

## COHOMOLOGY OF SYMPLECTIC REDUCTIONS OF GENERIC COADJOINT ORBITS

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ABSTRACT. Let  $\mathcal{O}_\lambda$  be a generic coadjoint orbit of a compact semi-simple Lie group  $K$ . Weight varieties are the symplectic reductions of  $\mathcal{O}_\lambda$  by the maximal torus  $T$  in  $K$ . We use a theorem of Tolman and Weitsman to compute the cohomology ring of these varieties. Our formula relies on a *Schubert basis* of the equivariant cohomology of  $\mathcal{O}_\lambda$ , and it makes explicit the dependence on  $\lambda$  and a parameter in  $\text{Lie}(T)^* =: \mathfrak{t}^*$ .

### 1. INTRODUCTION

Let  $K$  be a compact semisimple Lie group,  $T \subset K$  a maximal torus and  $\mathfrak{t} \subset \mathfrak{k}$  their Lie algebras. Pick a fundamental chamber in  $\mathfrak{t}^*$  and a point  $\lambda$  in the interior. Let  $\mathcal{O}_\lambda$  be the orbit of  $\lambda$  under the coadjoint representation of  $K$  on  $\mathfrak{t}^*$ .  $\mathcal{O}_\lambda$  is diffeomorphic to the flag variety  $K/T$ , and it has a naturally occurring symplectic form  $\omega$  known as the Kirillov-Kostant-Souriau form. The action of  $T$  on  $\mathcal{O}_\lambda$  is Hamiltonian, which means that there is an invariant map

$$\Phi : \mathcal{O}_\lambda \rightarrow \mathfrak{t}^*$$

satisfying  $\omega(X_\eta, \cdot) = d\Phi^\eta$ , where  $\eta \in \mathfrak{t}$ ,  $X_\eta$  is the vector field on  $\mathcal{O}_\lambda$  generated by  $\eta$ , and  $\Phi^\eta(m) = \Phi(m)(\eta)$  is defined by the natural pairing between  $\mathfrak{t}$  and  $\mathfrak{t}^*$ . We call  $\Phi$  a *moment map* for this action.

The image of  $\Phi$  is the convex hull of  $W \cdot \lambda$ , the Weyl group orbit of  $\lambda$ . Let  $\mu \in \Phi(\mathcal{O}_\lambda)$  be a regular value of  $\Phi$ . We define the *symplectic reduction* at  $\mu$  by

$$\Phi^{-1}(\mu)/T = \mathcal{O}_\lambda//T(\mu).$$

The goal of this note is to give a presentation of the cohomology<sup>1</sup> ring of  $\mathcal{O}_\lambda//T(\mu)$  in terms of the root system of  $K$ . We present  $H^*(\mathcal{O}_\lambda//T(\mu))$  as a quotient of the  $T$ -equivariant cohomology ring  $H_T^*(\mathcal{O}_\lambda)$  by a certain ideal. We rely on the following fundamental result.

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<sup>1</sup>Only cohomology with coefficients in the field  $\mathbb{Q}$  of rational numbers will be considered throughout this paper.

**Theorem 1.1** (Kirwan). *Let  $M$  be a compact symplectic manifold with a Hamiltonian  $T$  action, where  $T$  is a compact torus. If  $\mu \in \mathfrak{t}^*$  is a regular value of  $\Phi$ , then the restriction map in equivariant cohomology,*

$$\kappa : H_T^*(M) \rightarrow H_T^*(\Phi^{-1}(\mu)),$$

*is surjective.*

Since the  $T$  action is locally free on level sets of the moment map at regular values,  $H_T^*(\Phi^{-1}(\mu)) = H^*(M//T(\mu))$ . The resulting map  $\kappa : H_T^*(M) \rightarrow H^*(M//T(\mu))$  is called the *Kirwan map*. Kirwan’s result is of particular importance because the equivariant cohomology can be described in terms of the equivariant cohomology of the fixed point sets of the  $T$  action. In the case of isolated fixed points, this is just a sum of polynomial rings.

**Theorem 1.2** (Kirwan). *Let  $M$  be a compact Hamiltonian  $T$ -space. Let  $M^T$  denote the fixed point set of the  $T$  action. The restriction map*

$$i^* : H_T^*(M) \rightarrow H_T^*(M^T)$$

*is injective. In the case that  $M^T$  is a finite set of points,*

$$H_T^*(M^T) = \bigoplus_{p \in M^T} \mathbb{Q}[x_1, \dots, x_n]$$

*where  $n = \dim T$ .*

A presentation of the cohomology ring of the reduced space  $M//T(\mu)$  can be obtained by using the following description of the kernel of the Kirwan map, which is due to Tolman and Weitsman [TW]. If  $\alpha$  is in  $H_T^*(M)$  we denote

$$\text{supp}(\alpha) = \{p \in M^T : \alpha|_p \neq 0\}.$$

Fix an arbitrary inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{t}^*$ .

**Theorem 1.3** (Tolman-Weitsman). *The kernel of the Kirwan map  $\kappa$  is the ideal of  $H_T^*(M)$  generated by all  $\alpha \in H_T^*(M)$  with the property that there exists  $\xi \in \mathfrak{t}^*$  such that*

$$\Phi(\text{supp}(\alpha)) \subset \{x \in \mathfrak{t}^* | \langle \xi, x \rangle \leq \langle \xi, \mu \rangle\}.$$

*In other words,  $\ker \kappa$  consists of sums of equivariant cohomology classes  $\alpha$  with the property that all points of  $\text{supp}(\alpha)$  are mapped by  $\Phi$  to the same side of an affine hyperplane in  $\mathfrak{t}^*$  that passes through  $\mu$ .*

The  $T$ -equivariant cohomology ring of the coadjoint orbit  $\mathcal{O}_\lambda = K/T$  is well understood. Kostant and Kumar constructed in [KK] a basis  $\{x_w\}_{w \in W}$  of  $H_T^*(K/T)$  as a  $H_T^*(pt)$ -module, which we refer to as the *Schubert basis*. Let  $B$  be a Borel subgroup in  $G := K^{\mathbb{C}}$ , and let  $B_-$  be the opposite Borel subgroup. For any  $v \in W$ , let  $X_v = \overline{B_- \tilde{v} B} / B$ , where  $\tilde{v}$  is any choice of lift of  $v \in W$  in the normalizer of the torus. These *opposite Schubert varieties* are  $T$ -invariant subvarieties of  $G/B \cong K/T$ . The basis  $\{x_w\}$  is uniquely defined by the property that

$$\int_{X_v} x_w = \delta_{vw}.$$

Theorem 1.2 suggests the importance of knowing how to restrict the classes  $x_w$  to fixed points  $W \cdot \lambda$ . This formula was worked out for general  $K$  by S. Billey [Bi]. In

particular, it is easy to show that  $x_w|_v = 0$  if  $v \not\leq w$  in the Bruhat order.<sup>2</sup> In other words,

$$\text{supp}(x_w) = \{v\lambda : v \leq w\}.$$

To each  $\tau \in W$  we can associate the new basis

$$\{x_w^\tau = \tau \cdot x_{\tau^{-1}w}\}_{w \in W},$$

whose elements have the property

$$\text{supp}(x_w^\tau) = \{v\lambda : \tau^{-1}v \leq \tau^{-1}w\}.$$

Let  $\lambda_1, \dots, \lambda_l \in \mathfrak{t}^*$  denote the fundamental weights associated to the chosen fundamental chamber of  $\mathfrak{t}^*$ . Let  $\langle \cdot, \cdot \rangle$  be the restriction to  $\mathfrak{t}^*$  of a  $K$ -invariant product on  $\mathfrak{k}^*$ . Our main result is:

**Theorem 1.4.** *The cohomology ring  $H^*(\mathcal{O}_\lambda//T(\mu))$  is isomorphic to the quotient of  $H_T^*(K/T)$  by the ideal generated by*

$$\{x_v^\tau : \text{there exists } j \text{ such that } \langle \lambda_j, \tau^{-1}v\lambda \rangle \leq \langle \lambda_j, \tau^{-1}\mu \rangle\}.$$

*Remark 1.* One can take the description of  $H_T^*(K/T)$  (see, for instance, [Br]) and deduce a precise presentation of the cohomology ring  $H^*(\mathcal{O}_\lambda//T(\mu))$  in terms of generators and relations.

*Remark 2.* For  $K = SU(n)$  this result was proven by the first author in [Go1].

## 2. PRIMARY DESCRIPTION OF $\ker \kappa$

For any  $\xi \in \mathfrak{t}^*$  we denote by  $f_\xi$  the corresponding height function on  $\mathcal{O}_\lambda$ ,

$$f_\xi(x) = \langle \xi, x \rangle.$$

Under the pairing between  $\mathfrak{t}^*$  and  $\mathfrak{t}$ , the function  $f_\xi$  is a component of the moment map. In fact, it is well known that  $f_\xi$  is Morse-Bott for all  $\xi \in \mathfrak{t}^*$ . Denote by  $C \subset \mathfrak{t}^*$  the fundamental (positive) Weyl chamber, which can be described by

$$C = \{r_1\lambda_1 + \dots + r_l\lambda_l : \text{all } r_j > 0\},$$

and let  $\overline{C}$  be its closure.

**Lemma 2.1.** *Let  $\tau$  be in  $W$  and  $\xi$  in  $\overline{\tau C}$ . If  $\tau^{-1}v < \tau^{-1}w$  in the Bruhat order, then  $f_\xi(v\lambda) \leq f_\xi(w\lambda)$ .*

*Proof.* The result follows immediately from the fact that if  $\xi \in C$ , then the unstable manifold of  $f_\xi$  through  $v\lambda$  with respect to the Kähler metric on

$$\mathcal{O}_\lambda = K/T = G/B$$

is just the Bruhat cell  $B \cdot vB/B$  (see, for instance, [Ko]). □

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<sup>2</sup>The class  $x_w$  differs from the  $\xi^w$  constructed in [KK] by the relationship  $x_w := w_0 \cdot \xi^{w_0 w}$ , where  $w_0$  is the longest element of  $W$ .

The main result of this section is:

**Theorem 2.1.** *Suppose that  $\alpha \in H_T^*(\mathcal{O}_\lambda)$  has the property that*

$$\Phi(\text{supp}(\alpha)) \subset \{x \in \mathfrak{t}^* : \langle \xi, x \rangle \leq \langle \xi, \mu \rangle\}.$$

*Then  $\alpha$  can be decomposed as*

$$\alpha = \sum_{w \in W} a_w^\tau x_w^\tau$$

*with  $a_w^\tau \in H_T^*(pt)$ , such that if  $a_w^\tau \neq 0$ , then*

$$\Phi(\text{supp}(x_w^\tau)) \subset \{x \in \mathfrak{t}^* : \langle \xi, x \rangle \leq \langle \xi, \mu \rangle\}.$$

*Proof.* Take  $\tau \in W$  such that  $\xi \in \tau\overline{C}$ . Suppose that the decomposition of  $\alpha$  with respect to the basis  $\{x_w^\tau\}_{w \in W}$  is of the form

$$(1) \quad \alpha = \sum_{w \in W} a_w^\tau x_w^\tau + a_{v_1}^\tau x_{v_1}^\tau + \cdots + a_{v_r}^\tau x_{v_r}^\tau,$$

where the first sum contains only  $w$  with

$$\langle \xi, w\lambda \rangle \leq \langle \xi, \mu \rangle,$$

whereas

$$\langle \xi, v_j\lambda \rangle > \langle \xi, \mu \rangle, \quad a_{v_j}^\tau \in S(\mathfrak{t}^*), a_{v_j}^\tau \neq 0,$$

for any  $1 \leq j \leq l$ . We may assume that  $v_1$  has the property that there exists no  $j > 1$  with  $\tau^{-1}v_1 < \tau^{-1}v_j$ . Now let us evaluate both sides of (1) at  $v_1\lambda$ . Since

$$\langle \xi, w\lambda \rangle \leq \langle \xi, \mu \rangle < \langle \xi, v_1\lambda \rangle,$$

by Lemma 2.1 we must have

$$x_w^\tau|_{v_1\lambda} = 0$$

for any  $w$  corresponding to a term in the first sum in (1). It follows that

$$\alpha|_{v_1\lambda} = a_{v_1}^\tau x_{v_1}^\tau|_{v_1\lambda} \neq 0;$$

so  $v_1\lambda$  is in  $\text{supp}(\alpha)$  even though  $\langle \xi, v_1\lambda \rangle > \langle \xi, \mu \rangle$ . This is a contradiction. □

### 3. PROOF OF THE MAIN RESULT

We now prove Theorem 1.4. Let  $v$  and  $\tau$  in  $W$  be such that

$$(2) \quad \langle \lambda_j, \tau^{-1}v\lambda \rangle \leq \langle \lambda_j, \tau^{-1}\mu \rangle,$$

for some  $1 \leq j \leq l$ . We show that  $x_v^\tau$  is in the kernel of the Kirwan map

$$\kappa : H_T^*(\mathcal{O}_\lambda) \rightarrow H^*(\mathcal{O}_\lambda//T(\mu)).$$

Let  $\xi = \tau\lambda_j$  be in  $\tau\overline{C}$ . Note that if  $w \in \text{supp}(x_v^\tau)$ , then  $\tau^{-1}w \leq \tau^{-1}v$  implies by Lemma 2.1 that

$$\langle \xi, w\lambda \rangle \leq \langle \xi, v\lambda \rangle \leq \langle \xi, \mu \rangle.$$

Thus  $x_v^\tau \in \ker \kappa$ .

Now let us consider  $\alpha \in H_T^*(K/T)$  with the property that there exists  $\xi \in \mathfrak{t}^*$  with

$$\text{supp}(\alpha) \subset \{x \in \mathfrak{t}^* | \langle \xi, x \rangle \leq \langle \xi, \mu \rangle\}.$$

Take  $\tau \in W$  such that  $\xi \in \tau\overline{C}$ . By Theorem 2.2, we can decompose  $\alpha$  as

$$(3) \quad \alpha = \sum_{w \in W} a_w^\tau x_w^\tau$$

where  $a_w^\tau$  can be nonzero only if

$$\text{supp}(x_w^\tau) \subset \{x \in \mathfrak{t}^* \mid \langle \xi, x \rangle \leq \langle \xi, \mu \rangle\}.$$

In particular, if  $a_w^\tau \neq 0$ , then

$$(4) \quad \langle \xi, w\lambda \rangle \leq \langle \xi, \mu \rangle.$$

Since  $\xi$  is in  $\tau\overline{C}$ , it can be written as

$$(5) \quad \xi = \tau \sum_{j=1}^l r_j \lambda_j,$$

where all  $r_j$  are nonnegative. So (4) and (5) imply that there exists  $j \in \{1, \dots, l\}$  such that

$$\langle \tau \lambda_j, w\lambda \rangle \leq \langle \tau \lambda_j, \mu \rangle.$$

In other words, each nonzero term in the right-hand side of (3) is a multiple of a  $x_w^\tau$  of the type claimed in Theorem 1.4.  $\square$

*Remark.* It follows that, in the particular situation of generic coadjoint orbits, in order to cover the whole Tolman-Weitsman kernel of the Kirwan map it is sufficient to consider affine hyperplanes through  $\mu$  that are perpendicular to vectors of the type  $\tau \lambda_j$ , with  $\tau \in W$  and  $j \in \{1, \dots, l\}$ . But these are just the hyperplanes parallel to the walls of the moment polytope. This result concerning a “sufficient set of hyperplanes” has been proved by the first author in [Go2], for an *arbitrary* Hamiltonian torus action on a compact manifold.

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#### REFERENCES

- [Bi] S. Billey, *Kostant polynomials and the cohomology of  $G/B$* , Duke Math. J. **96** (1999) 205-224. MR 2000a:14060
- [Br] M. Brion, *Equivariant cohomology and equivariant intersection theory*, in *Representation Theory and Algebraic Geometry*, Kluwer Acad. Publ. (1998) 1-37. MR 99m:14005
- [Go1] R. F. Goldin, *The cohomology ring of weight varieties and polygon spaces*, Adv. in Math. **160** (2001) No. 2, 175-204. MR 2002f:53139
- [Go2] R. F. Goldin, *An effective algorithm for the cohomology ring of symplectic reductions*, Geom. and Funct. Anal., Vol. 12 (2002), 567-583. MR 2003m:53148
- [Ki] F. C. Kirwan, *Cohomology of Quotients in Complex and Algebraic Geometry*, Mathematical Notes **31**, Princeton University Press, Princeton, N. J. (1984). MR 86i:58050
- [Ko] R. R. Koehlerlakarta, *Integral homology of real flag manifolds and loop spaces of symmetric spaces*, Adv. in Math. **110** (1995) no. 1, 1-46. MR 96a:57066

- [KK] B. Kostant and S. Kumar, *The nil Hecke ring and cohomology of  $G/P$  for a Kac-Moody group  $G$* , Adv. in Math. **62** (1986) no. 3, 187-237. MR 88b:17025b
- [TW] S. Tolman and J. Weitsman, *The cohomology rings of symplectic quotients* Comm. Anal. Geom. **11** (2003), no. 4, 751–773.

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