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CANCELING BRANCH POINTS AND CUSPS ON PROJECTIONS OF KNOTTED SURFACES IN 4-SPACE

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Abstract. For a knotted surface in 4-space, its generic projection into 3-space has branch points as its singularities, and its successive projection into 2-space has fold points and cusps as its singularities. In this paper, we show that for non-orientable knotted surfaces, the numbers of branch points and cusps can be minimized by isotopy.

1. Introduction

By a *knotted surface* we mean a closed connected surface smoothly embedded in the 4-dimensional Euclidean space $\mathbf{R}^4$. Let $\pi_1\colon \mathbf{R}^4 \to \mathbf{R}^3$ and $\pi_2\colon \mathbf{R}^3 \to \mathbf{R}^2$ be orthogonal projections. For a knotted surface $F \subset \mathbf{R}^4$, its projections $\pi_1|_F$ into $\mathbf{R}^3$ and $\pi_2 \circ \pi_1|_F$ into $\mathbf{R}^2$ are known to play essential roles in the theory of knotted surfaces (for example, see [2, 3, 4, 5]).

It is well-known (see, for example, [4]) that for a knotted surface $F \subset \mathbf{R}^4$, its projection $\pi_1|_F$ into $\mathbf{R}^3$ has only branch points as its singularities, and its projection $\pi_2 \circ \pi_1|_F$ into $\mathbf{R}^2$ has only fold points and cusps as its singularities, provided that π_1 and π_2 are generic with respect to F (or if π_1 and π_2 are already fixed, we can isotope F slightly so that the projections have these properties). Here, a *singularity* is a point in F (or its corresponding image) where the differential of the map is not of rank 2. Note that branch points and cusps appear as discrete points, while the fold points appear as a 1-dimensional submanifold. Therefore, it is a fundamental problem whether the discrete singularities, branch points and cusps, can be eliminated by an isotopy of F. (Note that the fold points can never be eliminated, since F is compact.)

Let us orient $\mathbf{R}^4$ in a standard way. Then, it is known that the normal Euler number $e(F)$ of a knotted surface F is an isotopy invariant of F. For projections into $\mathbf{R}^3$, Banchoff [1] proved that for every branch point of a generic knotted surface projection into $\mathbf{R}^3$, one can define its sign, $+1$ or -1, using the orientation of $\mathbf{R}^4$,

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so that the number of positive branch points minus the number of negative ones is equal to the normal Euler number of a given knotted surface. This result implies that there must be at least $|e(F)|$ branch points on a projection of a knotted surface F, since the normal Euler number is an isotopy invariant. Then a natural question arises whether any knotted surface has a projection with this minimal number of branch points. Carter and Saito [3] showed that, in fact, this is always the case.

Theorem 1.1 (Carter and Saito). *Any knotted surface can be isotoped so that the number of branch points on its projection into* 3*-space is equal to the absolute value of its normal Euler number.*

Concerning cusps, Thom [9] (see also [6, 7]) proved that if a smooth map $f\colon M \to N$ of a closed surface M into an orientable surface N has only fold points and cusps as its singularities, then the number of cusps has the same parity as the Euler characteristic $\chi(M)$ of M. This implies that there must be at least 1 cusp on a projection of a knotted surface into $\mathbf{R}^2$ if the surface has an odd Euler characteristic. Carrara, Carter and Saito [2] showed the following.

Theorem 1.2 (Carrara, Carter and Saito). *Any knotted surface is isotopic to a knotted surface whose projection into* $\mathbf{R}^2$ *has no cusps or a single cusp depending on whether the Euler characteristic is even or odd, respectively.*

Note that in Theorems 1.1 and 1.2, the projection into $\mathbf{R}^3$ and that into $\mathbf{R}^2$ are treated separately. Therefore, it is natural to ask if it is possible to isotope a given knotted surface to another one F so that both the projections $\pi_1|_F$ into $\mathbf{R}^3$ and $\pi_2 \circ \pi_1|_F$ into $\mathbf{R}^2$ have the minimal numbers of singularities simultaneously.

For *orientable* knotted surfaces, Carrara, Carter and Saito [2] proved that this is always the case.

Theorem 1.3 (Carrara, Carter and Saito). *Any orientable knotted surface can be isotoped so that the projection into* $\mathbf{R}^3$ *has no branch points and the successive projection into* $\mathbf{R}^2$ *has no cusps.*

Moreover, they conjectured that the same would hold for non-orientable knotted surfaces as well [2, Conjecture 7.4.1]. The main purpose of this paper is to prove this conjecture affirmatively. More precisely, we prove the following.

Theorem 1.4. *Any non-orientable knotted surface can be isotoped so that the projection into* $\mathbf{R}^3$ *has the same number of branch points as the absolute value of its normal Euler number, and the successive projection into* $\mathbf{R}^2$ *has no cusps or a single cusp depending on whether the Euler characteristic is even or odd, respectively.*

Note that in the proof of Theorem 1.3, the following result due to Viro and Kamada [5] played an essential role: any orientable knotted surface in 4-space can be isotoped to a closed surface braid. However, it is known (see [8]) that a corresponding theorem does not hold for the non-orientable case. In fact, we shall prove our Theorem 1.4 in a somewhat elementary manner, without using the surface braid theory. Our strategy is to first eliminate branch points and then to eliminate cusps by pairs.

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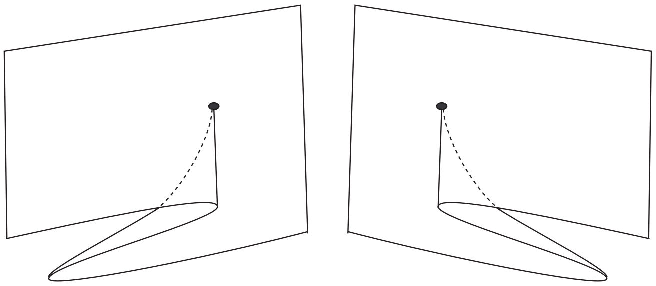

FIGURE 1. A type $\mathcal{Z}$ cusp (left) and a type $\mathcal{S}$ cusp (right)

2. TWO LEMMAS

In this section, we prepare two key lemmas. In the following, we always assume that a knotted surface F is generic enough with respect to the projections $\pi_1\colon \mathbf{R}^4 \to \mathbf{R}^3$ and $\pi_2\colon \mathbf{R}^3 \to \mathbf{R}^2$. In particular, the set of the branch points of $\pi_1|_F$ and that of the cusps of $\pi_2 \circ \pi_1|_F$ are disjoint to each other as subsets of F. (Recall that this is always realized by a small isotopy of F.) Furthermore, for a smooth map f of a surface into $\mathbf{R}^3$ or $\mathbf{R}^2$, $S(f)$ will denote the set of the singularities of f in the source surface. If it is a map into $\mathbf{R}^2$, then $C(f)$ will denote the set of the cusps. Note that if F is generic enough, then $S(\pi_1|_F)$ and $C(\pi_2 \circ \pi_1|_F)$ consist of finite numbers of points, and $S(\pi_2 \circ \pi_1|_F)$ is a smooth closed 1-dimensional submanifold of F.

Let us first introduce the following definition.

Definition 2.1. Let F be a knotted surface in $\mathbf{R}^4$. We say that a cusp c of $\pi_2 \circ \pi_1|_F$ is of *type* $\mathcal{Z}$ if the image of a neighborhood of c by $\pi_1|_F$ in $\mathbf{R}^3$ is as in the left of Figure 1, and it is of *type* $\mathcal{S}$ otherwise, i.e., if it is as in the right of Figure 1. (Here, we assume that the projection $\pi_2\colon \mathbf{R}^3 \to \mathbf{R}^2$ corresponds to the obvious projection to the sheet of paper.)

Lemma 2.2. *Let F be a knotted surface such that $\pi_2 \circ \pi_1|_F$ has a cusp c. Furthermore, let α be an embedded arc in F connecting c and a nonsingular point c' of $\pi_2 \circ \pi_1|_F$ such that* $\operatorname{Int}\alpha$ *intersects $S(\pi_2 \circ \pi_1|_F)$ transversely at a single point and α is disjoint from $S(\pi_1|_F)$. Then, by an isotopy whose support is contained in a small neighborhood of α, we can isotope F to F' so that the following conditions hold, where we naturally identify F and F' abstractly by the isotopy.*

1. *$S(\pi_1|_F) = S(\pi_1|_{F'})$.*
2. *$C(\pi_2 \circ \pi_1|_{F'}) = (C(\pi_2 \circ \pi_1|_F) \setminus \{c\}) \cup \{c'\}$.*
3. *The types of the cusps in $C(\pi_2 \circ \pi_1|_F) \setminus \{c\}$ remain the same, while the type of the cusp c' is different from that of the cusp c.*

The above lemma follows immediately from Figure 2. Note that $\pi_1(\alpha)$ may intersect the double point set or the triple points of $\pi_1|_F$; however, we can isotope α slightly so that it does not intersect the triple points and it intersects the double point set transversely at a finite number of points. Then, it is not difficult to see that a cusp can pass through the double point set by an isotopy of F (see, for example, [2, Figure 7]).

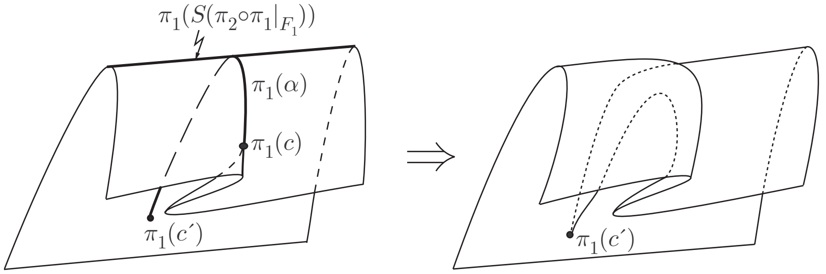

FIGURE 2. A cusp passing through a fold line

Lemma 2.3. *Let* $f\colon M \to \mathbf{R}^2$ *be a smooth map of a closed surface* M *which has only fold points and cusps as its singularities. Suppose that a closed curve* γ *in* M *intersects* $S(f)$ *transversely at a finite number of points. Then the number of intersection points is odd if and only if* γ *is orientation-reversing: i.e., if and only if* $\langle w_1(M), [\gamma]\rangle = 1$, *where* $w_1(M) \in H^1(M;\mathbf{Z}_2)$ *is the first Stiefel-Whitney class of* M, $[\gamma] \in H_1(M;\mathbf{Z}_2)$ *is the* $\mathbf{Z}_2$*-homology class represented by* γ, *and* $\langle\, ,\, \rangle$ *is the Kronecker product.*

Proof. Since $f|_{M\smallsetminus S(f)}$ is an immersion into $\mathbf{R}^2$, $M \smallsetminus S(f)$ is orientable. Let us orient $M \smallsetminus S(f)$ by pulling back the standard orientation of $\mathbf{R}^2$ by $f|_{M\smallsetminus S(f)}$. Note that this orientation is not coherent precisely along $S(f)$. Hence, the closed curve γ is orientation-reversing if and only if it intersects $S(f)$ at an odd number of points. This completes the proof. □

Remark 2.4. In fact, the above lemma is a direct consequence of Thom's result [9] which states that the Poincaré dual to the $\mathbf{Z}_2$-homology class represented by $S(f)$ coincides with $w_1(M)$.

3. PROOF OF THEOREM 1.4

Proof of Theorem 1.4. First, by an isotopy of F, we cancel branch points as in Theorem 1.1, so that the resulting surface F_1 projects to $\mathbf{R}^3$ with the minimum number of branch points.

Suppose that the projection into $\mathbf{R}^2$ of F_1 has two or more cusps. Let β be an embedded arc connecting two cusps c and c'. We may assume that it is disjoint from the branch points and the other cusps, and that it intersects $S(\pi_2 \circ \pi_1|_{F_1})$ transversely at a finite number of points. By applying Lemma 2.2 successively, we may assume that β does not intersect $S(\pi_2 \circ \pi_1|_{F_1})$. If the cusps, which are now close to each other, have distinct types, then we can cancel them by an isotopy as in Figure 3.

Suppose that they have the same type. Since F_1 is non-orientable by our assumption, by Lemma 2.3 there exists an embedded closed curve in F_1 passing through c such that it avoids the branch points and the other cusps, and it intersects $S(\pi_2 \circ \pi_1|_{F_1})$ transversely at an odd number of points. Then, by successively applying Lemma 2.2 along the embedded closed curve β, we may assume that the types of c and c' are different from each other. Therefore, they can be canceled by a further isotopy as in Figure 3.

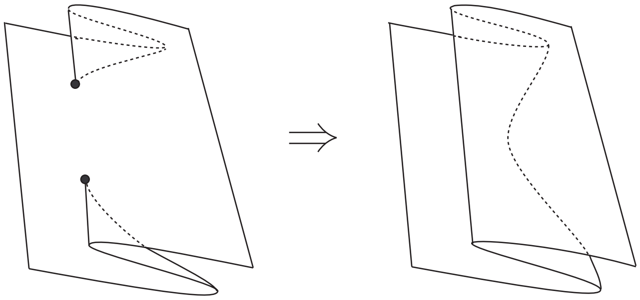

FIGURE 3. Canceling a pair of cusps

Hence, a pair of cusps can always be canceled without introducing any branch points or new cusps. Repeating this procedure finitely many times, we get a desired knotted surface isotopic to the original one. This completes the proof. ☐

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