PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 10, Pages 2939–2944
S 0002-9939(04)07448-9

ON FINITENESS OF THE SET OF INTERMEDIATE SUBFACTORS

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(Communicated by David R. Larson)

Abstract. For type II_1 factors $N \subset L$ with $[L : N] < \infty$, we show that the sets $\mathcal{L}_1 = \{M \in \mathcal{L}(N \subset L) \colon N' \cap L \subset M\}$ and $\mathcal{L}_2 = \{M \in \mathcal{L}(N \subset L) \colon N' \cap L = M' \cap L\}$ are finite. Moreover, $\mathcal{L}(N \subset L)$, the set of intermediate subfactors, is finite if and only if it is equal to $\mathcal{L}_1 \cup \mathcal{L}_2$. If N is an irreducible subfactor, then we recover a result of Y. Watatani.

0. Introduction

Let $N \subset L$ be an inclusion of II_1 factors. A subfactor M of L such that $N \subset M \subset L$ is called an intermediate subfactor. Intermediate subfactors inherit interesting rigidity properties. In ([B]), D. Bisch proves that if $N \subset L$ is a finite depth inclusion, then so are the two inclusions $N \subset M$ and $M \subset L$. Furthermore, he gives an abstract characterization of intermediate subfactors in terms of projections of $\langle L, e_N \rangle$. In general, the set of intermediate subfactors for an inclusion $N \subset L$ may be trivial, consisting of N and L. In this case, N is said to be a maximal subfactor of L. For example, for the inclusion $N \subset N \otimes M_p(\mathbb{C})$ where p is a prime number, there is no nontrivial intermediate subfactor. If p is not a prime, then there are an infinite number of intermediate subfactors. If G is a countable discrete group of outer automorphisms of a II_1 factor N, then any intermediate subfactor of the inclusion $N \subset N \rtimes G$ is of the form $N \rtimes H$ for some subgroup H of G. Thus, N is maximal if and only if G is cyclic of prime order. If N is irreducible, then any M with $N \subset M \subset L$ is automatically a factor and, as a result, the set of intermediate subfactors forms a lattice. If in addition $[L : N] < \infty$, Y. Watatani proved that the lattice is finite. This article deals mainly with the question of finiteness of the set of intermediate subfactors for finite-index-inclusion intermediate subfactors. It is worth mentioning that even in the case that $N' \cap L$ is abelian, the set of intermediate subfactors may not be finite (cf. Theorem 5.4 [TW]). Our main result states a necessary and sufficient condition, formulated in terms of the relative commutant $N' \cap L$, for the set of intermediate subfactors to be finite.

Received by the editors May 30, 2001 and, in revised form, August 30, 2001 and October 23, 2002.

2000 *Mathematics Subject Classification.* Primary 46L37.

Key words and phrases. Subfactors, von Neumann algebras, Jones index, lattice, relative commutants.

The first author's research was supported by an NSERC grant.

For an inclusion $N \subset L$ we let $\mathcal{L}(N \subset L)$ denote the set of intermediate subfactors, which in the case that N is an irreducible subfactor of L, but not in general, is a lattice under the operations $M_1 \wedge M_2 = M_1 \cap M_2$ and $M_1 \vee M_2 = (M_1 \cup M_2)''$ in $\mathcal{L}(N \subset L)$. In order to prove our main result (Theorem 1.7), we identify two finite subsets, $\mathcal{L}_1$ and $\mathcal{L}_2$, of $\mathcal{L}(N \subset L)$ such that: a) the two sets coincide and coincide with $\mathcal{L}(N \subset L)$ in the irreducible case, and hence recovering Watatani's theorem, we moreover prove that b) $\mathcal{L}(N \subset L)$ is finite if and only if $\mathcal{L}(N \subset L) = \mathcal{L}_1 \cup \mathcal{L}_2$. Intuitively, $\mathcal{L}_1$ consists of those intermediate subfactors that are close to N, and $\mathcal{L}_2$ consists of the ones close to L (see Theorem 1.3 for the definition). The perturbation techniques developed by E. Christensen ([C]) play an essential role in our analysis, much similar to the methods that we developed in ([KM]). The crucial observation is the fact that if M_1 and M_2 are intermediate subfactors for an inclusion $N \subset L$ of finite index, $[L : M_1] = [L : M_2]$, and the distance between M_1 and M_2 is sufficiently small, then they are unitarily equivalent, i.e., $M_2 = uM_1u^*$ for a unitary $u \in L$, and the unitary u can be chosen to be close to the identity of L.

1. THE CARDINALITY OF THE SET OF INTERMEDIATE SUBFACTORS

Throughout, $N \subset L$ are fixed II_1 factors, and $[L : N] < \infty$. Recall that using the trace on L, E. Christensen ([C]) defined a metric d on the set of von Neumann subalgebras of L. In many interesting situations, Christensen proved that if $d(M_1, M_2)$ is sufficiently small, then M_1 and M_2 are $*$-isomorphic via a unitary operator close to the identity of L. We are going to rely on the ideas, notation, and results of ([C]). The following function, γ, appears frequently in perturbation calculations:

$$\gamma(x) = 2^{1/4}x^{1/2}(1 - 2^{1/4}x^{1/2})^{-1}.$$

Recall that if $[L : N] < \infty$, then $N' \cap L$ is a finite-dimensional C^*-algebra. Thus, $\{\mathrm{tr}(p)\colon p \in N' \cap L, \text{projection}\}$ is a finite set. This fact is often used in subsequent arguments. Given $M \subset L$, $\langle L, e_M\rangle$ denotes Jones' basic construction and e_M the corresponding Jones' projection. We refer to ([J]) for basic notation and facts on index theory. Finally, we let $|A|$ denote the cardinality of the set A.

1.1. **Lemma.** *Let $[L : N] < \infty$, $M_1, M_2 \in \mathcal{L}(N \subset L)$, and $[L : M_1] = [L : M_2]$. Then there exists $\epsilon > 0$ such that if $d(M_1, M_2) < \epsilon$, then $M_2 = uM_1u^*$ for a unitary $u \in L$ with $\|u - 1\|_2 < 2\epsilon + 52\gamma(\epsilon)$.*

Proof. Let $[L : N] = c$. Then, for any $M \in \mathcal{L}(N \subset L)$, $[L : M] < c$. Let $\delta = \max\{10^{-6}, 10^{-4}c^{-1}\}$, such that $\min\{\gamma(\delta)^2, 26\gamma(\delta) + \delta\} < c^{-1}$. First note that if $d(M, N) < \delta$, then $[L : M] = [L : N]$ (see the last paragraph of the proof of Theorem 6 of [C]). By (Lemma 2.1, [C]) there exists a projection $e \in M' \cap \langle L, e_M\rangle''$. We have that $\mathrm{tr}(e) > [L : M]^{-1} > c^{-1}$. $|\,\mathrm{tr}(e_M) - \mathrm{tr}(q)| < \gamma(\delta)^2 < c^{-1}$. Note that the projection p in the discussion preceding (Lemma 4.1, [C]) is e_M and $\pi = E_M$ in our context. But for each projection e in $M' \cap \langle L, e_M\rangle''$ we have that $\mathrm{tr}(e) > [L : M]^{-1} > c^{-1}$. Whence, $\mathrm{tr}(e_M) = \mathrm{tr}(q)$ and hence $q \sim e_M$. Now by (Lemma 4.1, [C]) there exists a homomorphism Φ of N into M such that $\|\Phi(x) - E_M(x)\| < 26\gamma(\delta_1)$. Since $d(M, N) < \delta$,

$$\|\Phi(x) - x\|_2 \leq \|\Phi(x) - E_M(x)\|_2 + \|E_M(x) - x\|_2 < 26\gamma(\delta) + \delta.$$

Then, by (Theorem 3.1, [C]) there exists $v \in \langle N, \Phi(N)\rangle''$ such that $q = v^*v \in \Phi(N)'$, $r = vv^* \in N'$, and such that $\|1 - v\|_2$, $\|1 - q\|_2$, $\|1 - r\|_2$ are all less than

$26\gamma(\delta)+\delta$, and $q\Phi(x)=v^*xv$. Since q and r belong to the finite-dimensional algebra $M'\cap L$, our choice of the constant δ implies that $q=r=1$. Thus, v is a unitary in L that implements Φ, i.e., $\Phi(x)=vxv^*$ for each $x\in N$ (cf. Lemma 4.1, [C]). Since $v\in L$, $[L:\Phi(N)]=[L:vNv^*]=[L:N]=[L:M]$, but $\Phi(N)\subset M$. Hence, $\Phi(N)=M$ (cf. [J]). □

1.2. **Corollary.** *Let $M_1, M_2\in\mathcal{L}(N\subset L)$. Then there exists a unitary $u\in L$ such that $uM_2u^*=M_1$ and $uNu^*=N$.*

Proof. Let $\varphi\colon M_2\mapsto M_1$ be the isomorphism of Lemma 1.1. Note that N and $\varphi(N)$ are included in M_1, and $d(N,\varphi(N))<2\epsilon+52\gamma(\epsilon)$. Moreover, φ maps $N'\cap M_2$ onto $\varphi(N)'\cap M_1$. Extend φ to the algebra $\langle L,e_N\rangle$ by $\varphi(x)=vxv^*$ for all $x\in\langle L,e_N\rangle$. This is a trace-preserving isomorphism, and hence the minimal projections in $\varphi(N)'\cap M_1$ and those of $N'\cap M_2$ have the same set of trace values (up to permutations). Then, the argument of the preceding lemma can be applied to N and $\varphi(N)$ as subfactors of M_1 to get projections $q_1\in\varphi(N)\cap M_1$, $r_1\in N'\cap M_1$, and a partial isometry $v_1\in M_1$ such that $v_1v_1^*=q_1$ and $v_1^*v_1=r_1$, all of which are close to the identity in $\|\ \|_2$. Since $N'\cap M_i\subset N'\cap L$, a trace argument as before implies that $q_1=r_1=1$. Hence, v_1 is a unitary and $\varphi(N)=v_1Nv_1^*$. If v is the unitary of Lemma 1.1, then $u=v_1^*v$ is the desired unitary, i.e., $M_1=uM_2u^*$ and $uNu^*=N$. □

1.3. **Theorem.** *Let $N\subset L$ be II_1 factors such that $[L:N]<\infty$. Then,*
i) $\mathcal{L}_1=\{M\in\mathcal{L}(N\subset L)\colon N'\cap L\subset M\}$,
and
ii) $\mathcal{L}_2=\{M\in\mathcal{L}(N\subset L)\colon N'\cap L=M'\cap L\}$
are finite subsets of $\mathcal{L}(N\subset L)$.

Proof. The first part of the proof is the same for both i) and ii). For each $M\in\mathcal{L}(N\subset L)$, $e_M\in N'\cap\langle L,e_N\rangle$ with trace equal to $[L:M]^{-1}$. Since $N'\cap\langle L,e_N\rangle$ is finite dimensional, the set $\{[L:M]:M\in\mathcal{L}(N\subset L)\}$ must be finite. Hence, it suffices to show that for $c>1$, the intersections of $\mathcal{L}_1$ and $\mathcal{L}_2$ with the set $\{M\in\mathcal{L}(N\subset L):[L:M]=c\}$ is a finite set. If not there exists a sequence (K_n) in $\mathcal{L}_1$ (respectively in $\mathcal{L}_2$) such that $K_n\neq K_m$ if $n\neq m$ and $[L:K_n]=c$ for each n. Since $e_{K_n}\in N'\cap\langle L,e_N\rangle$, which is finite dimensional, the sequence (e_{K_n}) must have a limit point in $N'\cap\langle L,e_N\rangle$. Assume, without loss of generality, that the sequence (e_{K_n}) converges in the uniform topology to a projection $p\in N'\cap\langle L,e_N\rangle$, and such that $\mathrm{tr}(e_{K_n})=\mathrm{tr}(p)=c$. Then, $d(K_n,K_m)<\|e_{K_n}-e_{K_m}\|_2$, which shows that $d(K_n,K_m)$ can be made arbitrarily small by choosing m and n sufficiently large. Hence, by Lemma 1.1, there exists a unitary $u\in L$ such that $K_m=uK_nu^*$ for sufficiently large n and m such that $\|u-1\|<\epsilon$ for a given $\epsilon>0$. Thus, for each $z\in N\subset K_n$, $uzu^*=k$ for some $k\in K_m$. Whence,

$$E_{K_m}(u)z=kE_{K_m}(u),$$

which shows that $u^*E_{K_m}(u)z=zu^*E_{K_m}(u)$, i.e., $u^*E_{K_m}(u)\in N'\cap L$. At this point we consider the two cases separately.

i) Let $u^*E_{K_m}(u)=h$. Since $\|u-I\|_{\mathrm{tr}}$ can be made sufficiently small, and $\|h-I\|<2\|u-I\|_{\mathrm{tr}}$, we can choose ϵ such that

$$\|h-I\|<1.$$

Thus, the element h is invertible, and we have $u^*E_{K_m}(u)h^{-1} = I$. Since $K_m \in \mathcal{L}_1$, we have $N' \cap L \subset K_m$. Hence $h \in K_m$. Whence,

$$E_{K_m}(uh^{-1}) = u,$$

which shows that $u \in K_m$. Whence, $K_n = K_m$, which is a contradiction.

ii) Since $E_{K_m}(u) = uh$, the element $E_{K_m}(u)$ is invertible. Since $h \in K'_m \cap L = N' \cap L$, for any $x \in K_m$ we have

$$\begin{aligned} uxu^* &= E_{K_m}(u)h^{-1}xhE_{K_m}(u)^{-1} = E_{K_m}(u)h^{-1}hxE_{K_m}(u)^{-1} \\ &= E_{K_m}(u)xE_{K_m}(u)^{-1}. \end{aligned}$$

Thus $uxu^* \in K_m$, and it follows that $K_m = K_n$, which is in contradiction with the choice of K_n's. We conclude from this argument that $\mathcal{L}_1$ and $\mathcal{L}_2$ are not finite sets. ☐

The following corollary is a theorem of Y. Watatani ([W]).

1.4. **Corollary.** *Let* $[L : N] < \infty$ *and* $N' \cap L = \mathbb{C}$. *Then,* $\mathcal{L}at(N \subset L)$ *is finite.*

Proof. Observe that if $N' \cap L = \mathbb{C}$, then $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}at(N \subset L)$, and the corollary follows from Theorem 1.3. ☐

1.5. **Corollary.** *There exists an* $\epsilon > 0$ *such that if* $M_1, M_2 \in \mathcal{L}(N \subset L)$ *and* $d(M_1, M_2) < \epsilon$, *then* $M_2 = xM_1x^{-1}$ *for an invertible* $x \in N' \cap L$.

Proof. By (Theorem 5, [MT]), there exists a $\delta > 0$ such that $[L : M_1] = [L : M_2]$ when $d(M_1, M_2) < \delta$. Let ϵ be the minimum of δ and the constant given by Lemma 1.1. Then there exists $u \in L$ such that $M_1 = uM_2u^*$ and $\|u - I\| < 2\epsilon + 52\gamma(\epsilon)$. By the argument of Theorem 1.3, $uE_{M_2}(u) = x \in N' \cap L$. Now, choose ϵ sufficiently small such that $\|x - I\| < 1$. (In fact, $\|x - I\| < 2\epsilon + 104\gamma(\epsilon)$.) Then x is invertible, $x \in N' \cap L$, and

$$xM_2x^{-1} = uE_{M_2}(u)M_2E_{M_2}(u)^{-1}u^* = uM_2u^* = M_1. \qquad ☐$$

1.6. **Proposition.** *Let* $N \subset L$ *be an inclusion of* II_1 *factors such that* $[L : N] < \infty$. *Suppose that* $N' \cap L \neq \langle N' \cap M, M' \cap L\rangle''$. *Then* $|\mathcal{L}(N \subset L)|$ *is infinite.*

Proof. Choose a minimal projection $f \in N' \cap L$ that is not in $\langle N' \cap M, M' \cap L\rangle''$. Let p_1 and p_2 be distinct prime numbers larger than $[L : N]$. Consider the unitary

$$u = \exp(2\pi i/p_1)f + \exp(2\pi i/p_2)(1 - f) \in L.$$

Then u generates a group G of unitary elements of order p_1p_2.

Claim. The set $\{vMv^* : v \in G\}$ consists of distinct subfactors.

If not, there will be an element $v \in G$ such that $vMv^* = M$. Now $\mathrm{Ad}\, v$ is an outer automorphism of M. For otherwise, there exists a unitary $w \in M$ such that $\mathrm{Ad}\, v = \mathrm{Ad}\, w$. Then, $v^*w \in M' \cap L$ and since $w \in N' \cap M$ it follows that v and hence f, being a spectral projection of v, belongs to $\langle N' \cap M, M' \cap L\rangle''$, which is in contradiction with the choice of f. Let H be the subgroup of G generated by v. Then, $|H| > [L : N]$ by the choice of p_1 and p_2. Now the fixed point algebra M^H contains N, and $[M : M^H] = |H|$ (see [J], Example 2.3.3). Then, from the inclusions $N \subset M^H \subset M \subset L$ we have

$$[L : N] = [M^H : N][M : M^H][L : M] > |H| > [L : N],$$

which is a contradiction, and our claim is established.

We conclude that

$$|\mathcal{L}_L(N \subset L)| \geq p_1 p_2.$$

Since p_1 and p_2 can be chosen as large as we want, it follows that $|\mathcal{L}_L(N \subset L)| = \infty$. □

We are now ready to state our main result.

1.7. **Theorem.** *Let $N \subset L$ be an inclusion of II_1 factors such that $[L : N] < \infty$. Then $\mathcal{L}(N \subset L)$ is finite if and only if $\mathcal{L}(N \subset L) = \mathcal{L}_1 \cup \mathcal{L}_2$.*

Proof. The if part is just Theorem 1.3. Suppose there exists $M \in \mathcal{L}(N \subset L) \setminus \mathcal{L}_1 \cup \mathcal{L}_2$. Then we claim that $\mathcal{L}(N \subset L)$ is an infinite set. If

$$N' \cap L \neq \langle N' \cap M, M' \cap L \rangle'',$$

then the claim holds by Theorem 1.7. Hence, assume that

$$N' \cap L = \langle N' \cap M, M' \cap L \rangle''. \tag{$*$}$$

Let $\{p_1, p_2, \ldots, p_n\}$ and $\{q_1, q_2, \ldots, q_m\}$ be, respectively, the sets of minimal central projections of $N' \cap M$ and $M' \cap L$. Then n and m are larger than one. For if either n or m equals one, then $(*)$ implies that $M \in \mathcal{L}_1 \cup \mathcal{L}_2$, which is contrary to our assumption. Choose prime numbers $\{p_{jl} \colon 1 \leq j \leq n,\ 1 \leq l \leq m\}$ such that

i) $p_{11} > [L : N]$;
ii) $p_{jl} < p_{j+1l}$ and $p_{jl} < p_{jl+1}$ for all j and l.

Let,

$$u = \sum_{j,l} e^{(\frac{2\pi}{p_{jl}})i} p_j q_l.$$

Then, u is a unitary of order $\prod p_{jl}$ in $N' \cap L$. Hence, $N \subset u^k M u^{*^k} \subset L$. If the $u^k M u^{*^k}$'s, $1 \leq k \leq \prod p_{jl}$, were distinct, then $|\mathcal{L}(N \subset L)|$ must be infinite (for otherwise we can choose p_{ij}'s such that $\prod p_{ij}$ is larger than the cardinality of $\mathcal{L}(N \subset L)$). If not, there exists a positive integer k such that $\operatorname{Ad} u^k$ is an automorphism of M. We may assume that $k < p_{11}$ (for otherwise, by increasing p_{11} large enough we obtain a set of distinct intermediate subfactors whose cardinality is arbitrarily large, which is what we want). Moreover, $\operatorname{Ad} u^k$ is an outer automorphism of M. To see this suppose that $\operatorname{Ad} u^k = \operatorname{Ad} v$ for a unitary $v \in M$. If so, $v \in N' \cap M$ and $u^k = vw$ for some unitary $w \in M' \cap L$. Let $v = \sum_j x_j p_j$ with each $x_j \in N' \cap M$ and $w = \sum_l y_l q_l$ with each $y_l \in M' \cap L$. Then from $u^k = vw$, we obtain $x_j y_l = e^{(\frac{2k\pi}{p_{jl}})i}$. From this equation it follows that each x_j and each y_l are invertible with inverses respectively in $M' \cap L$ and $N' \cap M$. It is then easy to see that the elements x_j' and y_l' belong to the intersection of $N' \cap M$ with $M' \cap L$, which is trivial. Whence, x_j and y_l must be scalers. Whence, $x_j = e^{\theta_j i}$ and $y_l = e^{\beta_j i}$ for $1 \leq j \leq n$, $1 \leq l \leq m$, and $e^{(\theta_j + \beta_l)i} = e^{\frac{2k\pi}{p_{jl}} i}$. Hence,

$$\theta_j + \beta_l + 2r\pi = \frac{2k\pi}{p_{jl}},$$

which shows that $(\theta_j + \beta_l + 2r\pi)p_{jl} = 2k\pi$. By the choice of p_{jl}'s we must have $k > p_{1,1}$, which is a contradiction. Next, let r be the smallest power of u^k such that $\operatorname{Ad} u^{kr}$ is an inner automorphism M. Then $1 < r \leq \prod p_{jl}$ and the group H generated by $\operatorname{Ad} u^{k(r+1)}$ is a group of outer automorphisms of M. Moreover, the order of H divides $\prod p_{jl}$, and hence $|H| > [L : N]$. Now, the fixed point algebra

M^H contains N, and the argument of Proposition 1.6 can be repeated to arrive at a contradiction. The contradiction shows that $\{u^k M u^{*^k}\}$ consists of distinct intermediate subfactors, and since its cardinality can be made arbitrarily large, we conclude that $\mathcal{L}(N \subset L)$ is an infinite set, which is what we wanted to show. □

Define an equivalence relation on $\mathcal{L}(N \subset L)$ by $M_1 \sim M_2$ if there exists a unitary $u \in L$ such that $M_1 = uM_2u^*$ and $N = uNu^*$. Let $\mathcal{L}_N(N \subset L)$ be the corresponding quotient space. A second equivalence relation can be defined by $M_1 = uM_2u^*$, but u need not leave N invariant. Denote by $\mathcal{L}_L(N \subset L)$ the subsequent quotient. Let $|A|$ be the cardinality of the set A. Then we have the following theorem.

1.8. **Theorem.** *Let $N \subset L$ be an inclusion of II_1 factors such that $[L : N] < \infty$. Then $\mathcal{L}_N(N \subset L)$ and $\mathcal{L}_L(N \subset L)$ are finite sets. Moreover,*

$$|\mathcal{L}_L(N \subset L)| \leq |\mathcal{L}_N(N \subset L)| \leq |\mathcal{L}(N \subset L)|.$$

Proof. The finiteness of $\mathcal{L}_L(N \subset L)$ follows by using the type of argument given in Theorem 1.3 and by Lemma 1.1. Also, Corollary 1.2 shows that $\mathcal{L}_N(N \subset L)$ is finite. □

Acknowledgment

We would like to thank the referee for an extremely careful and pertinent review, which greatly improved the paper.

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