BASE-COVER PARACOMPACTNESS

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Abstract. Call a topological space $X$ base-cover paracompact if $X$ has an open base $\mathcal{B}$ such that every cover $\mathcal{C} \subset \mathcal{B}$ of $X$ contains a locally finite subcover. A subspace of the Sorgenfrey line is base-cover paracompact if and only if it is $F_\sigma$. The countable sequential fan is not base-cover paracompact. A paracompact space is locally compact if and only if its product with every compact space is base-cover paracompact.

Introduction

It was shown in [10] that the irrationals as a subspace of the Sorgenfrey line $S$ are not generalized left separated, although every $F_\sigma$ subspace of $S$ is. In [5] the $F_\sigma$ subspaces of $S$ were characterized as those subspaces that are continuous images of $S$. We define base-cover paracompactness and in Section 1 show that a subspace of $S$ is $F_\sigma$ if and only if it is base-cover paracompact, if and only if it is generalized left separated. In Section 2 we prove that the countable sequential fan is not base-cover metacompact, and base-cover paracompactness is not $F_\sigma$ hereditary in general, although it is hereditary with respect to open $F_\sigma$ subsets. Taking the product with a compact factor easily destroys base-cover paracompactness, and in Section 3 we show that a paracompact space is locally compact if and only if its product with every compact space is base-cover paracompact. In Section 4 related known properties including total paracompactness [13], base-base paracompactness and base paracompactness [23], [24], and the $D$-space property [10] are discussed.

Definition 0.1. A space $X$ is base-cover paracompact [21] (respectively base-cover hypocompact, base-cover metacompact [21]) if it has an open base $\mathcal{B}$ such that every cover $\mathcal{C} \subset \mathcal{B}$ of $X$ contains a locally finite (respectively star finite, point finite) subcover.

1. Base-cover paracompactness and subsets of the Sorgenfrey line

Recall that a $LOTS$ is a linearly ordered topological space $(X, \leq)$ for which the family of all open intervals forms a base. A space $(X, \leq)$ is a GO-space (i.e., a generalized ordered space) if it has a base of order-convex sets, [17].
Theorem 1.1. Suppose $(X, \leq)$ is a GO-space such that:
(a) the set $[x, \to)$ is open for each $x \in X$, and
(b) $X = \bigcup_{i<\omega} X_i$ where every nonempty relatively closed subset of each $X_i$ has a minimal element.

Then every $F_\sigma$ subspace of $X$ is base-cover hypocompact.

Proof. Since every $F_\sigma$-set satisfies (a) and (b), it is enough to prove that $X$ is base-cover hypocompact. Let $D = \{d_\alpha : \alpha < \kappa\}$ be a dense subset of $X$, and for each $\alpha \leq \kappa$, let $D_\alpha = \{d_\beta : \beta < \alpha\}$. For each non-isolated $x$, let $r(x, \alpha) = \inf((x, \to) \cap D_\alpha)$ (possibly a gap), and $B(x, \alpha) = [x, r(x, \alpha)]$. Let $\alpha(x) = \min\{\alpha : r(x, \alpha) = x\}$, and if $x \in X_i \setminus \bigcup_{k<i} X_k$, let $B_x = \{B(x, \alpha) : \alpha < \alpha(x), B(x, \alpha) \cap (\bigcup_{k<i} X_k) = \emptyset\}$. Conditions (a) and (b) imply that each $X_i$ is closed and hence $B_x$ is a base at $x$.

Suppose $C$ is a subcover of the base $(\bigcup\{B_x : x \text{ is not isolated}\}) \cup \{\{x : x \text{ is isolated}\}$ for each $x \in X$ let $C_x = \{C \in C : x \in C\}$. Call an element $C$ of $C_x$ nice (for $x$ and $C$) if $C \cap [x, \to)$ is maximal possible, that is, if $C' \in C_x$, then $C' \cap [x, \to) \subset C \cap [x, \to)$. If $C \subset [x, \to)$ for every $C \in C_x$, then every element of $C_x$ is nice. Else let $C_x^{-} = \{C \in C_x : C \cap [x, \to) \neq \emptyset\}$. Each element of $C_x^{-}$ is of the form $B(z, \alpha)$ for some $z \leq x$ and some $\alpha$. Let $\delta(x) = \min\{\alpha : B(z, \alpha) \in C_x^{-} \text{ for some } z\}$. Then every $B(z, \alpha)$ in $C_x^{-}$ with $\alpha = \delta(x)$ is nice, and $B(z, \alpha) \cap [x, \to) = [x, r(x, \delta(x))]$.

At stage $i$, form a family $\mathcal{C}_i \subset C$ that covers the points of $X_i$ not covered by $\bigcup_{k<i} \mathcal{C}_k$. First cover with one of its nice sets the minimal element of $X_i$ not yet covered, then cover the next minimal with one of its nice sets, etc. We will show that the cover $C' = \bigcup_{i<\omega} \mathcal{C}_i$ is star finite. Each $\mathcal{C}_i$ is well-ordered by the order in which its elements were added, which, together with the order in which the $\mathcal{C}_i$ were formed, defines a well-order on $C'$ denoted by $\prec$. Also define the following order: $C_1 \ll C_2$ in $C'$ if there is a $c_1 \in C_1$ such that $c_1 \ll C_2$ (i.e., $c_1 \ll c$ for every $c \in C_2$), but there is no $c' \in C_1$ such that $c' \ll C_2$. Every two distinct elements of $C'$ are $\ll$-comparable, for if $C_1 \ll C_2$, then $C_2$ was added to cover some $t \notin C_1$, and either $t < C_1$ and hence $C_2 \ll C_1$ (for otherwise $C_1$ would not be nice), or $t > C_1$ and $C_2 \gg C_1$ (since if $t_1$ justified the addition of $C_1$ into $C'$, then $t_1 < C_2$, for otherwise $C_1$ would not be nice). There could be no $C_1 \ll C_2 \ll C_3$ in $C'$ with $C_1 \ll C_2$ and $C_1 \cap C_3 \neq \emptyset$, for then $C_2 \setminus C_1 \subset C_2$; hence $C_2 \ll C_3$ and $C_3$ witnesses that $C_2$ was not nice. Fix a $C \in C'$. If $\prec$ and $\ll$ agree on a subfamily $C''$ of $C'$, then $C$ can meet at most one of its $\ll$-predecessors and at most two of its $\ll$-successors contained in $C''$. Thus, if $C$ meets infinitely many elements of $C'$, we may assume that there is a $\ll$-increasing, $\ll$-decreasing sequence $\{C_n : n < \omega\} \subset C'$ with $C \cap C_n \neq \emptyset$ for all $n$.

Since $C$ meets at most three elements of each $C_i$, we may assume that $C \ll C_n \ll C$, $(\bigcap_{k \leq n} C_k) \cap C = C \cap C_n \neq \emptyset$ and $C_n = B(z_n, \alpha_n)$ with $z_{n+1} < z_n$ for all $n$.

Case 1. Infinitely many $z_n$ come from the same $X_k$.

Then let $z = \inf\{z_n : n < \omega\}$. Condition (b) implies that $z \in X_k$ and $z$ is not isolated from the right. Fix a $C' \in C'$ with $z \ll C'$ and an $n$ with $z_n \in C' \ll B(z_n, \alpha_n)$. But then $B(z_{n+1}, \alpha_{n+1}) \subset C' \cup B(z_n, \alpha_n)$, a contradiction.

Case 2. Each $X_i$ contains only finitely many $z_n$.

Suppose $z_0 \notin X_i$. Fix an $n$ with $z_n \in X_j \setminus \bigcup_{k<j} X_k$ and $j > i$. Then $z_0 \notin B(z_{n+1}, \alpha_{n+1})$ and $B(z_n, \alpha_n) \cap B(z_0, \alpha_0) = \emptyset$, a contradiction.

Remark 1.2. The proof of the above theorem can be modified to show that $X$ is base-base ultraparacompact, i.e., $X$ has a base such that every subfamily which is a base has a disjoint subcover. The base defined above works (as does any base
consisting of convex, closed-and-open sets). Now we assume in addition that $C$ is a base, not only a cover. In the construction of $C_i$, if $x$ is the minimal point of $X_i$ not yet covered, cover $x$ with a basic set disjoint from the elements of $C'$ already defined. This is possible, since each $C_i$ is discrete and hence with a closed-and-open union. See also [23], where it is shown that the Sorgenfrey line is base-base paracompact, and Remark 1.3 in [10], where it is shown that spaces satisfying somewhat stronger conditions than $(a)$ and $(b)$ above (the Sorgenfrey line among them) are ultraparacompact (i.e., every open cover has a disjoint open refinement).

**Remark 1.3.** We cannot drop hypothesis $(a)$ or $(b)$ in the above results, since $\omega_1$ satisfies $(b)$ and, with its order reversed, $\omega_1$ satisfies $(a)$. We cannot replace $(a)$ above with the weaker condition $(a')$ from Theorem 1.4 below; thus we cannot drop $(c)$ in the latter theorem, since if we start with $\omega_1$ and for each limit $\beta < \omega_1$ add a sequence between $\beta$ and $\beta + 1$ decreasing to $\beta$, then the resulting LOT $S$ is not paracompact, satisfies $(a')$ and every closed subset has a minimal element.

**Theorem 1.4.** Let $(X, \leq)$ be a GO-space such that

(a') for each $x$, if $[x, \rightarrow]$ is open, then $x$ is isolated,

(b) $X = \bigcup_{j<\omega_X} X_i$, where every nonempty relatively closed subset of each $X_i$ has a minimal element, and

(c) $X$ has a dense $\sigma$-discrete set (where “discrete” means “discrete in $X$”).

Then every $F_\sigma$ subspace of $X$ is base-cover hypocompact.

**Proof.** The proof is similar to the proof of the preceding theorem, though not completely analogous. We change the construction of the base and the proof that nice elements of $C_x$ exist. Cases 1 and 2 are slightly different too.

Let $\bigcup_{m<\omega} D_m$ be a dense subset of $X$ where, for each $m$, $D_m$ is closed discrete and $D_m \subset D_{m+1}$. Let $I = \{x \in X \mid \{x\} \text{ is open}\}$ and $R = \{x \in X \mid I \setminus \{x, \rightarrow\} \text{ is open}\}$. By $(a')$, the set $\{x \in X \setminus I \mid \{x, \rightarrow\} \text{ is open}\}$ is empty. For each $x$ and each $m$, let $l(x, m) = \sup ((\rightarrow, x) \cap D_m)$ and $r(x, m) = \inf ((x, \rightarrow) \cap D_m)$ (possibly gaps). Define $B(x, m)$ as follows. If $x \in R$, then $B(x, m) = [x, r(x, m))$. If $x \in X \setminus (I \cup R)$, then $B(x, m) = (l(x, m), r(x, m))$.

Suppose $Y$ is an $F_\sigma$ set. By (b) we may assume that $Y = \bigcup_{j<\omega} Y_j$, where every relatively closed subset of each $Y_j$ has a minimal element. For each $x \in Y_1 \setminus \bigcup_{k<\omega} Y_k$ with $x \notin I$, let $B_x = \{B(x, m) : m \geq 1\}$. Then the family $B_Y = (\bigcup_{x \in Y \setminus I} B_x) \cup \{\{x\} : x \in I\}$ is a base for $Y$ in $X$ (i.e., elements of $B_Y$ are open in $X$ and their intersections with $Y$ form a base for $Y$). It is enough to show that if a family $C \subset B_Y$ covers $Y$, then there is a star finite subfamily of $C$ that covers $Y$. Define $C_x$ and $C_x^-$ as before. We need to show that if $C_x^- \neq \emptyset$, then it contains nice elements. Let $m(x) = \min \{m : B(z, m) \in C_x^- \text{ for some } z\}$. If $B(z, m) \in C_x^-$, then $z \leq r(x, m(x))$, for otherwise $r(x, m(x)) \leq y < z$ for some $y \in D_{m(x)}$ and $B(z, m) \subset B(y, m)$.)

If $z < r(x, m(x))$ and $B(z, m) \in C_x^-$, then $B(z, m) \cap [x, \rightarrow) \subset [x, r(x, m(x)))$. If $z < r(x, m(x))$ for every $B(z, m) \in C_x^-$, then any $B(z, m) \in C_x^-$ with $m = m(x)$ is nice, and $B(z, m) \cap [x, \rightarrow) = [x, r(x, m(x)))$. If $r(x, m(x)) \in X$, then there might be some elements of $C_x^-$ of the form $B(r(x, m(x)), j)$; then any $B(r(x, m(x)), j)$ in $C_x^-$ with minimal $j$ is nice.

The family $C'$ is constructed as before (using $Y_i$ in place of $X_i$). Now $\bigcup C'$ contains $Y$. We will show $C'$ is star finite. It is enough to show that there is no $C \in C'$ and a $\prec$-increasing, $\ll$-decreasing sequence $\{B(z_n, m_n) : n < \omega\} \subset C'$ each element of which meets $C$ and $\ll$-precedes $C$. We may assume Case 1: all $m_n$
coincide, say $m_n = M$, or \textbf{Case 2}: the $m_n$ form an increasing sequence. Case 1 and Case 2 each imply that $z_{n+1} \notin B(z_n, m_n)$, and hence $B(z_n, m_n) \subset (z_{n+1}, \rightarrow)$ for all $n$. In Case 1, by the definition of $B_{z_n}$ infinitely many $z_n$ come from the same $Y_k$ for some $k \leq M$, which leads to a contradiction as before. In Case 2, fix a large enough $N \geq 3$ such that there is a $d \in (z_3, z_1) \cap D_N$. Then $B(z_N, m_N) \subset B(z_N, N) \subset (\rightarrow, d) \subset (\rightarrow, z_1)$ and $B(z_0, m_0) \subset (z_1, \rightarrow)$, a contradiction. \hfill $\square$

\textbf{Remark 1.5.} We do not know if condition $(a')$ in the above theorem is essential. As we will see, any subset of the Sorgenfrey line that is not $F_\sigma$ shows that condition $(b)$ cannot be dropped in the two theorems above. Although the two proofs are similar, we do not see how to get one theorem to imply both theorems above. Condition $(c)$, which has been extensively used (see [2], [3], [9], [11], [12], [17]), could not be a hypothesis of any theorem that would imply the first one, since $\omega_1 + 1$ with the reverse order satisfies the conditions of the first theorem, $(a)$ and $(b)$, but does not satisfy $(c)$. We are left with conditions $(a)$, $(a')$ and $(b)$, and we gave an example (in Remark 1.3) that $(a')$ and $(b)$ would not be enough. The Sorgenfrey rationals show that the above results would become weaker if we replace assumption $(b)$ with the stronger one that every closed set has a minimal element. Although the Michael line is base-cover hypocompact [22], no GO-space homeomorphic to it satisfies $(b)$.

\textbf{Problem 1.6.} Characterize the base-cover para(hypo,meta)-compact GO-spaces.

\textbf{Definition 1.7} (E. van Douwen and W. Pfefer [10]). If $\preceq$ is a \textit{reflexive} binary relation on a set $X$ and $F \subset X$, we call an $m \in F$ an $\preceq$-\textit{minimal} element of $F$ if $x = m$ for each $x \in F$ with $x \preceq m$. The space $X$ is called a GLS (Generalized Left Separated) space if in addition $X$ is a topological space and

\begin{enumerate}
\item every nonempty closed subset of $X$ has a $\preceq$-minimal element, and
\item the set \{\(y \in X : x \preceq y\)\} is open for each \(x \in X\).
\end{enumerate}

Then $\preceq$ is called a GLS-relation on the space $X$. Note that the topology of $X$ is given without reference to $\preceq$, which is only required to be reflexive. For example, the countable sequential fan is a GLS-space witnessed by any well-founded relation in which the non-isolated point precedes all other points.

\textbf{Theorem 1.8.} For a subspace $X$ of the Sorgenfrey line $S$, the following are equivalent:

\begin{enumerate}
\item $(a)$ $X$ is $F_\sigma$,
\item $(b)$ $X$ is base-cover hypocompact,
\item $(c)$ $X$ is base-cover paracompact,
\item $(d)$ $X$ is base-cover metacompact,
\item $(e)$ $X$ is a continuous image of $S$,
\item $(f)$ $X$ is a GLS space.
\end{enumerate}

\textbf{Proof.} $(a) \rightarrow (b)$ follows from each of Theorem 1.4 and 1.3. $(b) \rightarrow (c) \rightarrow (d)$ is trivial. $(a) \leftrightarrow (e)$ is a result of D. Burke and J.T. Moore [9]. E.K. van Douwen and W. Pfefer proved in [10] that every finite power of $S$ is a GLS-space, and that an $F_\sigma$ subspace of a GLS-space is a GLS-space itself; thus $(a) \rightarrow (f)$. Although $(f) \rightarrow (a)$ was not proved in [10], it was proved that the Sorgenfrey irrationals are not a GLS-space, and the proof of $(f) \rightarrow (a)$ is an easy combination of that proof and the proof of $(d) \rightarrow (a)$ that follows.

Fix a base $B$ for $X$. For each $x \in X$ fix a $B_x \in B$ with $x \in B_x \subset [x, \rightarrow)$, and fix $n_x < \omega$ with $[x, x + \frac{1}{n_x}] \cap X \subset B_x$. Let $X_n = \{x \in X : n_x = n\}$. Clearly
Lemma 2.2. For each \( k \) (here) any base of half-open intervals does witness its base-base paracompactness.

Remark 1.9. No base \( \mathcal{B} = \{[x_\alpha, r_\alpha) : \alpha < 2^\omega\} \) with \(|\{r_\alpha : \alpha < 2^\omega\} \geq \omega_1\) can witness base-cover metacompactness of the Sorgenfrey line \( S \), though (23) and Remark 1.2 here) any base of half-open intervals does witness its base-base paracompactness. There is a \( k < \omega \) such that the set \( \{r_\alpha : r_\alpha - x_\alpha > \frac{1}{k}\} \) is uncountable, and hence has a two-sided limit point \( p \in S \). Fix a sequence \( r_\alpha, i < \omega, \) increasing to \( p \), and such that \( r_\alpha - x_\alpha > \frac{1}{k} \) for each \( i \). The family \( \mathcal{C}' = \{[x_\alpha, r_\alpha) : i < \omega\} \) covers the interval \( [x - \frac{1}{k}, x) \). Let \( \mathcal{C}'' \) be a subfamily of \( \mathcal{B} \) with \( \bigcup \mathcal{C}'' = S \setminus [x - \frac{1}{k}, x) \). The family \( \mathcal{C} = \mathcal{C}' \cup \mathcal{C}'' \) is a subcover of \( \mathcal{B} \). The only subcover of \( \mathcal{C} \) must contain infinitely many members of \( \mathcal{C}' \) and therefore cannot be point finite at \( x - \frac{1}{k} \).

The above remark shows that we do need a special base to show that the Sorgenfrey line \( S \) is base-cover hypocompact. A special base for \( S \) was already used by de Caux [6] to show that every finite power of \( S \) is hereditarily a D-space, answering a question of van Douwen and Pfeffer from [10]. It was shown in [10] that each GLS-space (thus each finite power of \( S \)) is a D-space, and it was observed that \( S \) itself is hereditarily a D-space. A space \( X \) is a D-space if for every open neighborhood assignment \( \{U_x : x \in X\} \) there is a closed discrete subspace \( D \) of \( X \) such that \( \{U_x : x \in D\} \) covers \( X \) (see also [4], [9], [12]).

Remark 1.10. The Cantor middle-third set minus all points of the form \( \frac{m}{2^n} \), where \( m \) is odd and \( i < \omega \), is not \( F_\sigma \) in the real line but is closed in the Sorgenfrey line.

Remark 1.11. There is a perfectly normal Lindel"of LOT S that is not base-cover metacompact: Take a non-\( F_\sigma \) subspace \( X \) of the Sorgenfrey line and split each \( x \in X \) into \( x^- \) and \( x^+ \) with \( x^- < x^+ \).

2. Subspaces and Unions of Base-Cover Paracompact Spaces

The countable sequential fan \( S_\omega \) is the union of countably many sequences converging to the same point, say \( S_\omega = \{0\} \cup \{x^j_i : i, j < \omega\} \), where all \( x^j_i \) are isolated, and neighborhoods of \( 0 \) have the form \( \{0\} \cup (\bigcup_{i < \omega} T^i) \), where each \( T^i \) contains a tail of the sequence \( S^i = \{x^i_j : j < \omega\} \). We need several lemmas before we prove that \( S_\omega \) is not base-cover metacompact. Recall that \( \omega^+ \) is the set of all functions from \( \omega \) to \( \omega \) partially pre-ordered by: \( f \leq^* g \) if \( f(n) \leq g(n) \) for all but finitely many \( n \). A set \( D \subset \omega^+ \) is dominating if it is cofinal in \( (\omega^+, \leq^*) \), i.e., for each \( f \in \omega^+ \) there is a \( g \in D \) with \( f \leq^* g \).

Lemma 2.1. If \( D = \bigcup_{n < \omega} D_n \subset \omega^+ \) is dominating, then some \( D_n \) is dominating.

Proof. This is a well-known diagonalization argument. If for each \( n \) there is an \( f_n \in \omega^+ \) which witnesses that \( D_n \) is not dominating, then define \( f \in \omega^+ \) by \( f(k) = \max \{f_n(k) : n \leq k\} \) for each \( k \). Then \( f_n(k) \leq f(k) \) for all \( k \geq n \); hence \( f_n \leq^* f \) for each \( n \), and \( f \) witnesses that \( D \) is not dominating.

Lemma 2.2. If \( D \) is dominating, then there is a \( g \in \omega^+ \) such that every neighborhood of \( g \) (in the product topology of \( \omega^+ \)) meets \( D \) in a dominating family.
Proof. Find $n_0$ such that all $f \in D$ with $f(0) = n_0$ form a dominating family (use the preceding lemma); then find $n_1$ such that all $f \in D$ with $f(0) = n_0$ and $f(1) = n_1$ form a dominating family, and so on. Let $g(i) = n_i$ for each $i < \omega$. □

Lemma 2.3. If $D$ is dominating, then there is an infinite $F \subset D$ such that if $F'$ is a finite subfamily of $F$, then $\min F'(i) > \min F(i)$ for all but finitely many $i < \omega$ (where $\min F(i) = \min \{ f(i) : f \in F \}$).

Proof. Let $g$ be as in the preceding lemma and for each $n < \omega$ pick an $f_n \in D$ with $f_n(0, n] = g(0, n]$ and $f_n(i) > g(i)$ for all but finitely many $i$. If $F = \{ f_n : n < \omega \}$, then $\min F(i) = g(i)$ for all $i$. If $F'$ is a finite subfamily of $F$, then $\min F'(i) > g(i)$ for all but finitely many $i$. □

Theorem 2.4. (a) The countable sequential fan $S_\omega$ is not base-cover meta compact.

(b) Base-cover para(hypo,meta)-compactness is not $F_\sigma$ hereditary in general.

Proof. (a) Let $B$ be a base for $S_\omega$. The family $B_0 = \{ B \in B : 0 \in B \}$ is a base at 0. For each $B \in B_0$ and $i < \omega$, let $f_B(i) = \min \{ j : x_j^i \in B \}$. Since $B_0$ is a base at 0, the family $D = \{ f_B : B \in B_0 \}$ is dominating in $\omega^\omega$ (it is cofinal in $\omega^\omega$, $\leq$) too). There is a subfamily $C_0$ of $B_0$ such that the set $F = \{ f_B : B \in C_0 \}$ satisfies the conclusion of the preceding lemma. Let $f(i) = \min F(i)$ for each $i$. The family $C = C_0 \cup \{ \{ x_j^i \} : f(i) \neq j \}$ is a subcover of $B$, since if $f(i) = j$, then $f_B(i) = j$ for some $B \in C_0$ and $x_j^i \in B$. If $C'$ is a finite subfamily of $C$, then $C' \cap 0$ is finite and there is an $i$ such that $\min \{ f_B(i) : B \in C' \cap C_0 \} > f(i)$; hence $x_{f(i)}^i$ is not covered by $C' \cap C_0$. It follows that $x_{f(i)}^i$ is not covered by $C'$, since $C' \setminus C_0 \subset \{ x_j^i \} : f(i) \neq j \}$.

(b) The Stone-Čech compactification of $S_\omega$ is base-cover hypocompact. □

Theorem 2.5. (a) A space is base-cover para(meta)-compact if it has an open cover the closures of the elements of which are base-cover para(meta)-compact and form a locally finite (point finite) cover.

(b) Every (normal) para(meta)-compact, locally base-cover para(meta)-compact space is base-cover para(meta)-compact.

(c) Every open $F_\sigma$ subspace of a (normal) base-cover para(meta)-compact space is base-cover para(meta)-compact.

(d) Every open subspace of a perfectly normal, base-cover para(meta)-compact space is base-cover para(meta)-compact.

Proof. (a) Fix an open cover $U = \{ U_\alpha : \alpha < \kappa \}$ of a space $X$ such that $\{ \overline{U_\alpha} : \alpha < \kappa \}$ is locally finite and each $\overline{U_\alpha}$ is base-cover paracompact. Let $B_\alpha^+$ be an open (relative to $\overline{U_\alpha}$) base of $\overline{U_\alpha}$, every subcover of which has a locally finite subcover. Let $B_\alpha = \{ B \in B_\alpha^+ : B \subset U_\alpha \}$. The family $B = \bigcup_{\alpha < \kappa} B_\alpha$ is an open base for $X$.

Suppose $C$ is a subcover of $B$. For each $\alpha < \kappa$, let $C_\alpha = C \cap B_\alpha$. We describe how to define families $C^-_\alpha \subset C_\alpha$ and closed sets $H_\alpha$ such that for all $\alpha < \kappa$:

(i) $H_\alpha = (\bigcup C_\alpha) \setminus (\bigcup_{\beta < \alpha} (\bigcup C_\beta)) \cup (\bigcup_{\alpha < \beta < \kappa} (\bigcup C_\beta)) \subset \bigcup C_\alpha$, and

(ii) $C^-_\alpha$ is locally finite at each point of $\overline{U_\alpha}$.

Suppose $\alpha < \kappa$ and $C^-_\beta$ were defined for all $\beta < \alpha$. A standard argument (using for limit $\alpha$ that $U$ is point-finite) shows that $X \setminus (\bigcup_{\beta < \alpha} (\bigcup C^-_\beta)) \cup (\bigcup_{\alpha < \beta < \kappa} (\bigcup C_\beta)) \subset \bigcup C_\alpha$; hence $H_\alpha$ is closed. The set $V_\alpha = \overline{U_\alpha} \setminus H_\alpha$ is open in $\overline{U_\alpha}$. Fix $C(V_\alpha) \subset B_\alpha^+$ with $\bigcup C(V_\alpha) = V_\alpha$, and let $C_\alpha^+ = C_\alpha \cup C(V_\alpha)$. Since $C_\alpha^+$ is a subcover of $B_\alpha^+$,
there is a locally finite subcover $C'_\alpha$ of $C'_\alpha^+$. The above conditions are satisfied with $C^-_\alpha = C'_\alpha \setminus \mathcal{C}(V_\alpha)$. Now the family $C' = \bigcup_{\alpha < \kappa} C^-_\alpha$ is a locally finite subcover of $\mathcal{C}$.

The other statement in part (a) has a similar proof.

The proof of $(a) \to (b)$ is easy (use also that every point finite open cover of a normal space has a shrinking). To prove $(a) \to (c)$, suppose $Y$ is an open $F_\sigma$ subspace of a base-cover paracompact space $X$. Then $Y = \bigcup_{i \in \omega} F_i$, where $F_i$ is closed and $F_i \subset \text{int} \ F_{i+1}$ for each $i$. Let $U_i = (\text{int} \ F_i) \setminus F_{i-2}$ (where $F_{-2} = F_{-1} = \emptyset$). It is easy to see that base-cover para(hypo,meta)-compactness is hereditary with respect to closed sets, and that $\{U_i : i < \omega\}$ is a cover of $Y$ that satisfies the premise in $(a)$.

**Remark 2.6.** Unlike most covering properties, for base-cover para(hypo,meta)-compactness “open hereditary” is not equivalent to “hereditary” even in perfectly normal spaces: the Sorgenfrey line is an example, as we see from Theorem [18].

**Question 3.2.** Is every $F_\sigma$ subspace of a perfectly normal, base-cover paracompact space base-cover paracompact? What if, in addition, the space is Lindelöf?

3. **Products of base-cover paracompact and compact spaces**

**Example 3.1.** The product $S \times (\omega + 1)$ of the Sorgenfrey line $S$ and the converging sequence $\omega + 1$ is not base-cover paracompact (even though $S$ is).

**Proof.** Suppose $\mathcal{B}$ is a base for $S \times (\omega + 1)$. For each $n < \omega$ fix $x_n$ and $r_n$ in $S$ and $B_n \in \mathcal{B}$ such that $\{x_n, r_n\} \times \{n\} \subset B_n \subset S \times \{n\}$ and $x_n < x_{n+1} < r_{n+1} < r_n$. If $F = \{\langle x_n, n \rangle : n < \omega\}$, then $F$ is closed and covered by $C' = \{B_n : n < \omega\}$. There is a subfamily $C''$ of $\mathcal{B}$ with $\bigcup C'' = X \setminus F$. Then $C' \cup C''$ is a subcover of $\mathcal{B}$. For each $n$ the only element of $C' \cup C''$ that contains $\langle x_n, n \rangle$ is $B_n$, and hence every subcover of $C' \cup C''$ must contain $C' = \bigcup C''$ is not locally finite at $(\sup_{n < \omega} x_n, \omega)$.

In a similar way one can show that $M \times (\omega + 1)$ and $L(\omega_1) \times (\omega + 1)$ are not base-cover paracompact, where $M$ is the Michael line and $L(\omega_1)$ is the one-point Lindelöfication of a discrete space of cardinality $\omega_1$, although $M$ and $L(\omega_1)$ are base-cover hypocompact, and $M \times (\omega + 1)$ is base-cover metacompact. It follows that perfect preimages of base-cover para(hypo)compact spaces need not be base-cover paracompact (recall that a perfect map is a continuous, closed map such that the preimage of every point is compact). Base-cover hypocompactness is not preserved by perfect maps in the forward direction (see Remark 1.3 in [22]).

**Question 3.3.** Is base-cover para(meta)compactness preserved by perfect maps?

**Question 3.4.** If $X \times (\omega + 1)$ is base-cover para(hypo)compact, then $X$ has what property $\mathcal{P}$ (such that the Sorgenfrey line and the Michael line do not have $\mathcal{P}$)? Is $X$ a paracompact $p$-space (i.e., the perfect preimage of a metric space)?

Recall that $A(\kappa)$ denotes the one-point compactification of a discrete space of cardinality $\kappa$. We think of $A(\kappa)$ as $\kappa + 1 = \{\alpha : \alpha < \kappa\}$ as a set, with all ordinals $\alpha < \kappa$ isolated and neighborhoods of $\kappa$ of the form $\kappa + 1$ minus a finite subset of $\kappa$. As usual, $\kappa^+$ denotes the least cardinal greater than $\kappa$. 

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Theorem 3.5. Suppose $X$ is a regular paracompact space and $\chi(X) = \kappa \geq \omega$. The following conditions are equivalent.

(a) $X$ is locally compact.

(a') $X$ is locally countably compact.

(b) $X \times A(\kappa^+)$ is base-cover paracompact.

(c) $X \times K$ is base-cover paracompact for every compact space $K$.

If $X$ is first countable, then the following condition can be added to the list:

(d) $X$ does not contain a closed copy of $T$, where $T = (\bigcup_{n<\omega} T_n) \cup \{\infty\}$, with each $T_n$ a countably infinite set consisting of isolated points, and basic neighborhoods of $\infty$ of the form $\{\infty\} \cup (\bigcup_{n \geq i} T_n)$, $i < \omega$.

Proof. The equivalence $(a) \leftrightarrow (a')$ is well known for paracompact spaces. If $X$ is paracompact, locally compact and $K$ is compact, then $X \times K$ is paracompact, locally compact, hence base-cover paracompact by Theorem 2.4. Thus, $(a) \rightarrow (c)$.

$(c) \rightarrow (b)$ is trivial. We show $(b) \rightarrow (a')$. If $X$ is not locally countably compact at a point $p$, then fix a local base $\{U_\beta : \beta < \kappa\}$ at $p$. Let $B$ be any base for the product $Z = X \times A(\kappa^+)$, and for each $\alpha < \kappa^+$ pick a $B_\alpha \in B$ and $\beta_\alpha < \kappa$ such that $(p, \alpha) \in \overline{U}_{\beta_\alpha} \times \{\alpha\} \subset B_\alpha \subset X \times \{\alpha\}$. There is a set $A \subset \kappa^+$ of cardinality $\kappa^+$ such that the $\beta_\alpha$ are the same, say $\beta_\alpha = \gamma$, for all $\alpha \in A$, and there is a countably infinite set $\{\alpha_n : n < \omega\} \subset A$. There is a countably infinite closed discrete subspace $\{x_n : n < \omega\}$ of $\overline{U}_\gamma$. The set $F = \{(x_n, \alpha_n) : n < \omega\}$ is closed in $Z$. If $C' = \{B_\alpha : n < \omega\}$ and $C''$ is a subfamily of $B$ with $\bigcup C'' = Z \setminus F$, then $\bigcup C' \cup \bigcup \{(\overline{U}_\gamma \times \{\alpha_n\}) : n < \omega\} \supset F$, and hence $C' \cup C''$ is a subcover of $B$. Any subcover of $C' \cup C''$ must contain $C'$, which is not locally finite at $(x, \kappa^+)$ for any $x \in \overline{U}_\gamma$.

It was observed in [15] that $(a') \leftrightarrow (d)$ for first countable spaces. \hfill \Box

Remark 3.6. The Sorgenfrey line and the Michael line are base-cover paracompact spaces that are not perfect preimages of a metric space. If $\mathbb{Q}$ denotes the rationals with the usual topology, then $\mathbb{Q} \times A(\omega_1)$ is a paracompact $p$-space that is not base-cover paracompact (since $\mathbb{Q}$ is not locally compact).

4. Some related classes of spaces

Definition 4.1. A space $X$ is totally paracompact [13] (totally metacompact) if every open base for $X$ contains a locally finite (point finite) subcover. A space $X$ is base-base paracompact [23] (base-base hypocompact, base-base metacompact) if $X$ has an open base $B$ such that every base $B' \subset B$ contains a locally finite (star finite, point finite) subcover. A space $X$ is base paracompact [23], [24] if it has an open base $B$ of cardinality equal to the weight of the space (i.e., $|B| = \chi(X)$) such that every open cover $C$ of $X$ has a locally finite open refinement $C' \subset B$.

For totally paracompact metric spaces, large and small inductive dimension coincide [13]. The Sorgenfrey line $S$ and the Michael line $M$ are not totally paracompact, or totally metacompact (see [25] (for the Michael line), [18], [19], [2]). The irrationals $\mathbb{P}$ with the usual topology are not totally para(meta)compact [7], [1] (Konstantinov); see also [16]. Thus total paracompactness is very restrictive, which led John Porter to define and study base-base paracompactness and base paracompactness [23], [24]. His proof that metrizable spaces are base-base paracompact worked for what we called base-cover paracompact spaces [21], [22]. Total paracompactness
and base-cover paracompactness each imply base-base paracompactness, which implies base-paracompactness \([23]\), which implies paracompactness. \(S, M\) and \(P\) are base-cover hypocompact (Theorem 1.8 here, and [22]): the countable sequential fan is totally paracompact. Base-base paracompact spaces are \(D\)-spaces, and Lindelöf spaces are base paracompact, but it is unknown if paracompactness implies base paracompactness, or if the latter implies base-base paracompactness [23], [24]. This is related to the following question of Eric van Douwen, versions of which have been discussed in [4], [6], [9], [10], [12], [26]: Is every Lindelöf space a \(D\)-space? The next question is implicit in [23] and should be compared with Theorem 1.8 here.

**Question 4.2.** Is every subspace of the Sorgenfrey line base-base paracompact?

Consistent examples of base-base paracompact subspaces of the Sorgenfrey line that are not \(F_\sigma\) are any Lusin subspace and, under MA, any uncountable subspace of cardinality less than the continuum; these spaces are Hurewicz (see [14] (for the reals), [18], [2], [20]), and thus totally paracompact [5]. Other questions implicit in [23] include: Is base-base paracompactness hereditary with respect to closed sets? If \(X\) is base-base paracompact and \(Y\) is compact, is \(X \times Y\) base-base paracompact? Is \(X\) base-base paracompact if it is paracompact and locally base-base paracompact?

I would like to thank my advisor at Auburn University, Gary Gruenhage, who was always ready to help during the preparation of my dissertation and this paper.

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