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GORENSTEIN INJECTIVE MODULES AND LOCAL COHOMOLOGY

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ABSTRACT. In this paper we assume that R is a Gorenstein Noetherian ring. We show that if $(R, \mathfrak{m})$ is also a local ring with Krull dimension d that is less than or equal to 2, then for any nonzero ideal $\mathfrak{a}$ of R, $H^d_{\mathfrak{a}}(R)$ is Gorenstein injective. We establish a relation between Gorenstein injective modules and local cohomology. In fact, we will show that if R is a Gorenstein ring, then for any R-module M its local cohomology modules can be calculated by means of a resolution of M by Gorenstein injective modules. Also we prove that if R is d-Gorenstein, M is a Gorenstein injective and $\mathfrak{a}$ is a nonzero ideal of R, then $\Gamma_{\mathfrak{a}}(M)$ is Gorenstein injective.

1. INTRODUCTION

Local cohomology is an important tool in algebraic geometry. A. Grothendieck defined and initiated the study of local cohomology, and in 1967, R. Hartshorne published "Local cohomology" (LNM 41), based on Grothendieck's 1961 lectures at Harvard. Since then local cohomology has been an effective tool in algebraic geometry and in commutative algebra. Suppose that M is an R-module and that $\mathfrak{a}$ is an ideal of R. Following [1], we define the i-th *local cohomology* of M with respect to $\mathfrak{a}$ by $H^i_{\mathfrak{a}}(M) = \varinjlim_n \mathrm{Ext}^i_R(\frac{R}{\mathfrak{a}^n}, M)$. Let

$$\mathcal{I}^\circ : 0 \xrightarrow{d^{-1}} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \to I^i \xrightarrow{d^i} I^{i+1} \to \dots$$

be an injective resolution of M, so that there is an R-homomorphism $\alpha : M \to I^0$ such that the sequence

$$0 \to M \xrightarrow{\alpha} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \to I^i \xrightarrow{d^i} I^{i+1} \to \dots$$

is exact. Again following [1], apply the functor $\Gamma_{\mathfrak{a}}(-)$ to the complex $\mathcal{I}^\circ$ to obtain

$$\Gamma_{\mathfrak{a}}(\mathcal{I}^0) : 0 \xrightarrow{\Gamma_{\mathfrak{a}}(d^{-1})} \Gamma_a(I^0) \xrightarrow{\Gamma_a(d^0)} \Gamma_a(I^1) \xrightarrow{\Gamma_{\mathfrak{a}}(d^1)} \cdots \to \Gamma_{\mathfrak{a}}(I^i) \xrightarrow{\Gamma_{\mathfrak{a}}(d^i)} \Gamma_{\mathfrak{a}}(I^{i+1}) \to \dots.$$

Then the i-th cohomology module of this complex is the i-th local cohomology of M with respect to $\mathfrak{a}$.

In 1966/67, M. Auslander defined the Gorenstein dimension of a module. By this definition the modules having Gorenstein dimension 0 are the Gorenstein projective

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modules. Auslander shows that over a commutative Gorenstein local ring R, a finitely generated module M is *Gorenstein projective* if and only if there is an exact sequence

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \ldots$$

of finitely generated projective modules such that $M = \mathrm{Ker}(P^0 \to P^1)$ and if $\mathrm{Hom}_R(-, P)$ is applied to the sequence above in which P is a projective module, we still have an exact sequence.

Finally, in 1995 E. Enochs and O. M. G. Jenda in [6] defined Gorenstein injective modules. They defined an R-module N to be *Gorenstein injective* if and only if there is an exact sequence

$$\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \ldots$$

of injective modules such that $N = \mathrm{Ker}(E^0 \to E^1)$ and such that for any injective R-module E, $\mathrm{Hom}_R(E, -)$ leaves the above complex exact. It should be noted that if N is Gorenstein injective, then the complex

$$0 \to N \to E^0 \to E^1 \to \ldots$$

is an injective resolution of N.

All rings considered in this paper will be nontrivial commutative Gorenstein rings unless an additional condition is considered. Our main aim in this paper is to establish some links between the two approaches mentioned above. We strongly believe that over Gorenstein rings, Gorenstein injective modules behave similarly to injective modules in many aspects. Since the class of Gorenstein injective modules is greater than the class of injective modules, in some situations, it may be convenient to use Gorenstein injective modules rather than injective modules. To clarify this point of view, we focus on local cohomology. In section 2, we will see that if R has finite Krull dimension d that is less than or equal to 2, then $H^d_{\mathfrak{a}}(R)$ is Gorenstein injective for any nonzero ideal $\mathfrak{a}$ of R. We also prove that if $(R, \mathfrak{m})$ is a local ring and $\dim R = d$, then for any Gorenstein projective R-module M, $H^d_{\mathfrak{m}}(M)$ is Gorenstein injective. In section 3, we prove that for any Gorenstein injective R-module M, $H^i_{\mathfrak{a}}(M) = 0$ for all ideals $\mathfrak{a}$ and all $i > 0$. Hence the local cohomology module of any R-module can be computed by means of a Gorenstein injective resolution of it. Moreover, we show that if R is d-Gorenstein, then $\Gamma_{\mathfrak{a}}(M)$ is Gorenstein injective. Finally, we find another lower bound for the vanishing of local cohomology.

2. Gorenstein injective modules and local cohomology

Throughout this section R is a commutative Noetherian Gorenstein ring, unless otherwise stated. Whenever R has finite Krull dimension, we will denote the dimension $\dim R$ by d.

Lemma 2.1. *Let R be Gorenstein, and let M be a Gorenstein injective R-module. Then there is an exact sequence $0 \to K \to E \to M \to 0$ such that E is injective and K is Gorenstein injective.*

Proof. See [11, Lemma 5.4.3]. □

The following lemma and proposition show that there exists a relation between injective modules and Gorenstein injective modules.

Lemma 2.2. *Let $(R, \mathfrak{m})$ be a regular local ring, and let M be a Gorenstein injective R-module. Then M is injective.*

Proof. Since M is Gorenstein injective, $\mathrm{Ext}^1_R(X, M) = 0$ for all R-modules X with finite injective dimension by [7, Proposition 11.2.5]. On the other hand, since R is a regular local ring, it has finite global dimension. Hence any R-module has finite injective dimension. So M is injective. □

We recall a definition from [4]:

Definition. An R-module N is *h-divisible* if N is a homomorphic image of some injective R-module.

Proposition 2.3. *Let M be a Gorenstein injective R-module. Then:*
(i) *M has a secondary representation.*
(ii) *If M is finitely generated, then M is Artinian.*

Proof. (i) Since M is Gorenstein injective, there is an exact sequence $0 \to K \to E \to M \to 0$ such that E is injective and K is a Gorenstein injective R-module by Lemma 2.1. The R-module E has a secondary representation, and $\mathrm{Att}(E)$ is a finite set by [1, 7.2.10 Exercise] and also M is h-divisible. So M has a secondary representation, and $\mathrm{Att}(M) \subseteq \mathrm{Att}(E)$ by [8, 4.1].

(ii) Since M is Gorenstein injective, there is an exact sequence

$$0 \to K \to E \to M \to 0$$

in which E is injective and K is Gorenstein injective by Lemma 2.1. By [1, 10.1.14, Reminder],

$$E = \bigoplus_{\substack{\mathfrak{p}_\alpha \in \operatorname{spec} R_0 \\ \alpha \in \Lambda}} E(\frac{R}{\mathfrak{p}_\alpha})^{t_{\mathfrak{p}_\alpha}},$$

where $t_{\mathfrak{p}_\alpha}$ is the number of copies of $E(\frac{R}{\mathfrak{p}_\alpha})$. By [5, Lemma 3.1] and [5, Proposition 3.2], we have

$$\mathrm{Hom}_R(E, M) \cong \bigoplus_{\substack{\mathfrak{m}_\alpha \in \max R \\ \alpha \in \Lambda}} \left(\mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}_\alpha}), M)\right)^{t_{\mathfrak{m}_\alpha}}.$$

In this decomposition a finite number of maximal ideals appear. Hence we have the short exact sequence

$$0 \to K \to \bigoplus_{\substack{\mathfrak{m}_\alpha \in \max R \\ \alpha \in \Lambda}} E\left(\frac{R}{\mathfrak{m}_\alpha}\right)^{t_{\mathfrak{m}_\alpha}} \to M \to 0.$$

It is known that any injective module $E(\frac{R}{\mathfrak{m}_\alpha})$ is Artinian, and so is M. □

Remark. Suppose that $(R, \mathfrak{m})$ is a local ring. It is known that for a nonzero Artinian R-module A, $\mathrm{Supp}(A)$ consists only of the maximal ideal. Put $\widehat{R} = \varprojlim_i \frac{R}{\mathfrak{m}^i}$. By the work of Sharp in [10], it follows that A has a natural structure as a module over $\widehat{R}$. It should be noted that $\widehat{R}$ is a complete local ring. Moreover, let $\psi : R \longrightarrow \widehat{R}$ denote the natural homomorphism. For any element $r \in R$ the multiplication by R on A has the same effect as the multiplication by $\psi(r) \in \widehat{R}$ on A as an $\widehat{R}$-module. Furthermore, a subset of A is an R-module if and only if it is an $\widehat{R}$-module.

Lemma 2.4. *Let $(R, \mathfrak{m})$ be a local ring, and let A be an R-module. If A is an h-divisible $\widehat{R}$-module, then A is an h-divisible R-module.*

Proof. By considering the definition and the fact that any injective $\widehat{R}$-module is an injective R-module, the proof is clear. $\square$

For an Artinian R-module A we put $\langle\mathfrak{m}\rangle A = \bigcap_{n\in N}\mathfrak{m}^n A$. Now we have the following proposition:

Proposition 2.5. *Let $(R,\mathfrak{m})$ be a local ring, and let $\mathfrak{a}$ be an ideal of R. Then $\sum_{n\in N}\langle\mathfrak{m}\rangle(0:_{E(\frac{R}{\mathfrak{m}})}\mathfrak{a}^n)$ is h-divisible.*

Proof. It will be convenient for us to write $E = E(\frac{R}{\mathfrak{m}})$. Since any Artinian R-module is an Artinian $\widehat{R}$-module in a natural way, E is an Artinian $\widehat{R}$-module. Now by considering Lemma 2.4, in order to prove the claim, it is enough to show that $\sum_{n\in N}\langle\mathfrak{m}\rangle(0:_E\mathfrak{a}^n)$ is an h-divisible $\widehat{R}$-module. So, we reduce the situation to the case that R is complete. Let $W = \{\mathfrak{p}\in \mathrm{Att}_R(E) | \dim(\frac{R}{\mathfrak{a}+\mathfrak{p}}) > 0\}$ and put $S = \bigcap_{\mathfrak{p}\in W}(R\setminus\mathfrak{p})$. Then there exists a natural homomorphism $\mathrm{Hom}_R(R_S,E)\longrightarrow E$, $f\longrightarrow f(1)$. The image of this homomorphism coincides with the S-component of E (i.e., $S(E) = \bigcap_{u\in S}(uE) = \sum_{n\in N}\langle\mathfrak{m}\rangle(0:_E\mathfrak{a}^n)$) by [9, Corollary 3.3] and [3, Remark 2.9]. Now we have the short exact sequence

$$\mathrm{Hom}_R(R_S,E)\longrightarrow \sum_{n\in N}\langle\mathfrak{m}\rangle(0:_E\mathfrak{a}^n)\longrightarrow 0.$$

Since R_S is a flat R-module, $\mathrm{Hom}_R(R_S,E)$ is an injective R-module. So, the module $\sum_{n\in N}\langle\mathfrak{m}\rangle(0:_E\mathfrak{a}^n)$ is h-divisible. $\square$

The above proposition leads us to the following theorem:

Theorem 2.6. *Let $(R,\mathfrak{m})$ be a Gorenstein local ring with Krull dimension d. If $d\leq 2$ and $\mathfrak{a}$ is a nonzero ideal of R, then $H^d_{\mathfrak{a}}(R)$ is Gorenstein injective.*

Proof. If $d=0$, the ring R is injective and the claim is obvious. For $d=1,2$ the proofs are similar; so let $d=2$. By [3, Theorem 3.2], there is an exact sequence

$$0\longrightarrow K=\sum_{n\in N}\langle\mathfrak{m}\rangle(0:_{E(\frac{R}{\mathfrak{m}})}\mathfrak{a}^n)\longrightarrow E(\frac{R}{\mathfrak{m}})\longrightarrow H^d_{\mathfrak{a}}(R)\longrightarrow 0$$

where K is h-divisible by Proposition 2.5. Therefore there exists a short exact sequence $0\longrightarrow K_1\longrightarrow E\longrightarrow K\longrightarrow 0$ such that E is injective. So, we have the exact sequence $0\longrightarrow K_1\longrightarrow E\longrightarrow E(\frac{R}{m})\longrightarrow H^d_{\mathfrak{a}}(R)\longrightarrow 0$. This sequence is a part of an injective resolution for K_1, and $H^d_{\mathfrak{a}}(R)$ is the 2-th cosyzygy of K_1. Thus by [7, Theorem 10.1.13], $H^d_{\mathfrak{a}}(R)$ is Gorenstein injective. $\square$

Theorem 2.7. *Let $(R,\mathfrak{m})$ be a Gorenstein local ring, and let M be a finitely generated Gorenstein projective R-module. Then $H^d_{\mathfrak{m}}(M)$ is Gorenstein injective.*

Proof. Since M is Gorenstein projective, by [7, Corollary 11.5.4], M is a maximal Cohen-Macaulay R-module; therefore, $\dim M = \dim R = d$. Since R is Gorenstein, the canonical module of R is R. So by [2, Theorem 3.3.10], $\mathrm{Hom}_R(M,R)$ is finitely generated maximal Cohen-Macaulay. By [7, Corollary 11.5.4], $\mathrm{Hom}_R(M,R)$ is Gorenstein projective, and by the local duality theorem we have

$$H^d_{\mathfrak{m}}(M)\cong \mathrm{Hom}_R(\mathrm{Hom}_R(M,R),E).$$

So by [7, Corollary 11.6.2], $H^d_{\mathfrak{m}}(M)$ is Gorenstein injective. $\square$

3. Local cohomology and Gorenstein injective resolution

In this section R is a Gorenstein ring. We show that for any R-module M, its local cohomology can be calculated by means of a Gorenstein injective resolution.

Theorem 3.1. *Let M be a Gorenstein injective R-module, and let $\mathfrak{a}$ be an ideal of R. Then $H^i_{\mathfrak{a}}(M) = 0$ for all $i > 0$.*

Proof. By Lemma 2.1 there exist the exact sequences

$$0 \to K_1 \to E_0 \to M \to 0 \quad \text{and}$$
$$0 \to K_{i+1} \to E_i \to K_i \to 0 \quad \text{for all } i$$

such that the E_i are injective and the K_i are Gorenstein injective for all i. Suppose that $\mathfrak{a}$ can be generated by t elements. By [1, 3.3.1 Theorem], $H^i_{\mathfrak{a}}(M) = 0$ for all $i > t$. First we prove that $H^t_{\mathfrak{a}}(M) = 0$. Application of the functor $H^i_{\mathfrak{a}}(-)$ yields the exact sequence

$$(\dagger) \qquad H^i_{\mathfrak{a}}(E_0) \to H^i_{\mathfrak{a}}(M) \to H^{i+1}_{\mathfrak{a}}(K_1).$$

If we consider $i = t$, then $H^t_{\mathfrak{a}}(E_0) = H^{t+1}_{\mathfrak{a}}(K_1) = 0$ by [1, 3.3.1 Theorem] and injectivity of E. So, $H^t_{\mathfrak{a}}(M) = 0$. In case $t = 1$, the proof is complete; hence we consider $t \geq 2$. Now we show that $H^{t-1}_{\mathfrak{a}}(M) = 0$. Put $i = t-1$. Replace $t-1$ by i in (†); then we have the exact sequence

$$H^{t-1}_{\mathfrak{a}}(E_0) \to H^{t-1}_{\mathfrak{a}}(M) \to H^t_{\mathfrak{a}}(K_1).$$

Replacing K_1 by M and repeating the same proof for $0 \to K_2 \to E_1 \to K_1 \to 0$, we obtain $H^t_{\mathfrak{a}}(K_1) = 0$. Since $t \geq 2$, by injectivity of E we have $H^{t-1}_{\mathfrak{a}}(E) = 0$; hence $H^{t-1}_{\mathfrak{a}}(M) = 0$. Now, by repeating this argument for $0 \to K_2 \to E_1 \to K_1 \to 0$ and $0 \to K_3 \to E_2 \to K_2 \to 0$, we get $H^{t-1}_{\mathfrak{a}}(K_1) = 0$. If $t = 2$, the proof is complete. So we suppose that $t > 2$, put $i = t-2$ in (†), and get $H^{t-2}_{\mathfrak{a}}(M) = 0$. By repeating this argument for $t > 3$ and $i = t-3, \ldots, 1$ and replacing $K_1, K_2, \ldots$ by M, we conclude that $H^i_{\mathfrak{a}}(M) = 0$ for all $i > 0$. □

We say that R is d-Gorenstein if R is a Gorenstein ring with Krull dimension d.

Theorem 3.2. *Let R be d-Gorenstein, let M be a Gorenstein injective R-module, and let $\mathfrak{a}$ be a nonzero ideal of R. Then $\Gamma_{\mathfrak{a}}(M)$ is Gorenstein injective.*

Proof. Since M is Gorenstein injective, by Lemma 2.1 we can construct an exact sequence

$$\cdots \to E_{i+1} \to E_i \to \cdots \to E_1 \to E_0 \to M \to 0$$

with all the E_i's injective and with each $K_i = \operatorname{Coker}(E_{i+1} \to E_i)$, $i \geq 1$, Gorenstein injective. Consider the short exact sequences $0 \to K_1 \to E_0 \to M \to 0$ and $0 \to K_{i+1} \to E_i \to K_i \to 0$ for all i.

Application of the functor $\Gamma_{\mathfrak{a}}(-)$ and Theorem 3.1 gives us the short exact sequences $0 \longrightarrow \Gamma_{\mathfrak{a}}(K_1) \to \Gamma_{\mathfrak{a}}(E_0) \to \Gamma_{\mathfrak{a}}(M) \to 0$ and $0 \to \Gamma_{\mathfrak{a}}(K_{i+1}) \to \Gamma_{\mathfrak{a}}(E_i) \to \Gamma_{\mathfrak{a}}(K_i) \to 0$ for all i.

By [1, 2.1.4 Proposition], $\Gamma_{\mathfrak{a}}(E_i)$ is injective. Then we have the following exact sequence for $\Gamma_{\mathfrak{a}}(M)$:

$$\cdots \to \Gamma_{\mathfrak{a}}(E_{i+1}) \to \Gamma_{\mathfrak{a}}(E_i) \to \cdots \to \Gamma_{\mathfrak{a}}(E_1) \to \Gamma_{\mathfrak{a}}(E_0) \to \Gamma_{\mathfrak{a}}(M) \to 0.$$

By [7, Theorem 10.1.13], $\Gamma_{\mathfrak{a}}(K_i)$ is Gorenstein injective for $i \geq d-1$. By [7, Theorem 10.1.4], and considering $0 \to \Gamma_{\mathfrak{a}}(K_{d-1}) \to \Gamma_{\mathfrak{a}}(E_{d-2}) \to \Gamma_{\mathfrak{a}}(K_{d-2}) \to 0$, we conclude that $\Gamma_{\mathfrak{a}}(K_{d-2})$ is Gorenstein injective. By repeating this argument for the other short exact sequences we conclude that $\Gamma_{\mathfrak{a}}(M)$ is Gorenstein injective. □

Let $\mathfrak{a}$ be a nonzero ideal of R and M be any R-module. Following [1, 2.2.1 Definitions], we let $D_{\mathfrak{a}}(M)$ denote the $\mathfrak{a}$-transform of M. This is sometimes defined as

$$D_{\mathfrak{a}}(M) = \varinjlim_{n} \mathrm{Hom}_R(\mathfrak{a}^n, M).$$

We have the following corollary:

Corollary 3.3. *Let R be d-Gorenstein. Let M be a Gorenstein injective R-module, and let $\mathfrak{a}$ be a nonzero ideal of R. Then $D_{\mathfrak{a}}(M)$ is Gorenstein injective.*

Proof. By [1, 2.2.4 Theorem] we have the exact sequence

$$0 \to \Gamma_{\mathfrak{a}}(M) \to M \to D_{\mathfrak{a}}(M) \to H^1_{\mathfrak{a}}(M) \to 0.$$

Since M is Gorenstein injective, $H^1_{\mathfrak{a}}(M) = 0$ by Theorem 3.1. Now we have the short exact sequence

$$0 \to \Gamma_{\mathfrak{a}}(M) \to M \to D_{\mathfrak{a}}(M) \to 0.$$

By [7, Theorem 10.1.4] and Theorem 3.2 we conclude that $D_{\mathfrak{a}}(M)$ is Gorenstein injective. □

Definition (See [1, 4.1.1 Definition])**.** We say that an R-module M is $\Gamma_{\mathfrak{a}}$-*acyclic* precisely when $H^i_{\mathfrak{a}}(M) = 0$ for all $i > 0$. Also, a complex

$$\mathcal{A}^\circ : 0 \overset{d^{-1}}{\to} A^0 \overset{d^0}{\to} A^1 \overset{d^1}{\to} \cdots \to A^i \overset{d^i}{\to} A^{i+1} \to \ldots$$

is said to be a $\Gamma_{\mathfrak{a}}$-*acyclic resolution* of M if $A^0, A^1, \ldots, A^i, \ldots$ are $\Gamma_{\mathfrak{a}}$-acyclic and there is an R-homomorphism $\alpha : M \to A^0$ such that the sequence

$$0 \to M \overset{\alpha}{\to} A^0 \overset{d^0}{\to} A^1 \overset{d^1}{\to} \cdots \to A^i \overset{d^i}{\to} A^{i+1} \to \ldots$$

is exact.

From ([1], 4.1.2 Exercise) we have that if

$$\mathcal{A}^\circ : 0 \overset{d^{-1}}{\to} A^0 \overset{d^0}{\to} A^1 \overset{d^1}{\to} \cdots \to A^i \overset{d^i}{\to} A^{i+1} \to \ldots$$

is a $\Gamma_{\mathfrak{a}}$-acyclic resolution for an R-module M and apply $\Gamma_{\mathfrak{a}}(-)$ on this resolution, then the i-th cohomology module of the complex $\Gamma_{\mathfrak{a}}(\mathcal{A}^\circ)$ is the i-th local cohomology module M with respect to $\mathfrak{a}$.

Remark. By Theorem 3.1, we conclude that every Gorenstein injective R-module is $\Gamma_{\mathfrak{a}}$-acyclic. Hence, for a given R-module M, its local cohomology modules with respect to an ideal $\mathfrak{a}$ of R can be calculated by means of a resolution of M by Gorenstein injective modules as follows. Let

$$0 \overset{d^{-1}}{\to} G^0 \overset{d^0}{\to} G^1 \overset{d^1}{\to} \ldots \to G^{i-1} \overset{d^{i-1}}{\to} G^i \overset{d^i}{\to} G^{i+1} \to \ldots$$

be a Gorenstein injective resolution of M so that $G^0, G^1, \ldots, G^i, \ldots$ are all Gorenstein injective R-modules and there is an $\alpha : M \longrightarrow G^0$ such that the sequence

$$0 \to M \overset{\alpha}{\to} G^0 \overset{d^0}{\to} G^1 \overset{d^1}{\to} \ldots \to G^{i-1} \overset{d^{i-1}}{\to} G^i \overset{d^i}{\to} G^{i+1} \to \ldots$$

is exact. Then

$$H^i_{\mathfrak{a}}(M) \simeq \frac{\mathrm{Ker}(\Gamma_{\mathfrak{a}}(d^i))}{\mathrm{Im}(\Gamma_{\mathfrak{a}}(d^{i-1})}.$$

Definition. Let n be a natural number. We say that an R-module M has *Gorenstein injective dimension* less than or equal to n if M has the Gorenstein injective resolution

$$0 \to M \to G^0 \to G^1 \to G^2 \cdots \to G^{m-1} \to G^m \to 0$$

such that $m \leq n$. We denote the Gorenstein injective dimension of M by Gorinj-dim M.

Remark. One can easily show that Gorinj-dim $M \leq \mathrm{injdim} M$. Then we can show that there is a lower bound for vanishing of the local cohomology. We have the following corollary:

Corollary 3.4. *Let M be an R-module, and let $\mathfrak{a}$ be a nonzero ideal of R. Then $H^i_{\mathfrak{a}}(M) = 0$ for all $i >$* Gorinj-dim *M.*

Proof. In view of the above remarks the proof is clear. □

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