ON FINITENESS OF THE SET
OF INTERMEDIATE SUBFACTORS

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Abstract. For type $II_1$ factors $N \subset L$ with $[L : N] < \infty$, we show that the sets
$L_1 = \{M \in \mathcal{L}(N \subset L) : N' \cap L \subset M\}$ and $L_2 = \{M \in \mathcal{L}(N \subset L) : N' \cap L = M' \cap L\}$ are finite. Moreover, $\mathcal{L}(N \subset L)$, the set of intermediate subfactors, is finite if and only if it is equal to $L_1 \cup L_2$. If $N$ is an irreducible subfactor, then we recover a result of Y. Watatani.

0. Introduction

Let $N \subset L$ be an inclusion of $II_1$ factors. A subfactor $M$ of $L$ such that $N \subset M \subset L$ is called an intermediate subfactor. Intermediate subfactors inherit interesting rigidity properties. In [B], D. Bisch proves that if $N \subset L$ is a finite depth inclusion, then so are the two inclusions $N \subset M$ and $M \subset L$. Furthermore, he gives an abstract characterization of intermediate subfactors in terms of projections of $\langle L, e_N \rangle$. In general, the set of intermediate subfactors for an inclusion $N \subset L$ may be trivial, consisting of $N$ and $L$. In this case, $N$ is said to be a maximal subfactor of $L$. For example, for the inclusion $N \subset N \otimes M_p(\mathbb{C})$ where $p$ is a prime number, there is no nontrivial intermediate subfactor. If $p$ is not a prime, then there are an infinite number of intermediate subfactors. If $G$ is a countable discrete group of outer automorphisms of a $II_1$ factor $N$, then any intermediate subfactor of the inclusion $N \subset N \rtimes G$ is of the form $N \rtimes H$ for some subgroup $H$ of $G$. Thus, $N$ is maximal if and only if $G$ is cyclic of prime order. If $N$ is irreducible, then any $M$ with $N \subset M \subset L$ is automatically a factor and, as a result, the set of intermediate subfactors forms a lattice. If in addition $[L : N] < \infty$, Y. Watatani proved that the lattice is finite. This article deals mainly with the question of finiteness of the set of intermediate subfactors for finite-index-inclusion intermediate subfactors. It is worth mentioning that even in the case that $N' \cap L$ is abelian, the set of intermediate subfactors may not be finite (cf. Theorem 5.4 [TW]). Our main result states a necessary and sufficient condition, formulated in terms of the relative commutant $N' \cap L$, for the set of intermediate subfactors to be finite.
For an inclusion $N \subset L$ we let $\mathcal{L}(N \subset L)$ denote the set of intermediate subfactors, which in the case that $N$ is an irreducible subfactor of $L$, but not in general, is a lattice under the operations $M_1 \wedge M_2 = M_1 \cap M_2$ and $M_1 \vee M_2 = (M_1 \cup M_2)^{\prime \prime}$ in $\mathcal{L}(N \subset L)$. In order to prove our main result (Theorem 1.7), we identify two finite subsets, $\mathcal{L}_1$ and $\mathcal{L}_2$, of $\mathcal{L}(N \subset L)$ such that: a) the two sets coincide and coincide with $\mathcal{L}(N \subset L)$ in the irreducible case, and hence recovering Watatani’s theorem, we moreover prove that b) $\mathcal{L}(N \subset L)$ is finite if and only if $\mathcal{L}(N \subset L) = \mathcal{L}_1 \cup \mathcal{L}_2$.

1. The cardinality of the set of intermediate subfactors

Throughout, $N \subset L$ are fixed $II_1$ factors, and $[L : N] < \infty$. Recall that using the trace on $L$, E. Christensen ([C]) defined a metric $d$ on the set of von Neumann subalgebras of $L$. In many interesting situations, Christensen proved that if $d(M_1, M_2)$ is sufficiently small, then $M_1$ and $M_2$ are $*$-isomorphic via a unitary operator close to the identity of $L$. We are going to rely on the ideas, notation, and results of ([C]). The following function, $\gamma$, appears frequently in perturbation calculations:

$$\gamma(x) = 2^{1/4}x^{1/2}(1 - 2^{1/4}x^{1/2})^{-1}.$$ 

Recall that if $[L : N] < \infty$, then $N' \cap L$ is a finite-dimensional $C^*$-algebra. Thus, $\{\text{tr}(p) : p \in N' \cap L, \text{projection}\}$ is a finite set. This fact is often used in subsequent arguments. Given $M \subset L$, $\langle L, e_M \rangle$ denotes Jones’ basic construction and $e_M$ the corresponding Jones’ projection. We refer to ([J]) for basic notation and facts on index theory. Finally, we let $|A|$ denote the cardinality of the set $A$.

1.1. Lemma. Let $[L : N] < \infty$, $M_1, M_2 \in \mathcal{L}(N \subset L)$, and $[L : M_1] = [L : M_2]$. Then there exists $\epsilon > 0$ such that if $d(M_1, M_2) < \epsilon$, then $M_2 = uM_1u^*$ for a unitary $u \in L$ with $\|u - 1\|_2 < 2\epsilon + 52\gamma(\epsilon)$.

Proof. Let $[L : N] = c$. Then, for any $M \in \mathcal{L}(N \subset L)$, $[L : M] < c$. Let $\delta = \max\{10^{-6}, 10^{-4}\epsilon^{-1}\}$, such that $\min\{\gamma(\delta)^2, 26\gamma(\delta) + \delta\} < \epsilon^{-1}$. First note that if $d(M, N) < \delta$, then $[L : M] = [L : N]$ (see the last paragraph of the proof of Theorem 6 of [C]). By (Lemma 2.1, [C]) there exists a projection $e \in M' \cap (\langle L, e_M \rangle)^{\prime \prime}$. We have that $\text{tr}(e) > [L : M]^{-1} > \epsilon^{-1}$, $|\text{tr}(e_M) - \text{tr}(q)| < \gamma(\delta)^2 < \epsilon^{-1}$. Note that the projection $p$ in the discussion preceding (Lemma 4.1, [C]) is $e_M$ and $\pi = E_M$ in our context. But for each projection $e$ in $M' \cap (\langle L, e_M \rangle)^{\prime \prime}$ we have that $\text{tr}(e) > [L : M]^{-1} > \epsilon^{-1}$. Whence, $\text{tr}(e_M) = \text{tr}(q)$ and hence $q \sim e_M$. Now by (Lemma 4.1, [C]) there exists a homomorphism $\Phi$ of $N$ into $M$ such that $\|\Phi(x) - E_M(x)\| < 26\gamma(\delta)$. Since $d(M, N) < \delta$,

$$\|\Phi(x) - x\|_2 \leq \|\Phi(x) - E_M(x)\|_2 + \|E_M(x) - x\|_2 < 26\gamma(\delta) + \delta.$$ 

Then, by (Theorem 3.1, [C]) there exists $v \in \langle N, \Phi(N) \rangle^{\prime \prime}$ such that $q = v^*v \in \Phi(N)'$, $r = vv^* \in N'$, and such that $\|1 - v\|_2, \|1 - q\|_2, \|1 - r\|_2$ are all less than...
26\gamma(\delta) + \delta$, and $q\Phi(x) = v^* xv$. Since $q$ and $r$ belong to the finite-dimensional algebra $M' \cap L$, our choice of the constant $\delta$ implies that $q = r = 1$. Thus, $v$ is a unitary in $L$ that implements $\Phi$, i.e., $\Phi(x) = v^* xv$ for each $x \in N$ (cf. Lemma 4.1, \(\square\)). Since $v \in L$, $[L : \Phi(N)] = [L : vNv^*] = [L : N] = [L : M]$, but $\Phi(N) \subset M$. Hence, $\Phi(N) = M$ (cf. \(\square\)).

1.2. Corollary. Let $M_1, M_2 \in \mathcal{L}(N \subset L)$. Then there exists a unitary $u \in L$ such that $uM_2u^* = M_1$ and $uNu^* = N$.

Proof. Let $\varphi: M_2 \mapsto M_1$ be the isomorphism of Lemma 1.1. Note that $N$ and $\varphi(N)$ are included in $M_1$, and $d(N, \varphi(N)) < 2\epsilon + 52\gamma(\epsilon)$. Moreover, $\varphi$ maps $N' \cap M_2$ onto $\varphi(N)' \cap M_1$. Extend $\varphi$ to the algebra $\langle L, e_N \rangle$ by $\varphi(x) = v^* xv$ for all $x \in \langle L, e_N \rangle$. This is a trace-preserving isomorphism, and hence the minimal projections in $\varphi(N)' \cap M_1$ and those of $N' \cap M_2$ have the same set of trace values (up to permutations). Then, the argument of the preceding lemma can be applied to $N$ and $\varphi(N)$ as subfactors of $M_1$ to get projections $q_1 \in \varphi(N) \cap M_1$, $r_1 \in N' \cap M_1$, and a partial isometry $v_1 \in M_1$ such that $v_1v_1^* = q_1$ and $v_1^*v_1 = r_1$, all of which are close to the identity in $\| \cdot \|_2$. Since $N' \cap M_1 \subset N' \cap L$, a trace argument as before implies that $q_1 = r_1 = 1$. Hence, $v_1$ is a unitary and $\varphi(N) = v_1Nv_1^*$. If $v$ is the unitary of Lemma 1.1, then $u = v_1^*v$ is the desired unitary, i.e., $M_1 = uM_2u^*$ and $uNu^* = N$. \(\square\)

1.3. Theorem. Let $N \subset L$ be $II_1$ factors such that $[L : N] < \infty$. Then,

i) $\mathcal{L}_1 = \{M \in \mathcal{L}(N \subset L) : N' \cap L \subset M\}$,

and

ii) $\mathcal{L}_2 = \{M \in \mathcal{L}(N \subset L) : N' \cap L = M' \cap L\}$

are finite subsets of $\mathcal{L}(N \subset L)$.

Proof. The first part of the proof is the same for both i) and ii). For each $M \in \mathcal{L}(N \subset L)$, $e_M \in N' \cap \langle L, e_N \rangle$ with trace equal to $[L : M]^{-1}$. Since $N' \cap \langle L, e_N \rangle$ is finite dimensional, the set $\{[L : M] : M \in \mathcal{L}(N \subset L)\}$ must be finite. Hence, it suffices to show that for $c > 1$, the intersections of $\mathcal{L}_1$ and $\mathcal{L}_2$ with the set $\{M \in \mathcal{L}(N \subset L) : [L : M] = c\}$ is a finite set. If there does not exist a sequence $(K_n)$ in $\mathcal{L}_1$ (respectively in $\mathcal{L}_2$) such that $K_n \neq K_m$ if $n \neq m$ and $[L : K_n] = c$ for each $n$. Since $e_{K_n} \in N' \cap \langle L, e_N \rangle$, which is finite dimensional, the sequence $(e_{K_n})$ must have a limit point in $N' \cap \langle L, e_N \rangle$. Assume, without loss of generality, that the sequence $(e_{K_n})$ converges in the uniform topology to a projection $p \in N' \cap \langle L, e_N \rangle$, and such that $\text{tr}(e_{K_n}) = \text{tr}(p) = c$. Then, $d(K_n, K_m) < \|e_{K_n} - e_{K_m}\|_2$, which shows that $d(K_n, K_m)$ can be made arbitrarily small by choosing $m$ and $n$ sufficiently large. Hence, by Lemma 1.1 there exists a unitary $u \in L$ such that $K_m = uK_nu^*$ for sufficiently large $n$ and $m$ such that $\|u - 1\| < \epsilon$ for a given $\epsilon > 0$. Thus, for each $z \in N \subset K_n$, $uzu^* = k$ for some $k \in K_m$. Whence,

$E_{K_m}(u)z = ke_{K_m}(u)$,

which shows that $u^*E_{K_m}(u)z = zu^* E_{K_m}(u)$, i.e., $u^*E_{K_m}(u) \in N' \cap L$. At this point we consider the two cases separately.

i) Let $u^*E_{K_m}(u) = h$. Since $\|u - I\|_{tr}$ can be made sufficiently small, and $\|h - I\| < 2\|u - I\|_{tr}$, we can choose $\epsilon$ such that $\|h - I\| < 1$. 

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Thus, the element $h$ is invertible, and we have $u^* E_{K_m}(u)h^{-1} = I$. Since $K_m \in \mathcal{L}_1$, we have $N' \cap L \subset K_m$. Hence $h \in K_m$. Whence,

$$E_{K_m}(uh^{-1}) = u,$$

which shows that $u \in K_m$. Whence, $K_n = K_m$, which is a contradiction.

ii) Since $E_{K_m}(u) = uh$, the element $E_{K_m}(u)$ is invertible. Since $h \in K'_m \cap L = N' \cap L$, for any $x \in K_m$ we have

$$uxu^* = E_{K_m}(u)h^{-1}xhE_{K_m}(u)^{-1} = E_{K_m}(u)h^{-1}hxE_{K_m}(u)^{-1}$$

Thus $uxu^* \in K_m$, and it follows that $K_m = K_n$, which is in contradiction with the choice of $K_n$’s. We conclude from this argument that $\mathcal{L}_1$ and $\mathcal{L}_2$ are not finite sets. 

The following corollary is a theorem of Y. Watatani ([W]).

1.4. Corollary. Let $[L : N] < \infty$ and $N' \cap L = \mathbb{C}$. Then, $\text{Lat}(N \subset L)$ is finite.

Proof. Observe that if $N' \cap L = \mathbb{C}$, then $\mathcal{L}_1 = \mathcal{L}_2 = \text{Lat}(N \subset L)$, and the corollary follows from Theorem 1.3.

1.5. Corollary. There exists an $\epsilon > 0$ such that if $M_1, M_2 \in \mathcal{L}(N \subset L)$ and $d(M_1, M_2) < \epsilon$, then $M_2 = xM_1 x^{-1}$ for an invertible $x \in N' \cap L$.

Proof. By (Theorem 5, [MT]), there exists a $\delta > 0$ such that $[L : M_1] = [L : M_2]$ when $d(M_1, M_2) < \delta$. Let $\epsilon$ be the minimum of $\delta$ and the constant given by Lemma 1.3. Then there exists $u \in L$ such that $M_1 = uM_2u^*$ and $\|u - I\| < 2\epsilon + 52\gamma(\epsilon)$. By the argument of Theorem 1.3, $uE_{M_2}(u) = x \in N' \cap L$. Now, choose $\epsilon$ sufficiently small such that $\|x - I\| < 1$. (In fact, $\|x - I\| < 2\epsilon + 104\gamma(\epsilon)$.) Then $x$ is invertible, $x \in N' \cap L$, and

$$xM_2x^{-1} = uE_{M_2}(u)M_2E_{M_2}(u)^{-1}u^* = uM_2u^* = M_1.$$ 

1.6. Proposition. Let $N \subset L$ be an inclusion of II$_1$ factors such that $[L : N] < \infty$. Suppose that $N' \cap L \neq \langle N' \cap M, M' \cap L \rangle''$. Then $|\mathcal{L}(N \subset L)|$ is infinite.

Proof. Choose a minimal projection $f \in N' \cap L$ that is not in $\langle N' \cap M, M' \cap L \rangle''$. Let $p_1$ and $p_2$ be distinct prime numbers larger than $[L : N]$. Consider the unitary $u = \exp(2\pi i/p_1)f + \exp(2\pi i/p_2)(1 - f) \in L$.

Then $u$ generates a group $G$ of unitary elements of order $p_1p_2$.

Claim. The set $\{vMv^* : v \in G\}$ consists of distinct subfactors.

If not, there will be an element $v \in G$ such that $vMv^* = M$. Now $\text{Ad} v$ is an outer automorphism of $M$. For otherwise, there exists unitary $w \in M$ such that $\text{Ad} v = \text{Ad} w$. Then, $v^*w \in M' \cap L$ and since $w \in N' \cap M$ it follows that $v$ and hence $f$, being a spectral projection of $v$, belongs to $\langle N' \cap M, M' \cap L \rangle''$, which is in contradiction with the choice of $f$. Let $H$ be the subgroup of $G$ generated by $v$. Then, $|H| > [L : N]$ by the choice of $p_1$ and $p_2$. Now the fixed point algebra $M^H$ contains $N$, and $[M : M^H] = |H|$ (see [J], Example 2.3.3). Then, from the inclusions $N \subset M^H \subset M \subset L$ we have $[L : N] = [M^H : N][M : M^H][L : M] > |H| > [L : N]$, which is a contradiction, and our claim is established.
We conclude that
|\mathcal{L}_L(N \subset L)| \geq p_1p_2.

Since \( p_1 \) and \( p_2 \) can be chosen as large as we want, it follows that \( |\mathcal{L}_L(N \subset L)| = \infty. \)

We are now ready to state our main result.

1.7. Theorem. Let \( N \subset L \) be an inclusion of II\(_1\) factors such that \([L : N] < \infty.\) Then \( \mathcal{L}(N \subset L) \) is finite if and only if \( \mathcal{L}(N \subset L) = \mathcal{L}_1 \cup \mathcal{L}_2. \)

Proof. The if part is just Theorem 1.3. Suppose there exists \( M \in \mathcal{L}(N \subset L) \setminus \mathcal{L}_1 \cup \mathcal{L}_2. \) Then we claim that \( \mathcal{L}(N \subset L) \) is an infinite set. If
\[ N' \cap L \neq \langle N' \cap M, M' \cap L \rangle', \]
then the claim holds by Theorem 1.7. Hence, assume that
\[(*) \quad N' \cap L = \langle N' \cap M, M' \cap L \rangle'. \]

Let \( \{p_1, p_2, \ldots, p_m\} \) and \( \{q_1, q_2, \ldots, q_n\} \) be, respectively, the sets of minimal central projections of \( N' \cap M \) and \( M' \cap L. \) Then \( n \) and \( m \) are larger than one. For if either \( n \) or \( m \) equals one, then \( (*) \) implies that \( M \in \mathcal{L}_1 \cup \mathcal{L}_2, \) which is contrary to our assumption. Choose prime numbers \( \{p_{jl} : 1 \leq j \leq n, 1 \leq l \leq m\} \) such that
\begin{enumerate}[i)]
  \item \( p_{11} > [L : N]; \)
  \item \( p_{jl} < p_{j+1,l} \) and \( p_{jl} < p_{j+1,l+1} \) for all \( j \) and \( l. \)
\end{enumerate}

Let,
\[ u = \sum_{j,l} e^{(\frac{2\pi}{p_{jl}})i}p_{jl}. \]

Then, \( u \) is a unitary of order \( \prod p_{jl} \) in \( N' \cap L. \) Hence, \( N \subset u^kM u^k \subset L. \) If the \( u^kM u^k \)'s, \( 1 \leq k \leq \prod p_{jl}, \) were distinct, then \( |\mathcal{L}(N \subset L)| \) must be infinite (for otherwise we can choose \( p_{jl}'s \) such that \( \prod p_{jl} \) is larger than the cardinality of \( \mathcal{L}(N \subset L) \)). If not, there exists a positive integer \( k \) such that \( \text{Ad } u^k \) is an automorphism of \( M. \) We may assume that \( k < p_{11} \) (for otherwise, by increasing \( p_{11} \) large enough we obtain a set of distinct intermediate subfactors whose cardinality is arbitrarily large, which is what we want). Moreover, \( \text{Ad } u^k \) is an outer automorphism of \( M. \) To see this suppose that \( \text{Ad } u^k = \text{Ad } v \) for a unitary \( v \in M. \) If so, \( v \in N' \cap M \) and \( u^k = vv' \) for some unitary \( w \in M' \cap L. \) Let \( v = \sum_j x_j p_j \) with each \( x_j \in N' \cap M \) and \( w = \sum_l y_l q_l \) with each \( y_l \in M' \cap L. \) Then from \( u^k = vv' \), we obtain \( x_j y_l = e^{(\frac{2k\pi}{p_{jl}})i} \). From this equation it follows that each \( x_j \) and each \( y_l \) are invertible with inverses respectively in \( M' \cap L \) and \( N' \cap M. \) It is then easy to see that the elements \( x_j' \) and \( y_l' \) belong to the intersection of \( N' \cap M \times M' \cap L, \) which is trivial. Whence, \( x_j \) and \( y_l \) must be scalars. Whence, \( x_j = e^{\theta_j i} \) and \( y_l = e^{\beta_l i} \) for \( 1 \leq j \leq n, 1 \leq l \leq m, \) and \( e^{\theta_j + \beta_l} = e^{\frac{2k\pi}{p_{jl}}}. \) Hence,
\[ \theta_j + \beta_l + 2r\pi = \frac{2k\pi}{p_{jl}}, \]
which shows that \( (\theta_j + \beta_l + 2r\pi)p_{jl} = 2k\pi. \) By the choice of \( p_{jl}'s \) we must have \( k > p_{1,1}, \) which is a contradiction. Next, let \( r \) be the smallest power of \( u^k \) such that \( \text{Ad } u^{kr} \) is an inner automorphism \( M. \) Then \( 1 < r \leq \prod p_{jl} \) and the group \( H \) generated by \( \text{Ad } u^{k(r+1)} \) is a group of outer automorphisms of \( M. \) Moreover, the order of \( H \) divides \( \prod p_{jl}, \) and hence \( |H| > [L : N]. \) Now, the fixed point algebra
$M^H$ contains $N$, and the argument of Proposition 1.6 can be repeated to arrive at a contradiction. The contradiction shows that $\{u^kM^u^k\}$ consists of distinct intermediate subfactors, and since its cardinality can be made arbitrarily large, we conclude that $L(N \subset L)$ is an infinite set, which is what we wanted to show. \qed

Define an equivalence relation on $L(N \subset L)$ by $M_1 \sim M_2$ if there exists a unitary $u \in L$ such that $M_1 = uM_2u^*$ and $N = uNu^*$. Let $L_N(N \subset L)$ be the corresponding quotient space. A second equivalence relation can be defined by $M_1 = uM_2u^*$, but $u$ need not leave $N$ invariant. Denote by $L_L(N \subset L)$ the subsequent quotient. Let $|A|$ be the cardinality of the set $A$. Then we have the following theorem.

1.8. Theorem. Let $N \subset L$ be an inclusion of $I_1$ factors such that $[L : N] < \infty$. Then $L_N(N \subset L)$ and $L_L(N \subset L)$ are finite sets. Moreover,

$$|L_L(N \subset L)| \leq |L_N(N \subset L)| \leq |L(N \subset L)|.$$ 

Proof. The finiteness of $L_L(N \subset L)$ follows by using the type of argument given in Theorem 1.3 and by Lemma 1.1. Also, Corollary 1.2 shows that $L_N(N \subset L)$ is finite. \qed

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