ell_1}} x$ since $L(E,\ell_1) = L_{wsc}(E,\ell_1)$.

The proof of (2) is similar. □

Proposition 2.6. *Let E and F be Banach spaces. Consider the following statements:*

(1) *$L(E,F)$ is reflexive.*
(2) *E is reflexive and $L(E,F) = K(E,F)$.*
(3) *$\overline{B}_E$ is w_{EF}-compact.*

Then (2) $\Leftrightarrow$ (3). Also, (2) $\Rightarrow$ (1) when F is a reflexive space. If, furthermore, either E or F has the compact approximation property, then (1) $\Rightarrow$ (2).

Proof. (2) $\Rightarrow$ (3) Follows from Proposition 2.4.

(3) $\Rightarrow$ (2) The reflexivity of E is a consequence of $w \preceq w_{EF}$. Since every $T \in L(E,F)$ is w_{EF}-continuous, $T(B_E)$ is relatively compact in F.

If F is reflexive, (2) $\Rightarrow$ (1) is just part of Theorem 1 of [11].

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(1) ⇒ (2) Under the condition (1), both E and F are reflexive spaces since E^* and F are isomorphic to complemented subspaces of $L(E,F)$. Then, if E or F has the compact approximation property, we apply Theorem 2.1 of [13] to complete the proof. □

Recall that a completely regular Hausdorff space X is a k-space when a subset G of X is open if and only if its intersection with each compact subset K of X is open in K. The space (E, w_{EF}) is completely regular since it is a locally convex space.

If $F = \mathbb{C}$, it is known that (E, w_{EF}) is a k-space if and only if $\dim E < \infty$ (since w_{EF} coincides with the weak topology). In general we have:

Proposition 2.7. *Let E and F be complex Banach spaces such that (E, w_{EF}) is a k-space. If $L(E,F) = K(E,F)$ (respectively, $L(E,F) = L_{wsc}(E,F)$), then E is reflexive (resp., reflexive or Schur).*

Proof. Assume $L(E,F) = K(E,F)$. By Proposition 2.4, the weak topology and the w_{EF} topology coincide on the bounded subsets of E. So, if bw denotes the finest topology in E that induces the weak topology on every bounded subset of E, it is obvious that $w_{EF} \preceq bw$ and that a subset K of E is w_{EF}-compact if and only if it is bw-compact. So, given a bw-closed subset C of E we have that $C \cap K$ is w_{EF}-closed for every w_{EF}-compact K and, by our hypothesis, C is w_{EF}-closed. This implies that $w_{EF} = bw$; so bw is a locally convex topology, and this has been shown to imply that E is reflexive (see [10], Theorem 3.7).

In the case $L(E,F) = L_{wsc}(E,F)$, we replace in the above argument bw by kw, the finest topology agreeing with the weak topology on weakly compact sets which was introduced in [9], to come up with $w_{EF} = kw$, which in turn implies E to be a reflexive or Schur space by Theorem 2.9 in [9]. □

Let $P_{w_{EF}}({}^nE, F)$ be the space of all w_{EF}-continuous n-homogeneous polynomials from E into F. The set $P_{w_{EF}}(E,F) = \bigoplus_{n=0}^{\infty} P_{w_{EF}}({}^nE, F)$ of all w_{EF}-continuous polynomials from E into F is a complex algebra, since for given $Q \in P_{w_{EF}}({}^nE, F)$ and $P \in P_{w_{EF}}({}^mE, F)$, the polynomial PQ is also w_{EF}-continuous. Indeed, given $\epsilon > 0$ there exist $T_1, T_2, ..., T_k \in L(E,F)$ such that $||Q(x)|| < \sqrt{\epsilon}$ and $||P(x)|| < \sqrt{\epsilon}$ whenever $\sup\limits_{1 \le i \le k} ||T_i(x)|| < 1$. So $\sup\limits_{1 \le i \le k} ||T_i(x)|| < 1 \Rightarrow ||Q(x)P(x)|| \le ||Q(x)||||P(x)|| < \epsilon$. Therefore, PQ is w_{EF}-continuous since w_{EF} is a locally convex topology.

Now we may construct w_{EF}-continuous polynomials just by multiplying continuous functionals on E by w_{EF}-continuous polynomials from E into F.

If E is a subspace of F, we have $P_{w_{EF}}({}^nE, F) = P({}^nE, F)$, and if $F = \mathbb{C}$, we have $P_{w_{E\mathbb{C}}}({}^nE) = P_w({}^nE)$ where $P_w({}^nE)$ denotes the space of all weakly continuous n-homogeneous polynomials. It is well known that $P_f({}^nE) = P_w({}^nE)$.

Clearly, $\mathbb{P}_f({}^nE, F) \subset P_{w_{EF}}({}^nE, F)$. If $F = \mathbb{C}$, we have $\mathbb{P}_f({}^nE) = P_w({}^nE)$, and it is natural to ask whether $\mathbb{P}_f({}^nE, F) = P_{w_{EF}}({}^nE, F)$ for a Banach algebra F. This is not true in general. For instance, in case $E = c_0$ and $F = \ell_\infty$, since $c_0 \subset \ell_\infty$, every element of $P({}^nc_0, \ell_\infty)$ is $w_{c_0\ell_\infty}$-continuous. If $P : c_0 \longrightarrow \ell_\infty$ is defined by $P((x_k)_{k\in\mathbb{N}}) = a \cdot (\sum\limits_{k=0}^{\infty} \frac{1}{2^k} x_k^n)$ where $a = (a_j)_{j\in\mathbb{N}} \in c_0,\ a \neq 0$, it is clear that $P \in P({}^nc_0, \ell_\infty)$ and $P \notin \mathbb{P}_f({}^nc_0, \ell_\infty)$ (cf. Example 2.3 of [12]).

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Proposition 2.8. *Let E be a complex Banach space. The following statements are equivalent:*

(1) *E is a finite-dimensional space.*
(2) *$\mathbb{P}_f(^nE, F) = P_{w_{EF}}(^nE, F)$ for every $n \in \mathbb{N}$ and for every complex Banach algebra with identity F.*
(3) *$\mathbb{P}_f(^nE, F) = P_{w_{EF}}(^nE, F)$ for some $n \in \mathbb{N}$ and for every complex Banach algebra with identity F.*

Proof. If E is a finite-dimensional space, the w_{EF} topology coincides with the norm topology on E, and by using Proposition 2.3 of [12] we get (1) $\Rightarrow$ (2). It is obvious that (2) $\Rightarrow$ (3).

Let us show that (3) $\Rightarrow$ (1). Fix $e \in E$ with $||e|| = 1$ and $\varphi \in E'$ such that $\varphi(e) = 1$, and consider the complex Banach algebra $F = (E, \odot)$ defined at the very beginning of §1 for the given Banach space E. In this case, the w_{EF} topology coincides with the norm topology on E, and so $P_{w_{EF}}(^nE, F) = P(^nE, F)$ for all $n \in \mathbb{N}$. If E is an infinite-dimensional space, then there exists a biorthogonal system $\{x_k, f_k : k \in \mathbb{N}\}$ such that $\{x_k : k \in \mathbb{N}\} \subset E$, $\{f_k : k \in \mathbb{N}\} \subset E'$ and $f_k(x_j) = \delta_{kj}$. It is clear that

$$Q(x) := \sum_{k=1}^{\infty} \frac{1}{2^k ||f_k||} f_k^n(x), \qquad x \in E,$$

defines a continuous n-homogeneous polynomial in E. To complete the proof it is enough to show that the continuous n-homogeneous polynomial $P : E \longrightarrow F$ defined by $P(x) := Q(x) \cdot e$ for all $x \in E$ does not belong to $\mathbb{P}_f(^nE, F)$. Indeed, if $P \in \mathbb{P}_f(^nE, F)$, we have $Q \in P_f(^nE)$ by Remark 2.8 since $\varphi \circ P = Q$. Then $Q = \sum_{i=1}^{m} \psi_i^n$ where $\{\psi_1, \dots, \psi_m\} \subset E'$, and then for each $x \in E$ we have

$$dQ(x) = \sum_{i=1}^{m} n\psi_i^{n-1}(x)\psi = \sum_{k=1}^{\infty} \frac{n}{2^k ||f_k||} f_k^{n-1}(x) f_k.$$

So, for every $j \in \mathbb{N}$, $dQ(x_j) = \frac{n}{2^j ||f_j||} \cdot f_j = \sum_{i=1}^{m} n\psi_i^{n-1}(x_j)\psi_i$, and this means that $\{f_k : k \in \mathbb{N}\}$ is a subset of the finite-dimensional subspace of E' spanned by $\{\psi_1, \dots, \psi_m\}$. This contradicts the linear independence of the set $\{f_k : k \in \mathbb{N}\}$. □

Proposition 1.6 implies that the Aron-Berner extension of polynomials in $\mathbb{P}_f(^nE, F)$ is $w_{E''F''}$-continuous. This holds for all w_{EF}-continuous polynomials:

Proposition 2.9. *The Aron-Berner extension of a w_{EF}-continuous polynomial is $w_{E''F''}$-continuous.*

Proof. By the polarization formula the multilinear mapping associated to a w_{EF}-continuous polynomial is also w_{EF}-continuous. Let $A \in L(^nE, F)$ be w_{EF}-continuous. Then there are $T_{i,j} \in L(E, F)$, $i = 1, \dots, n$, $j = 1, \dots, j(i)$, such that

$$||A(x_1, \dots, x_n)|| \leq \sup_{1 \leq j \leq j(1)} ||T_{1,j}(x_1)|| \cdot \ldots \cdot \sup_{1 \leq j \leq j(n)} ||T_{n,j}(x_n)||.$$

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Since the norm in F'' is weak* lower semicontinuous, each of the functions $x'' \in E'' \mapsto \sup_{1\le j\le j(i)} ||T''_{i,j}(x'')||$ is weak* lower semicontinuous on E''. Therefore, if $(z_\alpha) \subset E''$ weak* converges to $x'' \in E''$, then by ([4], IV, §6, Prop. 4)

$$\underline{\lim}_\alpha \sup_{1\le j\le j(i)} ||T''_{i,j}(z_\alpha)|| = \sup_{1\le j\le j(i)} ||T''_{i,j}(x'')||, \qquad i = 1, \dots, n.$$

Since for fixed f in the unit ball of F', and $a_1, \dots, a_{k-1} \in E$ and $a''_{k+1}, \dots, a''_n \in E''$, the mapping

$$x'' \in E'' \mapsto |f \circ \tilde{A}(a_1, \dots, a_{k-1}, x'', a''_{k+1}, \dots, a''_n)|$$

is weak* continuous on E'', we can prove that for a net $(x_\alpha) \subset E$ weak* convergent to $x'' \in E''$,

$$\begin{aligned} |f \circ \tilde{A}(a_1, \dots, a_{k-1}, x'', a''_{k+1}, \dots, a''_n)| &= \lim_\alpha |f \circ \tilde{A}(a_1, \dots, a_{k-1}, x_\alpha, a''_{k+1}, \dots, a''_n)| \\ \le \sup_{1\le j\le j(1)} ||T_{1,j}(a_1)|| \cdot \ldots \cdot \underline{\lim}_\alpha &\sup_{1\le j\le j(k)} ||T_{k,j}(x_\alpha)|| \cdot \ldots \cdot \sup_{1\le j\le j(n)} ||T''_{n,j}(a''_n)|| \\ = \sup_{1\le j\le j(1)} ||T_{1,j}(a_1)|| \cdot \ldots \cdot &\sup_{1\le j\le j(k)} ||T''_{k,j}(x'')|| \cdot \ldots \cdot \sup_{1\le j\le j(n)} ||T''_{n,j}(a''_n)||. \end{aligned}$$

To check this we begin with the last variable and go backwards until the first one, and, finally, we obtain

$$|f \circ \tilde{A}(a''_1, \dots, a''_n)| \le \sup_{1\le j\le j(1)} ||T''_{1,j}(a''_1)|| \cdot \ldots \cdot \sup_{1\le j\le j(n)} ||T''_{n,j}(a''_n)||.$$

This shows the $w_{E''F''}$-continuity of $\tilde{A}$. □

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