

ON EXTENSION OF ISOMETRIES BETWEEN UNIT SPHERES OF AL_p -SPACES ($0 < p < \infty$)

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ABSTRACT. In this paper, we study the extension of isometries between unit spheres of atomic AL_p -spaces ($0 < p < \infty$, $p \neq 2$). We find a condition under which an isometry T between unit spheres can be linearly isometrically extended. Moreover, we prove that every onto isometry between unit spheres of atomic AL_p -spaces ($0 < p < \infty$, $p \neq 2$) can be linearly isometrically extended to the whole space.

1. INTRODUCTION

Throughout this paper, we denote by \mathbb{N} and \mathbb{R} the sets of positive integers and of reals. We use the standard notation of vector lattices, and for all other notation we follow the definitions of the papers [1], [2], and [12]. This paper mainly discusses the atomic AL_p -spaces on \mathbb{R} with positive values of p that is not equal to two; therefore, E and F are used to denote these spaces unless otherwise stated.

The unit spheres of atomic AL_p -spaces are frequently discussed; we use $S(E)$ and $S(F)$ to denote the unit spheres of E and F . Moreover, T denotes an isometry from $S(E)$ to $S(F)$.

A mapping $T: E \rightarrow F$ is said to be an isometry if

$$(1.1) \quad \|Tx - Ty\| = \|x - y\| \quad (\forall x, y \in E).$$

For $1 \leq p < +\infty$, an AL_p -space E is a Banach lattice in which

$$\|x + y\|^p = \|x\|^p + \|y\|^p \quad (\forall x, y \in E \text{ with } |x| \wedge |y| = \theta).$$

For the cases that $0 < p < 1$, an AL_p -space E means a p -homogeneous F -lattice, and the last equality turns into

$$\|x + y\| = \|x\| + \|y\| \quad (\forall x, y \in E \text{ with } |x| \wedge |y| = \theta).$$

A complete disjoint system $\{e_\lambda: \lambda \in \Lambda\}$ of E is a pairwise disjoint collection of elements of E_+ , the positive cone of E . That is, a complete disjoint system satisfies

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the following two conditions:

- (1) for each pair $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$, $e_\lambda \wedge e_\mu = \theta$, and
- (2) for each $u \in E$ with $u \wedge e_\lambda = \theta$ for every $\lambda \in \Lambda$, the equality $u = \theta$ holds.

An atom e of a vector lattice E is a nonzero element of E with the property that the conditions $\theta \leq y, z \leq |e|$ and $y \wedge z = \theta$ imply $y = \theta$ or $z = \theta$.

If an element e is an atom of an Archimedean vector lattice E , then the principal ideal E_e generated by e is a projection band B_e (see [1, Theorem 2.16]).

An atomic vector lattice is a vector lattice E having a complete disjoint system consisting of atoms of E . In this case, we have $x = \sup_{\lambda \in \Lambda} \{P_{e_\lambda}(x)\}$, where P_{e_λ} is a band projection with respect to B_{e_λ} (see [1, Theorem 2.15]). An atomic AL_p -space is in fact linearly isometric to $l_p(\Gamma)$, where Γ may be any complete disjoint system of atoms of norm one.

In the following sections, we use $\{e_\gamma\}_{\gamma \in \Gamma}$ to denote a norm-one complete disjoint system of E , B_{Te_γ} to denote a principal projection band generated by Te_γ , and P_γ to denote a principal band projection from E onto B_{Te_γ} ($\gamma \in \Gamma$).

P. Mankiewicz ([9]) proved in 1972 that an isometry mapping an open connected subset of a normed space E onto an open subset of another normed space F can be extended to an affine isometry from E onto F . In 1987, D. Tingley in [13] raised the following problem:

Let $S(E)$ be the unit sphere of a Banach space E , and let $S(F)$ be the unit sphere of another Banach space F . Let $T: S(E) \rightarrow S(F)$ be isometric. Does there exist any isometry $\tilde{T}: E \rightarrow F$ such that $\tilde{T}|_{S(E)} = T$?

In the same paper, Tingley proved that if E and F are finite-dimensional Banach spaces and $T: S(E) \rightarrow S(F)$ is a *surjective* isometry, then $T(-x) = -T(x)$ for all $x \in S(E)$. Ma Yumei proved the same result for the infinite-dimensional space l_1 in [10]. During the past decade, Professor Ding Guanggui ([3]-[5]) and a group of his students including Wang Risheng ([14]-[20]), Xiao Yuhui ([21, 22]), and Zhan Dapeng ([23, 24]) had been working on this topic and had obtained many significant results. A number of these works only considered the *surjective* mappings between two spaces of the same type. However, Professor Ding Guanggui had discussed the extension of isometries between unit spheres of spaces of different type in [3]. Moreover, he also first discussed the mapping extension problem on Hilbert spaces without assuming the surjectivity of T in [4], where he showed that a 1-Lipschitz mapping between the unit spheres of two Hilbert spaces can be extended to a real linear isometry of the whole space.

For real AL_p -spaces E and F with $p > 1$, Xiao Yuhui proved in [21, Theorem 1] that every surjective isometry between unit spheres $S(E)$ of E and $S(F)$ of F can be linearly isometrically extended to the whole space. For complex AL_p -spaces E and F with $p > 1$, he proved in [21, Theorem 2] that for every surjective isometry between unit spheres $S(E)$ and $S(F)$ there exist an additive isometry \tilde{T} from E into F and two mutually complementary subspaces X and Y of E such that the restriction of \tilde{T} to $S(E)$ is T , that to X is linear and that to Y is conjugate linear.

However, a crucial result, Corollary 3, which stated that T maps every atom in $S(E)$ to an atom in $S(F)$, was not proved in the paper [21]. For the cases $p > 1$, Professor Ding Guanggui proved the assertion in [5], and gave the representation of a surjective isometry $T: S(E) \rightarrow S(F)$ by a method that is different from what Xiao Yuhui has in [21]. In our study in this paper, we shall drop the *surjectivity*

assumption for the mappings, and mainly study the case that $T: S(E) \rightarrow S(F)$ is an *injective* isometry. We find a condition under which an isometry $T: S(E) \rightarrow S(F)$ can be extended to a linear isometry $\tilde{T}: E \rightarrow F$. Therefore, we generalize the results of Xiao Yuhui in [21].

2. SOME IMPORTANT LEMMAS

Lemma 2.1 ([11]). *Let ξ and η be two real numbers. Then:*

- (i) $|\xi + \eta|^p + |\xi - \eta|^p \geq 2(|\xi|^p + |\eta|^p)$ for every real number $p \geq 2$;
- (ii) $|\xi + \eta|^p + |\xi - \eta|^p \leq 2(|\xi|^p + |\eta|^p)$ for every real number p , $0 < p \leq 2$;
- (iii) $|\xi - \eta|^p \leq |\xi|^p + |\eta|^p$ for every real number p , $0 < p < 1$.

Moreover, if $p \neq 2$, equality can only hold in the above inequalities if $\xi = 0$ or $\eta = 0$.

Corollary 2.1. *Let f and g be two elements of $l_p(\Gamma)$. Then:*

- (i) for the cases that $p \geq 1$ and $p \neq 2$, we have

$$(2.2) \quad \|f + g\|^p + \|f - g\|^p = 2(\|f\|^p + \|g\|^p)$$

if and only if $|f| \wedge |g| = \theta$;

- (ii) for the cases that $0 < p < 1$, we have

$$(2.3) \quad \|f - g\| = \|f\| + \|g\|$$

if and only if $|f| \wedge |g| = \theta$;

$$(2.4) \quad \|f - g\| + \|f + g\| = 2(\|f\| + \|g\|)$$

if and only if $|f| \wedge |g| = \theta$.

Lemma 2.2. *Let γ be an arbitrary element of Γ . Suppose that we have*

- (i) $P_\gamma(T(S(E))) \subseteq \text{span}(Te_\gamma)$ when $p \neq 1$, and
- (ii) $T(-e_\gamma) \in \text{span}(Te_\gamma)$ when $p = 1$.

Then, $T(-e_\gamma) = -Te_\gamma$.

Proof. We consider first the case that $p = 1$. Note that the property

$$(2.5) \quad \|u\| = \|v\| \text{ implies } v = \pm u$$

holds in every one-dimensional subspace of AL -spaces. Since T is isometric, $\|T(-e_\gamma)\| = \|Te_\gamma\| = \|e_\gamma\| = 1$. Combining (2.5), the fact that $\dim(\text{span}(Te_\gamma)) = 1$ and $T(-e_\gamma) \in \text{span}(Te_\gamma)$, we conclude that $T(-e_\gamma) = \pm Te_\gamma$. Suppose that $T(-e_\gamma) = Te_\gamma$. Then $0 = \|Te_\gamma - T(-e_\gamma)\| = \|2e_\gamma\|$, and thus $e_\gamma = \theta$, which is a contradiction. Therefore, we have

$$(2.6) \quad T(-e_\gamma) = -Te_\gamma.$$

We consider now the case that $p \neq 1$. Because of the projection property of AL_p -spaces, we may assume that $P = I - P_\gamma$, where I is the identity mapping on F . It is obvious that P is a band projection from F onto the orthogonal complement of B_{Te_γ} . Therefore, we have $T(-e_\gamma) = P_\gamma(T(-e_\gamma)) + P(T(-e_\gamma))$.

Since $P_\gamma(T(-e_\gamma)) \in \text{span}(Te_\gamma)$, we conclude that there exists a real number a_γ such that $P_\gamma(T(-e_\gamma)) = a_\gamma Te_\gamma$.

For the cases that $p = \beta < 1$, since T is isometric, we have

$$\begin{aligned} \|T(-e_\gamma) - Te_\gamma\| &= \|-e_\gamma - e_\gamma\| = 2^\beta \\ &= \|P_\gamma(T(-e_\gamma)) + P(T(-e_\gamma)) - Te_\gamma\| = \|a_\gamma Te_\gamma - Te_\gamma\| + \|P(T(-e_\gamma))\| \\ &= |a_\gamma - 1|^\beta + \|T(-e_\gamma)\| - \|P_\gamma(T(-e_\gamma))\| = |a_\gamma - 1|^\beta + 1 - |a_\gamma|^\beta. \end{aligned}$$

Therefore, we have the following equality: $|a_\gamma - 1|^\beta - |a_\gamma|^\beta = 2^\beta - 1$.

We know that the function $h(t) = |1 - t|^\beta - |t|^\beta - 2^\beta + 1$ has two zeros in $[-1, 1]$, $t_1 = -1$, $t_2 \in (0, \frac{1}{2})$.

Suppose that $0 \leq a_\gamma = t_2 < \frac{1}{2}$. Now let $T(a_\gamma e_\gamma) = b_\gamma Te_\gamma + P(T(a_\gamma e_\gamma))$. Then $|b_\gamma| < \frac{1}{2}$ and

$$\begin{aligned} (1 - a_\gamma)^\beta &= \|a_\gamma e_\gamma - e_\gamma\| = \|T(a_\gamma e_\gamma) - Te_\gamma\| \\ &= \|b_\gamma Te_\gamma + P(T(a_\gamma e_\gamma)) - Te_\gamma\| = \|b_\gamma Te_\gamma - Te_\gamma\| + \|P(T(a_\gamma e_\gamma))\| \\ &= (1 - b_\gamma)^\beta + \|P_\gamma(T(-e_\gamma))\| = (1 - b_\gamma)^\beta + a_\gamma^\beta - |b_\gamma|^\beta, \end{aligned}$$

and so

$$(\Delta) \quad (1 - a_\gamma)^\beta - a_\gamma^\beta = (1 - b_\gamma)^\beta - |b_\gamma|^\beta.$$

Notice that the function $u(t) = (1 - t)^\beta - t^\beta$ is strictly decreasing on $[0, \frac{1}{2}]$ but the function $v(t) = (1 - t)^\beta - (-t)^\beta$ is strictly increasing on $[-\frac{1}{2}, 0]$. If $b_\gamma < 0$, then

$$(1 - a_\gamma)^\beta - a_\gamma^\beta = u(a_\gamma) > u(0) = 1 = v(0) > v(b_\gamma) = (1 - b_\gamma)^\beta - (-b_\gamma)^\beta,$$

contradicting $u(a_\gamma) = v(b_\gamma)$ by (Δ) , and so $b_\gamma \geq 0$. Therefore, $b_\gamma = a_\gamma$, and so $P(T(a_\gamma e_\gamma)) = \theta$. We now have $T(a_\gamma e_\gamma) = a_\gamma Te_\gamma$.

On the other hand, we have

$$\begin{aligned} (1 + a_\gamma)^\beta &= \|-e_\gamma - a_\gamma e_\gamma\| = \|T(-e_\gamma) - T(a_\gamma e_\gamma)\| \\ &= \|a_\gamma Te_\gamma + P(T(-e_\gamma)) - T(a_\gamma e_\gamma)\| = \|P(T(-e_\gamma))\| = 1 - a_\gamma^\beta, \end{aligned}$$

that is, $(1 + a_\gamma)^\beta + a_\gamma^\beta = 1$, and so $a_\gamma = 0$, contradicting the assumption $a_\gamma > 0$. It follows that $a_\gamma = t_1 = -1$, implying that $\|P_\gamma(T(-e_\gamma))\| = |a_\gamma|^\beta = 1$.

Since

$$1 = \|T(-e_\gamma)\| = \|P_\gamma(T(-e_\gamma))\| + \|P(T(-e_\gamma))\| = 1 + \|P(T(-e_\gamma))\|,$$

we derive that $P(T(-e_\gamma)) = \theta$, and thus prove (2.6) for this case.

Now suppose that $p > 1$. Since we have

$$\begin{aligned} \|T(-e_\gamma) - Te_\gamma\|^p &= \|-e_\gamma - e_\gamma\|^p = 2^p \\ &= \|P_\gamma(T(-e_\gamma)) + P(T(-e_\gamma)) - Te_\gamma\|^p = \|a_\gamma Te_\gamma - Te_\gamma\|^p + \|P(T(-e_\gamma))\|^p \\ &= |a_\gamma - 1|^p + \|T(-e_\gamma)\|^p - \|P_\gamma(T(-e_\gamma))\|^p = |a_\gamma - 1|^p + 1 - |a_\gamma|^p, \end{aligned}$$

the equality (2.6) is easily obtained using arguments similar to those of the proof for the cases that $p < 1$. \square

Lemma 2.3. *Let $p = \beta$ be a real number that is less than or equal to one, n a positive integer, and $\{e_1, e_2, \dots, e_n\}$ a subset of $S(E)$ satisfying the condition that the equality $|e_i| \wedge |e_j| = \theta$ holds for each distinct pair of indices i and j . Suppose that $x = \sum_{i=1}^n \lambda_i e_i$, where λ_i are nonzero real numbers that satisfy the equation $\sum_{i=1}^n |\lambda_i|^\beta = 1$.*

Suppose that $T: S(E) \rightarrow S(F)$ is an isometric embedding satisfying

- (i) $P_i(T(S(E))) \subseteq \text{span}(Te_i)$ when $0 < p < 1$,
- (ii) $P_i(T(S(E))) \subseteq \text{span}(Te_i)$ and $T(-e_i) \in \text{span}(Te_i)$ when $p = 1$

for all $1 \leq i \leq n$. Then, we have $T(x) = \sum_{i=1}^n \lambda_i Te_i$.

Proof. By the assumptions of the lemma, and Lemma 2.2, it is easy to derive that

$$(2.7) \quad T(-e_i) = -Te_i \quad (1 \leq i \leq n).$$

Since T is isometric, the elements of $\{e_1, e_2, \dots, e_n\}$ are pairwise disjoint, and the equality (2.7) holds, we conclude by (2.2) and (2.4) of Corollary 2.1 that

$$\begin{aligned} \|Te_i + Te_j\| + \|Te_i - Te_j\| &= \|e_i + e_j\| + \|e_i - e_j\| \\ &= 2(\|e_i\| + \|e_j\|) = 2(\|Te_i\| + \|Te_j\|) \quad (\forall 1 \leq i, j \leq n, i \neq j). \end{aligned}$$

As a result, we derive, again by applying (2.2) and (2.4) of Corollary 2.1, that the equality $|Te_i| \wedge |Te_j| = \theta$ holds for every pair of distinct indices i and j . It follows that the elements of the set $\{Te_i\}_{i=1}^n$ are mutually disjoint, and thus $\{B_{Te_i}\}_{i=1}^n$ is mutually disjoint.

Since F is order complete, it possesses the projection property. Therefore, we may assume that $P = I - \sum_{i=1}^n P_i$.

Since $|Te_i| \wedge |Te_j| = \theta$, it follows that $\sum_{i=1}^n P_i$ is the band projection of F onto the band generated by $\{Te_i\}_{i=1}^n$. Thus P is the band projection of F onto the orthogonal complement of $\{Te_i\}_{i=1}^n$.

Since $T: S(E) \rightarrow S(F)$ is isometric, by defining $\varepsilon_i = \frac{\lambda_i}{|\lambda_i|}$ for every $i, 1 \leq i \leq n$, we have

$$(2.8) \quad \begin{aligned} \|Tx - T(\varepsilon_i e_i)\| &= \|x - \varepsilon_i e_i\| = \sum_{j=1, j \neq i}^n |\lambda_j|^\beta + |\lambda_i - \varepsilon_i|^\beta \\ &= 1 - |\lambda_i|^\beta + |\lambda_i - \varepsilon_i|^\beta = 1 - |\lambda_i|^\beta + (1 - |\lambda_i|)^\beta. \end{aligned}$$

On the other hand, we may assume, by the fact that $P_i(T(S(E))) \subseteq \text{span}(Te_i)$ and the equality (2.7), that $P_i(Tx) = \lambda'_i T(\varepsilon_i e_i)$ for every $1 \leq i \leq n$. Since we have $|\lambda'_i|^\beta = \|P_i(Tx)\| \leq \|Tx\| = \|x\| = 1$, and $|\lambda'_i| \leq 1$ for every $1 \leq i \leq n$, we conclude that

$$(2.9) \quad \begin{aligned} \|Tx - T(\varepsilon_i e_i)\| &= \left\| \sum_{j=1, j \neq i}^n P_j(Tx) + P(Tx) + P_i(Tx) - T(\varepsilon_i e_i) \right\| \\ &= \|Tx\| - \|P_i(Tx)\| + \|P_i(Tx) - T(\varepsilon_i e_i)\| \\ &= 1 - |\lambda'_i|^\beta + \|\lambda'_i T(\varepsilon_i e_i) - T(\varepsilon_i e_i)\| = 1 - |\lambda'_i|^\beta + |\lambda'_i - 1|^\beta \\ &\geq 1 - |\lambda'_i|^\beta + (1 - |\lambda'_i|)^\beta. \end{aligned}$$

It follows by (2.8) and (2.9) that

$$|\lambda_i|^\beta - (1 - |\lambda_i|)^\beta \leq |\lambda'_i|^\beta - (1 - |\lambda'_i|)^\beta.$$

For the case that $\beta = 1$, we have $2|\lambda_i| - 1 \leq 2|\lambda'_i| - 1$, and thus $|\lambda_i| \leq |\lambda'_i|$. For the cases that $\beta < 1$, it is well known that the function $f(t) = t^\beta - (1 - t)^\beta$ is increasing on the interval $[0, 1]$. Therefore, the equality $|\lambda_i| \leq |\lambda'_i|$ holds for every $1 \leq i \leq n$.

However, we have

$$\sum_{i=1}^n |\lambda_i|^\beta = \|x\| = \|Tx\| = \sum_{i=1}^n \|P_i(Tx)\| + \|P(Tx)\| = \sum_{i=1}^n |\lambda'_i|^\beta + \|P(Tx)\|.$$

Therefore, we conclude that $\|P_i(Tx)\| = |\lambda'_i|^\beta = |\lambda_i|^\beta$ and $P(Tx) = \theta$. Noting that (2.5) holds in every one-dimensional subspace of a β -homogeneous F^* -space, it follows from $P_i(T(S(E))) \subseteq \text{span}(Te_i)$ and (2.7) that

$$P_i(Tx) = \pm |\lambda_i| T(\varepsilon_i e_i) \quad (1 \leq i \leq n).$$

Suppose that, in the last equation, there exists at least one equality whose right-hand side bears a negative sign. Then, we may assume that $P_j(Tx) = -|\lambda_j| T(\varepsilon_j e_j)$ for some $\lambda_j \neq 0$ ($1 \leq j \leq n$). By this assumption, we derive that

$$Tx + T(\varepsilon_j e_j) = \sum_{i=1, i \neq j}^n P_i(Tx) + T(\varepsilon_j e_j) - |\lambda_j| T(\varepsilon_j e_j),$$

and thus

$$\begin{aligned} \|Tx + T(\varepsilon_j e_j)\| &= \left\| \sum_{i=1, i \neq j}^n P_i(Tx) \right\| + (1 - |\lambda_j|)^\beta \|T(\varepsilon_j e_j)\| \\ &= 1 - |\lambda_j|^\beta + (1 - |\lambda_j|)^\beta. \end{aligned}$$

However, since T is isometric and the equality (2.7) holds, we obtain that

$$\begin{aligned} \|Tx + T(\varepsilon_j e_j)\| &= \|Tx - T(-\varepsilon_j e_j)\| = \|x + \varepsilon_j e_j\| \\ &= \sum_{i=1, i \neq j}^n \|\lambda_i e_i\| + \|(\lambda_j + \varepsilon_j) e_j\| = 1 - |\lambda_j|^\beta + |\lambda_j + \varepsilon_j|^\beta \\ &= 1 - |\lambda_j|^\beta + (1 + |\lambda_j|)^\beta. \end{aligned}$$

As a result, we derive that $\lambda_j = 0$ holds for the last two equalities, a contradiction. Therefore, we conclude that $P_i(Tx) = |\lambda_i| T(\varepsilon_i e_i)$, $1 \leq i \leq n$. We obtain by (2.7) that $P_i(Tx) = \lambda_i T e_i$, $1 \leq i \leq n$, thus proving the lemma. \square

Example 1. In (ii) of Lemma 2.3, the condition $P_\gamma(T(S(E))) \subseteq \text{span}(Te_\gamma)$ need not imply $T(-e_\gamma) \in \text{span}(Te_\gamma)$. For instance, we define a mapping $T : S(l_1^2) \rightarrow S(l_1^3)$ for any $x = a_1 e_1 + a_2 e_2 \in S(l_1^2)$ by

$$Tx = \begin{cases} a_1 e'_1 + a_2 e'_3 & \text{if } a_1 \geq 0, \\ a_1 e'_2 + a_2 e'_3 & \text{if } a_1 < 0, \end{cases}$$

where $\{e_1 = (1, 0), e_2 = (0, 1)\}$ is a standard basis of l_1^2 , and $\{e'_1 = (1, 0, 0), e'_2 = (0, 1, 0), e'_3 = (0, 0, 1)\}$ is a standard basis of l_1^3 . It is easy to see that T is isometric with the property that $P_n(T(S(l_1^2))) \subseteq \text{span}(Te_n)$ for $n = 1, 2$. But $T(-e_1) = -e'_2 \notin \text{span}(Te_1) = \text{span}(e'_1)$.

Lemma 2.4. Let p be a real number satisfying the conditions $p > 1$ and $p \neq 2$, and let the element x and the e_i 's be as in Lemma 2.3. Suppose that the isometric embedding $T : S(E) \rightarrow S(F)$ satisfies $P_i(T(S(E))) \subseteq \text{span}(Te_i)$ for all $1 \leq i \leq n$ ($n \in \mathbb{N}$). Then $T(x) = \sum_{i=1}^n \lambda_i T e_i$.

Proof. We need only to note that AL_p -norms satisfy the p -additivity and that the equality (2.2) of Corollary 2.1 holds. The method used in the proof of Lemma 2.3 can be applied here for the AL_p -spaces satisfying $p > 1$, $p \neq 2$. Indeed, as in the proof of the last lemma, we conclude that

$$\begin{aligned} \|Tx - T(\varepsilon_i e_i)\|^p &= \|x - \varepsilon_i e_i\|^p = \sum_{j=1, j \neq i}^n |\lambda_j|^p + |\lambda_i - \varepsilon_i|^p \\ &= 1 - |\lambda_i|^p + |\lambda_i - \varepsilon_i|^p = 1 - |\lambda_i|^p + (1 - |\lambda_i|)^p, \end{aligned}$$

and that

$$\begin{aligned} \|Tx - T(\varepsilon_i e_i)\|^p &= \left\| \sum_{j=1, j \neq i}^n P_j(Tx) + P(Tx) + P_i(Tx) - T(\varepsilon_i e_i) \right\|^p \\ &= \|Tx\|^p - \|P_i(Tx)\|^p + \|P_i(Tx) - T(\varepsilon_i e_i)\|^p \\ &= 1 - |\lambda'_i|^p + \|\lambda'_i T(\varepsilon_i e_i) - T(\varepsilon_i e_i)\|^p = 1 - |\lambda'_i|^p + |\lambda'_i - 1|^p \\ &\geq 1 - |\lambda'_i|^p + (1 - |\lambda'_i|)^p. \end{aligned}$$

It follows that

$$|\lambda_i|^p - (1 - |\lambda_i|)^p \leq |\lambda'_i|^p - (1 - |\lambda'_i|)^p,$$

and thus $|\lambda_i| \leq |\lambda'_i|$. However, since

$$\sum_{i=1}^n \|\lambda_i e_i\|^p = \|x\|^p = \|Tx\|^p = \sum_{i=1}^n \|P_i(Tx)\|^p + \|P(Tx)\|^p,$$

we conclude that $\|P_i(Tx)\| = |\lambda'_i| = |\lambda_i|$ and $P(Tx) = \theta$.

The remainder of the proof is similar to the corresponding part of Lemma 2.3 and shall be omitted here. \square

3. THE MAIN RESULTS

Theorem 3.1. *Let $0 < p < \infty$ with $p \neq 2$. An isometry $T: S(E) \rightarrow S(F)$ can be linearly isometrically extended if and only if, for every $\gamma \in \Gamma$,*

- (i) $P_\gamma(T(S(E))) \subseteq \text{span}(Te_\gamma)$ when $p \neq 1$, and
- (ii) $P_\gamma(T(S(E))) \subseteq \text{span}(Te_\gamma)$ and $T(-e_\gamma) \in \text{span}(Te_\gamma)$ when $p = 1$.

Proof. First we show the sufficiency. Suppose that $x = \sum a_\gamma e_\gamma \in S(E)$. It is well known that E is linearly isometric to $l_p(\Gamma)$. By the characterization of elements in $l_p(\Gamma)$, the set $\{\gamma \in \Gamma : a_\gamma \neq 0\} := I_x$ is countable. By the continuity of isometries, Lemma 2.3, and Lemma 2.4, we conclude that

$$(3.10) \quad Tx = \sum_{i \in I_x} a_i T e_i = \sum a_\gamma T e_\gamma \quad (\forall x = \sum a_\gamma e_\gamma \in S(E)).$$

Define the operator \tilde{T} on E by

$$(3.11) \quad \tilde{T}(x) = \begin{cases} \|x\|T\left(\frac{x}{\|x\|}\right) & \text{if } x \neq \theta, \\ \theta & \text{if } x = \theta \quad (\forall x \in E). \end{cases}$$

Suppose that $x = \sum a_\gamma e_\gamma$, $y = \sum b_\gamma e_\gamma \in E$ and $s, t \in \mathbb{R}$. By virtue of (3.10) and (3.11), we obtain that

$$\tilde{T}(x) = \|x\|T\left(\frac{x}{\|x\|}\right) = \|x\| \sum \frac{a_\gamma}{\|x\|} T e_\gamma = \sum a_\gamma T e_\gamma.$$

Therefore, we have

$$\tilde{T}(sx + ty) = \sum (sa_\gamma + tb_\gamma)Te_\gamma = s \sum a_\gamma Te_\gamma + t \sum b_\gamma Te_\gamma = s\tilde{T}(x) + t\tilde{T}(y).$$

The last equation shows that \tilde{T} is a linear isometry from E into F , and that \tilde{T} is a linearly isometric extension of T .

Now we prove the necessity. Suppose that T can be extended to a linear isometry from E to F . Following the method used in the proof of Lemma 2.3, it is not hard to show that $\{B_{Te_\gamma}\}_{\gamma \in \Gamma}$ is mutually disjoint.

The remaining part of the proof is trivial and thus is omitted. \square

Throughout the remainder of this paper we use E and F to denote complex atomic AL_p -spaces.

Theorem 3.2. *Let $0 < p < \infty$ with $p \neq 2$. Let E and F be complex atomic AL_p -spaces. Suppose that, for every $\gamma \in \Gamma$, we have*

- (i) $P_\gamma(T(S(E))) \subseteq \text{span}(Te_\gamma)$ when $p \neq 1$, and
- (ii) $P_\gamma(T(S(E))) \subseteq \text{span}(Te_\gamma)$ and $T(-e_\gamma) \in \text{span}(Te_\gamma)$ when $p = 1$.

Then, there exist an additive isometry \tilde{T} from E into F and two mutually complementary subspaces X and Y of E such that

- (1) *the restriction of \tilde{T} to $S(E)$ is T ,*
- (2) *that to X is linear, and*
- (3) *that to Y is conjugate linear.*

To show Theorem 3.2, we need the following lemma.

Lemma 3.1. *Let $D = \{\lambda \in \mathbb{C}: |\lambda| = 1\}$. Suppose that T is as in Theorem 3.2. Then, for every $\gamma \in \Gamma$,*

$$\text{either } T(\lambda e_\gamma) = \lambda Te_\gamma \text{ or } T(\lambda e_\gamma) = \bar{\lambda} Te_\gamma$$

for any $\lambda \in D$.

Proof. The method Xiao Yuanhui used in [21, Lemma 7] is valid for this assertion. \square

Proof of Theorem 3.2. By combining Lemma 2.3, Lemma 2.4, Lemma 3.1 and Theorem 3.1, the proof that Xiao Yuanhui introduced in [21, Theorem 2] can be easily modified to prove the assertion. \square

Remark 3.1. For the case of $p = 2$, Theorem 3.1 and Theorem 3.2 are also true (cf. [4]).

Corollary 3.1. *If the mapping T carries some atom $e \in S(E)$ to an atom $Te \in S(F)$, then the assertions of Theorems 3.1 and 3.2 remain true.*

Proof. It is sufficient to consider the real atomic AL_p -spaces. Since an AL_p -space is an Archimedean F -lattice and Te_γ is an atom for each $\gamma \in \Gamma$, it follows that $B_{Te_\gamma} = \text{span}(Te_\gamma)$ for any $\gamma \in \Gamma$, and that $P_\gamma(T(S(E))) \subseteq B_{Te_\gamma} = \text{span}(Te_\gamma)$ for any $\gamma \in \Gamma$. The last equality is the condition required in Theorems 3.1 and 3.2, thus proving the assertion. \square

Corollary 3.2. *If $T: S(E) \rightarrow S(F)$ is a surjective isometry, then the assertions of Theorems 3.1 and 3.2 remain true.*

Proof. Again it is sufficient to consider the real case. The complex case can be obtained by applying Xiao's method of proof [21, Theorem 2].

To show that

$$(3.12) \quad T(-e_\gamma) = -Te_\gamma \quad (\forall \gamma \in \Gamma),$$

we consider three cases: $0 < p < 1$, $p = 1$ and $p > 1$.

First we consider the case that $0 < p < 1$. Since T is isometric and the elements of $\{e_\gamma\}_{\gamma \in \Gamma}$ are mutually disjoint, it follows by (2.3) of Corollary 2.1 that, for any $\gamma, \tau \in \Gamma$, $\gamma \neq \tau$,

$$(3.13) \quad \|T(\pm e_\gamma) - Te_\tau\| = \|\pm e_\gamma - e_\tau\| = \|\pm e_\gamma\| + \|e_\tau\| = \|T(\pm e_\gamma)\| + \|Te_\tau\|.$$

Consequently, we have $|T(\pm e_\gamma)| \wedge |Te_\tau| = \theta$ ($\gamma, \tau \in \Gamma$, $\gamma \neq \tau$), and the elements of $\{Te_\tau\}_{\tau \in \Gamma}$ are mutually disjoint. Therefore, $\{B_{Te_\tau}\}_{\tau \in \Gamma}$ are mutually disjoint.

Suppose that $\gamma \in \Gamma$. We show first that

$$(*) \quad Tw \in B_{Te_\gamma} \Rightarrow w = \pm e_\gamma \quad \text{for any } w \in S(E).$$

Indeed, by the preceding proof, we have $|Tw| \wedge |Te_\tau| = \theta$ ($\tau \in \Gamma$, $\tau \neq \gamma$), and thus $|w| \wedge |e_\tau| = \theta$ ($\tau \in \Gamma$, $\tau \neq \gamma$). Therefore, we conclude that $w \in \text{span}(e_\gamma)$. The conclusion that $w = \pm e_\gamma$ follows immediately by the fact that $w \in S(E)$.

In order to show that $T(-e_\gamma) \in B_{Te_\gamma}$, we assume, without loss of generality, that P' is the band projection of F onto the band generated by $\{Te_\tau\}_{\tau \in \Gamma, \tau \neq \gamma}$, and that $P = I - P_\gamma - P'$. Since F is order complete, it follows that P is a band projection from F onto the orthogonal complement of $\{Te_\tau\}_{\tau \in \Gamma}$. By (3.13) and (2.3) of Corollary 2.1, we derive that $|T(-e_\gamma)| \wedge |Te_\tau| = \theta$ ($\tau \in \Gamma$, $\tau \neq \gamma$).

Therefore, we obtain that $P'(T(-e_\gamma)) = \theta$, and that $T(-e_\gamma) = P_\gamma(T(-e_\gamma)) + P(T(-e_\gamma))$.

If $P(T(-e_\gamma)) \neq \theta$, since T is onto, there exists $y \in S(E)$ such that $Ty = \frac{P(T(-e_\gamma))}{\|P(T(-e_\gamma))\|^{1/p}}$. Note that $|Ty| \wedge |Te_\tau| = \theta$ for all $\tau \in \Gamma$. It follows that $|y| \wedge |e_\tau| = \theta$ for all $\tau \in \Gamma$; thus $y = \theta$, and $P(T(-e_\gamma)) = \theta$. Therefore, $T(-e_\gamma) = P_\gamma(T(-e_\gamma)) \in B_{Te_\gamma}$.

Next we show that $T(-e_\gamma) = -Te_\gamma$. Obviously, we have $T(-e_\gamma) + Te_\gamma \in B_{Te_\gamma}$. Suppose that we have the contrary; that is, suppose that $T(-e_\gamma) + Te_\gamma \neq \theta$.

Let $\alpha = \frac{1}{\|T(-e_\gamma) + Te_\gamma\|^{1/p}}$. Then, it follows that $\alpha(T(-e_\gamma) + Te_\gamma) \in S(F) \cap B_{Te_\gamma}$. By the fact that the mapping T is onto, there exists $y \in S(E)$ with $Ty = \alpha(T(-e_\gamma) + Te_\gamma)$. Similar to the preceding proof, we conclude that $y = \pm e_\gamma$. Therefore, we have $(1 - \alpha)T(\pm e_\gamma) = T(\mp e_\gamma)$. By the fact that T is isometric, we deduce that $|1 - \alpha| = 1$. This last result implies that $\alpha = 2$, since α is not equal to zero. Therefore, we conclude that $-T(\pm e_\gamma) = T(\mp e_\gamma)$. That is, $T(-e_\gamma) + Te_\gamma = \theta$. As a result, we obtain a contradiction. Therefore, we conclude that $T(-e_\gamma) + Te_\gamma = \theta$. That is, $T(-e_\gamma) = -Te_\gamma$.

For the case $p = 1$, Ma Yumei proved in [10] that the condition that $T: S(E) \rightarrow S(F)$ is a surjective isometry implies that the equality $T(-x) = -Tx$ holds for every $x \in S(E)$.

For $p > 1$, for all $x \in S(E)$ we have

$$\|Tx + (-T(-x))\| = \|Tx - T(-x)\| = \|2x\| = \|Tx\| + \|-T(-x)\|.$$

Since the space F in this case is strictly convex, there exists a constant $c > 0$ for which $Tx = c(-T(-x))$. The conclusion that $c = 1$ follows immediately, because T is isometric. Therefore, $T(-x) = -Tx$ for all $x \in S(E)$.

Summing up the above discussion, we conclude that the equality (3.12) holds for every positive number $p \neq 2$.

Similarly to the proof of Lemma 2.3, it follows by (2.2) and (2.3) of Corollary 2.1 and (3.12) that $\{B_{Te_\gamma}\}_{\gamma \in \Gamma}$ are mutually disjoint.

For a number p satisfying $0 < p < \infty$, $p \neq 2$, and an arbitrary $\gamma \in \Gamma$, we now show that $B_{Te_\gamma} = \text{span}(Te_\gamma)$.

For an element $u \in B_{Te_\gamma}$, if $u = \theta$, then $u \in \text{span}(Te_\gamma)$. Therefore, we may assume that $\theta \neq u$. Let

$$u_0 = \begin{cases} \frac{u}{\|u\|^{1/p}} & \text{if } 0 < p < 1, \\ \frac{u}{\|u\|} & \text{if } p \geq 1. \end{cases}$$

Then, we have $u_0 \in S(F) \cap B_{Te_\gamma}$. Note that the equality (*) also holds for the cases that $p \geq 1$. Since T is a surjective isometry, it follows by (*) that there exists $x_0 \in S(E)$ such that $Tx_0 = u_0$; thus $x_0 = \pm e_\gamma$. Therefore, we conclude that $u = \|u\|^{1/p}u_0 = \|u\|^{1/p}Tx_0 = \|u\|^{1/p}T(\pm e_\gamma) = \pm\|u\|^{1/p}Te_\gamma \in \text{span}(Te_\gamma)$ for $0 < p < 1$, and that $u = \pm\|u\|Te_\gamma \in \text{span}(Te_\gamma)$ for $p \geq 1$.

It follows easily by Theorem 3.1 that T can be linearly isometrically extended to E , and this proves the theorem. \square

Remark 3.2. For the cases that $p > 1$, Corollary 3.2 is just a result of Xiao [21, Theorem 1]. In the proof of Corollary 3.2, we have actually shown that if T is surjective, then it maps atoms of $S(E)$ into atoms of $S(F)$ in a way that is different from what Professor Ding Guanggui had described in [5]. The converse is not true, however. For example, let $T: S(l_1) \rightarrow S(l_1)$ be defined by $Tx = \sum \lambda_i e_{2i-1}$ for all $x = \sum \lambda_i e_i \in S(l_1)$, where $\{e_i\}_{i \in \mathbb{N}}$ is a standard basis. Obviously, T maps atoms of $S(l_1)$ into atoms of $S(l_1)$, but T is not surjective.

Remark 3.3. We can see in the proofs of Lemma 2.3 and Lemma 2.4 that an isometry $T: S(E) \rightarrow S(F)$ can be linearly isometrically extended if and only if, for any $\gamma \in \Gamma$, there exists $\{a_{\gamma, \gamma'}\}_{\gamma' \in \Gamma'_\gamma} \subset \mathbb{R}$ with $\sum_{\gamma' \in \Gamma'_\gamma} |a_{\gamma, \gamma'}|^p = 1$ such that

$$Tx = \sum_{\gamma \in \Gamma} \sum_{\gamma' \in \Gamma'_\gamma} \lambda_\gamma a_{\gamma, \gamma'} e'_{\gamma'} \quad (\forall x = \sum_{\gamma \in \Gamma} \lambda_\gamma e_\gamma \in S(E)),$$

where $\{e'_{\gamma'}\}_{\gamma' \in \Gamma'}$ is a norm-one complete disjoint system of F , and $\Gamma'_\gamma = \{\gamma' \in \Gamma' : e'_{\gamma'} \in B_{Te_\gamma}\}$.

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