TRIANGULAR $G_a$ ACTIONS ON $\mathbb{C}^4$

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(Communicated by Bernd Ulrich)

Abstract. Every locally trivial action of the additive group of complex numbers on four-dimensional complex affine space that is given by a triangular derivation is conjugate to a translation. A criterion for a proper action on complex affine $n$-space to be locally trivial is given, along with an example showing that the hypotheses of the criterion are sharp.

1. Introduction

Let $G_a$ denote the additive group of complex numbers, and $X$ a complex affine variety. By an action of $G_a$ on $X$ we will mean an algebraic action. It is well known that every such action can be realized as the exponential of some locally nilpotent derivation $D$ of the coordinate ring $\mathbb{C}[X]$ and that every locally nilpotent derivation gives rise to an action. The ring $C_0$ of $G_a$ invariants in $\mathbb{C}[X]$ is equal to the ring of constants of the generating derivation.

Given an action $\sigma : G_a \times X \to X$, let $\tilde{\sigma} : G_a \times X \to X \times X$ denote the graph morphism and $\tilde{\sigma} : \mathbb{C}[X] \to \mathbb{C}[X,t]$ (resp. $\tilde{\sigma} : \mathbb{C}[X \times X] \to \mathbb{C}[X,t]$) denote the induced maps on coordinate rings.

The action is said to be proper if $\tilde{\sigma}$ is a proper morphism (i.e., if $\mathbb{C}[X,t]$ is integral over the image of $\tilde{\sigma}$). The action is said to be equivariantly trivial if there is a variety $Y$ for which $X$ is $G_a$ equivariantly isomorphic to $G_a \times Y$, the action on $G_a \times Y$ being given by $g \cdot (y, h) = (y, g + h)$. The action is locally trivial if there is a cover of $X$ by $G_a$ stable affine open subsets $X_i$ on which the action is equivariantly trivial. Equivariant triviality of an action on $X$ is equivalent with the existence of a regular function $s \in \mathbb{C}[X]$ for which $Ds = 1$. Such a function is called a slice and, if one exists, $\mathbb{C}[X] = C_0[s]$. If $X$ is factorial, i.e., its coordinate ring is a unique factorization domain, then local triviality is equivalent with the intersection of the kernel and image of $D$ generating the unit ideal in $\mathbb{C}[X]$.

The affine cancellation problem can be phrased in terms of $G_a$ actions on $X = \mathbb{C}^{n+1}$. If the action is equivariantly trivial, is then $Y \cong \mathbb{C}^n$? The answer is affirmative for $n = 2$, and for $n = 3$ provided the ring of invariants contains a coordinate function [18, Cor. 4.5.5]. It has recently been shown that the ring of $G_a$ invariants is finitely generated for actions on $\mathbb{C}^4$ whose generating derivation is triangulable (triangulable actions) [2]. These positive results suggest that a more complete understanding of actions on $\mathbb{C}^4$ is within reach. In section 1 we show that
locally trivial triangulable actions on $\mathbb{C}^4$ are in fact equivariantly trivial, admitting a geometric quotient isomorphic to $\mathbb{C}^3$. Thus the example of Winkelmann [13] of a locally trivial, but not equivariantly trivial, triangular action on $\mathbb{C}^5$ is optimal.

Locally trivial actions are proper, and proper actions on $\mathbb{C}^n$ are locally trivial provided that $\mathbb{C}[X]$ is a flat ring extension of $C_0$ [3, Theorem 2.8]. This need not always be the case as shown in [5]. On the other hand, Holmann [12] showed that any proper holomorphic action on a complex manifold admits a quotient that is a manifold, while Popp [11, Lecture 3] showed that this quotient admits the structure of an algebraic space if the action is algebraic and the manifold is a smooth variety. Based on these results, we give in section 3 a ring-theoretic criterion for a proper action on $\mathbb{C}^n$ to be locally trivial and indicate where the hypotheses fail for the example in [5] of a nonlocally trivial proper action on $\mathbb{C}^5$.

2. Locally trivial triangular actions on $\mathbb{C}^4$

From [3, Theorem 2.8] we know in general that the quotient of a locally trivial action on an affine factorial variety $X$ exists as a quasiaffine variety $Y \subset \text{Spec} R$, where $R$ is the subring of $C_0$ constructed as follows: Let $\delta(a_1), \ldots, \delta(a_n) \in C_0$ generate the unit ideal in $\mathbb{C}[X]$, and set $R_i = \mathbb{C}[X, \frac{1}{\delta(a_i)}]^{G_i}$. Note that $\mathbb{C}[X, \frac{1}{\delta(a_i)}] = R_i[\frac{1}{\delta(a_i)}]$ so that $R_i$ is a finitely generated $\mathbb{C}$ algebra, say $R_i = \mathbb{C}[b_{i1}, \ldots, b_{im}, \frac{1}{\delta(a_i)}]$, with $b_{ij} \in C_0$. The ring $R = \mathbb{C}[b_{ij}, \delta(a_i) \mid 1 \leq i \leq n, 1 \leq j \leq m]$ is the required subring of $C_0$.

It is easy to see that $C_0$ is the factorial closure of $R$ (i.e., the intersection of all unique factorization domains containing $R$), and we ask whether $C_0$ is the integral closure of $R$. Of course a positive answer would solve Hilbert’s 14th problem for locally trivial $G_a$ actions. Since $Y$ is a geometric quotient, $C_0$ is the ring of global sections of its structure sheaf. With $I$ denoting the ideal defining the complement of $Y$ in $\text{Spec} R$, and $F$ the quotient field of $R$, the ring $C_0$ is isomorphic to $T_I R = \bigcup_{n \geq 0} \{ \alpha \in F \mid \alpha I^n \subset R \}$, the ideal transform of $R$ with respect to $I$. A fuller discussion of these notions can be found in [6].

Consider a locally trivial $G_a$ action on $\mathbb{C}^4$ generated by the locally nilpotent derivation of $\mathbb{C}[x_1, x_2, x_3, x_4]$ defined by $\delta$

\[
\begin{align*}
x_4 & \mapsto p(x_1, x_2, x_3), \\
x_3 & \mapsto q(x_1, x_2), \\
x_2 & \mapsto r(x_1), \\
x_1 & \mapsto 0.
\end{align*}
\]

It was recently shown [2] that $C_0$ is finitely generated for any triangular action on $\mathbb{C}^4$. In the special case under consideration, we show that $Y \cong \text{Spec} C_0$. Since the quotient $Y$ is then affine, the action is equivariantly trivial (locally trivial actions with quotient $Y$ correspond to elements of $H^1(Y, O(Y))$, which is 0 with $Y$ affine), and van Rossum’s thesis [10] then shows that $Y \cong \mathbb{C}^3$.

**Theorem 2.1.** Let $G_a$ act locally trivially on $X = \mathbb{C}^4$ via a triangular derivation as above. Then the action is equivariantly trivial with quotient isomorphic to $\mathbb{C}^3$.

**Proof.** Set $Z = \text{Spec} C_0$, and denote by $\pi : X \to Z$ the $G_a$ equivariant morphism induced by the ring inclusion $C_0 \hookrightarrow \mathbb{C}[x_1, x_2, x_3, x_4]$. By hypothesis, $x_1 \in C_0$ and is prime, so that for each $\lambda \in \mathbb{C}$, $\pi \lambda$, the restriction of $\pi$ to the hyperplane $X_\lambda$
defined by \( x_1 - \lambda \), is a \( G_a \) equivariant morphism to the surface \( Z_\lambda \subset Z \) defined there by \( x_1 - \lambda \). The assertion is proved by showing that \( \pi_\lambda \) is surjective for all \( \lambda \).

It suffices to consider only those \( \lambda \) for which \( x_1 - \lambda \) divides \( r(x_1) \), since otherwise \( \frac{x_1}{r(x_1)} \) defines a slice on \( X_\lambda \). Assume that \( r(0) = 0 \), and we consider \( \pi_0 : X_0 \to Z_0 \). Note that \( X_0 \cong \mathbb{C}^3 \) so that the action on \( X_0 \) has a slice and \( \pi_0 \) is an orbit map, but \( Z_0 \) is not a priori the quotient (which is the open image of \( \pi_0 \)). Indeed, \( Z_0 \cong \text{Spec} \ C_0/(x_1) \), and \( C_0/(x_1) \subset [C[X]/(x_1)]^{G_a} \), the latter being the ring of functions defined on the image of \( \pi_0 \) in \( Y_0 \), and \( X_0 \to \text{Spec} \ [C[X]/(x_1)]^{G_a} \) is clearly surjective. Denote by \( I \) the ideal in \( C_0/(x_1) \) defining \( Y_0 - \text{im}(\pi_0) \). We have \( T_1(C_0/(x_1)) \cong [C[X]/(x_1)]^{G_a} \cong \mathbb{C}^2 \), a polynomial ring in two variables.

Lemma 2.5 of \([2]\) can be interpreted in this context as saying that \( C_0/(x_1) = R[u] \), where \( u \) is transcendental over some subring \( R \), i.e., that the algorithm \([1]\) to construct the ring of invariants terminates with the adjunction of an element transcendental over a subring. Denote by \( \hat{R} \) the integral closure of \( R \) in its quotient field. Note that \( \hat{R}[u] \subset T_1(R[u]) \) since both are integrally closed. Moreover, \( \hat{R} \) is Dedekind and rational since the quotient field, \( qf(\hat{R}[u]) \cong \mathbb{C}^2 \cong qf(R[u]) \). By the generalized Luroth theorem, \( qf(R) \) is rational. Since \( \hat{R} \) is a subring of a polynomial ring, \( \hat{R} \cong \mathbb{C}^1 \).

Note that \( T_1(\hat{R}[u]) \cong T_1(R[u]) \cong \mathbb{C}^2 \). If \( h_T(\hat{R}[u]) = 1 \), then \( \hat{R}[u] \) is principal, say \( \hat{R}[u] = (f) \). But then \( f^{-1} \in \mathbb{C}^2 \), a contradiction. Thus \( h_T(\hat{R}[u]) = 2 \), which implies that \( T_1(\hat{R}[u]) = R[u] = \hat{R}[u] \). Finally, we obtain the chain of surjections:

\[
X_0 \to \text{Spec} \ [C[X]/(x_1)]^{G_a} = \text{Spec} \ R[u] \to \text{Spec} \ R[u] = Z_0.
\]

That the quotient is isomorphic to \( \mathbb{C}^3 \) follows from a special case of \([10]\) Cor. 4.5.5. \( \square \)

Remark 2.2. It appears to be true that every fixed point free action on \( \mathbb{C}^3 \) has a slice (S. Kaliman, preprint), while Winkelmann produced an example of a locally trivial triangular derivation on \( \mathbb{C}^3 \) that has no slice, and there is an example of a triangular proper action on \( \mathbb{C}^5 \) that is not locally trivial. The situation for \( \mathbb{C}^4 \) is not so clear, but the next section proposes an avenue of attack on the proper triangular case.

3. Proper actions

Consider a proper \( G_a \) action on \( X = \mathbb{C}^n \) generated by the locally nilpotent derivation \( D \). Assume that the ring of invariants \( C_0 \) is finitely generated defining the affine variety \( Y = \text{Spec} \ C_0 \). Let \( \pi : \mathbb{C}^n \to Y \) as above be the morphism induced by the ring inclusion \( C_0 \subset \mathbb{C}[x_1,...,x_n] \), and let \( I \) denote the ideal \( C_0 \cap \text{im}D \) (\( I = C_0 \) if and only of the action is equivariantly trivial). Assuming that the action is not equivariantly trivial, in particular \( n \geq 4 \), denote by \( Z \) the closed subset of \( Y \) defined by \( I \). From \([7]\) we know that every irreducible component of \( Z \) has codimension two and that \( \pi_{|X-I^{-1}Z} : X - \pi^{-1}Z \to Y - Z \) is a principal \( G_a \) bundle. The action is locally trivial if and only if \( \pi^{-1}Z = \emptyset \).

From Holmann \([12]\) we know that the space of orbits carries the structure of an analytic space \( X/G_a \) (in fact, \( X/G_a \) is a manifold) and from Popp \([11]\) that \( X/G_a \) is an algebraic space. The simplicity of our context enables us to make this even more explicit. The orbit \( G_a x \) of any point \( x \in X \) is isomorphic to a line. As such it is a coordinate line in some coordinate system, \( (x_1,...,x_n) \) for \( X \), say...
$G_a x$ is the $x_1$-axis, and we can take $x$ to be the origin. If $H_x$ is the hyperplane $x_1 = 0$, then it is clear that the morphism $G_a \times H_x \cong \mathbb{C}^n \to X = \mathbb{C}^n$ given by $\rho : (\lambda, y) \mapsto \sigma(\lambda, y)$ is étale in an affine neighborhood $U$ of $(\lambda, x)$ (the principal open subset defined by the Jacobian determinant $d$ of the regular mapping). Indeed, $\rho$ is $G_a$ equivariant with respect to the action on $G_a \times H_1$ given by $(\mu, (\lambda, y)) \mapsto (\mu + \lambda, y)$. Thus $d \in \mathbb{C}[G_a \times H_x]^{G_a} = \mathbb{C}[H_x]$, and $U = G_a \times U_x$ with $U_x$ the principal open subset of $H_x$ defined by $d$. Identifying $U_x$ with the 0 section of the trivial $G_a$ bundle, and therefore the quotient of $U$ with respect to the $G_a$ action, the restriction $\rho|_{U_x} : U_x \to X/G_a$ gives an étale morphism. The images of finitely many such morphisms $\rho|_{U_{x_i}} : U_{x_i} \to X/G_a$ cover $X/G_a$. That $\Pi_i U_{x_i} \to X/G_a$ provides an affine étale covering making $X/G_a$ an algebraic space is explained in [11, p. 39].

This description of the quotient as an algebraic space uses the complex structure. An alternative realization of the quotient of a variety $X$ by a proper action of an algebraic group $G$ as an algebraic space, valid in any characteristic, is given by Seshadri. Indeed, the construction is similar, differing in that Seshadri builds a variety $Z$ finite over $X$ from affine varieties analogous to $U$ above. The action of $G$ extends to $Z$ and is locally trivial. The quotient $W$ is separated but need not be quasiprojective. However, $\mathbb{C}(Z)/\mathbb{C}(X)$ is Galois with group $\Gamma$, $\Gamma$ acts on $W$, and the quotient of $X$ by $G$ is the algebraic space $W/\Gamma$. For the purposes of this paper the first description of the quotient is more convenient. For example, we can give a nice description of the stalks of the structure sheaf of the algebraic space $X/G_a$.

For a local ring $R$ with maximal ideal $m$, denote by $R^h$ the henselization of $R$ and by $\overline{R}$ its completion at $m$. Recall that for $R$ equal to the localization of an affine domain, $R^h$ is the algebraic closure of $R$ in $\overline{R}$.

**Proposition 3.1.** Let $z \in X/G_a$. Then $O_{z,X/G_a} \equiv \lim (O(U \times X/G_a \times X)^{G_a}) \equiv \lim (O(U \times X/G_a \times X))^{G_a} \equiv (\mathbb{C}[x_2, ..., x_n](x_2, ..., x_n))^h$, where the limit is taken over all étale open subsets $U \to X/G_a$ of $X/G_a$.

**Proof.** The isomorphism between $O_{z,X/G_a}$ and $(\mathbb{C}[x_2, ..., x_n](x_2, ..., x_n))^h$ is clear from the above construction of $X/G_a$. It is also clear that $\lim (O(U \times X/G_a \times X)^{G_a})$ maps injectively into $[\lim (O(U \times X/G_a \times X))^{G_a}]$. Take $\overline{h} \in (\lim (O(U \times X/G_a \times X))^{G_a}$. Because the action is proper, we can find an étale open $V \to X/G_a$ of $X/G_a$ with $V$ affine and $V \times X/G_a$ equivariantly isomorphic to $V \times \mathbb{C}$ with $\overline{h}$ represented by some element $h \in O(V \times \mathbb{C})$. Because $\overline{h}$ is $G_a$ invariant, for each $\lambda \in G_a$ there is an open subset $V_\lambda \times \mathbb{C} \subset V \times \mathbb{C}$ with $\lambda(h)|_{V_\lambda \times \mathbb{C}} = h|_{V_\lambda \times \mathbb{C}}$. But then the cyclic subgroup of $G_a$ generated by $\lambda$ stabilizes $h$ on $V_\lambda \times \mathbb{C}$. The stabilizer of $h$ is an algebraic subset of $G_a$ and is therefore all of $G_a$ for any $\lambda \neq 0$. Thus $h \in O(V \times \mathbb{C})^{G_a}$ and the image of $h$ in $\lim (O(U \times X/G_a \times X)^{G_a})$ is the desired preimage of $\overline{h}$. \[\square\]

**Example 1.** The action on $X = \mathbb{C}^5$ determined by the locally nilpotent derivation of $\mathbb{C}[x_1, x_2, y_1, y_2, z]$, namely

$$
\delta : x_2 \mapsto x_1 \mapsto 0, \quad y_2 \mapsto y_1 \mapsto 0, \quad z \mapsto (1 + x_1 y_2^2),
$$

is proper. Its quotient is an algebraic space that is not a scheme [3]. In particular, $W$, as in Seshadri’s construction above, is not quasiprojective.
The ring of invariants \( C_0 \) is generated by the five polynomials

\[
\begin{align*}
c_1 &= x_1, \\
c_2 &= x_2, \\
c_3 &= x_1 y_2 - x_2 y_1, \\
c_4 &= 3 y_1 z - x_1 y_2^3 - 3 y_2, \\
c_5 &= \frac{x_1^2 c_4 + c_5^3 + 3 x_1 c_5}{y_1}.
\end{align*}
\]

Set \( Y = \text{Spec } C[c_1, c_2, c_3, c_4, c_5] \), and let \( \pi : X \to Y \) be the morphism defined by the rings inclusion. One checks that the \( C_0 \) ideal \( \sqrt{C_0 \cap \text{im}(\delta)} = (c_1, c_2, c_3) \) has height 2, and that \( [C_0 \cap \text{im}(\delta)]C[X] = (x_1, y_1) \). The singular locus \( S \) of \( Y \) is one dimensional, properly contained in the zeros locus \( Z \) of \((c_1, c_2, c_3)\), and \( \pi(\pi^{-1}(Z)) \subset S \). \( \pi|_{X - \pi^{-1}(Z)} \) is a quotient morphism, but fibers over points in \( S \) are all two dimensional.

In general, for a proper action with finitely generated \( C_0 \), the universal property for geometric quotients yields a morphism of algebraic spaces \( \pi : X/G_a \to Y \) that is an isomorphism outside of a closed subset of codimension 2 in \( X/G_a \) and \( Y \) (the zero loci of \((c_1, c_2)\) in the respective spaces). Note that \( C[Y] \) is a unique factorization domain (UFD), so that if \( X/G_a \) had the structure of a variety, \( \pi \) would be an isomorphism into its image [15] Prop. 1, p. 289. In our example, however, the completions (and henselizations) of the local rings over points in \( S \) do not retain the unique factorization domain property.

To see this, we rely on the paper [15] where it is shown that the localization of the UFD \( A = C[X, Y, Z, T]/(XY - ZT + X^3 + Y^3) \) at the maximal ideal generated by the classes of \( X, Y, Z, T \) does not remain a UFD upon completion. In fact, the completion \( \hat{A} \) is shown to be isomorphic to \( C[[X, Y, Z, T]]/(XY - ZT) \), i.e., \( X^3 + Y^3 \in (XY - ZT) \) \( C[[X, Y, Z, T]] \). In the example above, \( C[Y] \cong C[c_1, c_2, c_3, c_4, c_5]/(c_2 c_5 - c_2^2 c_4 - c_3^3 - 3 c_1 c_5) \). With a simple change of variables (replacing \( c_3 \) by \( 3 c_3 \)) and the observation that \( c_3^2 + c_3^3 \in (c_2 c_5 - c_1 c_5)[C[c_1, c_2, c_3, c_4, c_5]] \), we can realize the completion of \( C[Y] \) at the maximal ideal \((c_1, c_2, c_3, c_4, c_5)\) as isomorphic to \( C[c_1, c_2, c_3, c_4, c_5]/(c_2 c_5 - c_1 K) \) for some \( K \).

**Lemma 1.** Let \( A \) be the localization of a finitely generated domain over \( C \) at the maximal ideal \( m \). Then the henselization \( \hat{A} \) of \( A \) is a unique factorization domain if and only if the completion \( \hat{A} \) of \( A \) is.

**Proof.** In this context, the henselization and strict henselization of \( A \) are equal and \( \hat{A} \cong \hat{A} \) [15] p. 38. From [3] (proof of Theorem 1), we have \( C(A^h) = C(A) \), where \( C(-) \) denotes the divisor class group. \( \square \)

**Lemma 2.** Let \( G_a \) act properly on \( X = C^n \) with geometric quotient the algebraic space \( X/G_a \). Assume that \( C_0 = C[X]^{G_a} \) is finitely generated defining the affine variety \( Y \). Denote by \( q \) the quotient morphism \( X \to Y \) induced by the ring inclusion, by \( \pi \) the quotient morphism \( X/G_a \to Y \), and by \( \hat{\pi} \) the canonical morphism \( X/G_a \to Y \). For \( z \in X/G_a \), let \( K_z \) be the quotient field of the stalk at \( z \) of the structure sheaf of \( X/G_a \), and let \( F_{\hat{\pi}(z)} \) be the quotient field of the henselization of \( C_0, m_{\hat{\pi}(z)} \). Then \( \hat{\pi} \) induces an isomorphism of \( F_{\hat{\pi}(z)} \) and \( K_z \).

**Proof.** If the action is locally trivial in the Zariski topology, then there is nothing to show (note that since \( Y \) is a variety, \( O_{\hat{\pi}(z), Y} \) is the henselization of \( C_0, m_{\hat{\pi}(z)} \)). If the
action is not Zariski locally trivial, then it is nevertheless geometrically irreducible in codimension one (GICO) \[7\], i.e., the intersection of the kernel and image of the generating derivation \(\delta\) lies in no height one prime ideal of \(\mathcal{O}[X]\) (or of \(\mathcal{O}_0\)). As a consequence, there is a closed subset \(Z\) of codimension precisely 2 in \(Y\) so that \(\pi^{-1}(Z)\) has codimension 2 in \(X\), and \(q: X - \pi^{-1}(Z) \to Y - Z\) is a principal \(G_a\) bundle, locally trivial in the Zariski topology with quotient \(Y - Z\) \[7\] (and Theorem 4.2 below) below. By the uniqueness of geometric quotients, \(Y - Z \cong X/G_a - \pi^{-1}Z\).

The result follows by appealing to an affine étale covering of \(X/G_a\) enabling the reduction to an affine neighborhood \(U\) of \(z\). A rational function on \(U\) representing an element of \(K_z\) is clearly in the function field of \(Y\) at \(\pi(z)\).

**Theorem 3.2.** Consider a proper action of \(G_a\) on \(X = \mathbb{C}^n\) with finitely generated ring of invariants \(C_0\) defining the affine variety \(Y\) and morphism \(q: X \to Y\). The action is locally trivial with quasifinite quotient if and only if for each \(x \in X\) the completion of the local ring of \(q(x)\) on \(Y\) is a unique factorization domain.

**Proof.** If the action is locally trivial, then \(q\) is a flat morphism (in the Zariski topology) whose image is in the smooth locus of \(Y\).

To prove the converse, the argument is essentially that of \[15\] p. 289, Proposition 1. Let \(y = q(x)\), \(m\) the corresponding maximal ideal of \(C_0\) and \(A = C_0/m\). If \(y\) is a smooth point of \(Y\), then the action is locally trivial in an affine neighborhood of \(q^{-1}(y)\) \[7\]; so assume that \(Y\) is singular at \(y\). From Lemma 3.2, we know that \(A^h\) is a unique factorization domain.

Since the action is proper, \(X/G_a\) exists as an algebraic space. Let \(\pi(z) = y\) for \(z \in X/G_a\), and denote by \(B = O_{z,X}/G_a\). Since \(X/G_a\) is smooth but \(Y\) is singular at \(y\), we can view \(A^h\) as a proper subring of \(B\). If \(\varphi \in B - A\), from Lemma 3.3, \(\varphi = \frac{r}{s}\), \(r, s \in A^h\). By choosing a suitable affine neighborhood of \(y\), we can assume that there is a morphism of affine varieties \(\pi: V \to W\), subvarieties \(V_1, W_2\) of \(W\) (the zero loci of \(r\) and \(s\)), whose intersection has pure codimension 2 in \(W\), and a subvariety \(V_1\) of \(V\) (the zero locus of \(s\)) of codimension 1. Since \(r = \varphi s\), every component of \(V_1\) maps to \(W_1 \cap W_2\). In particular, the set of points at which \(\pi\) is not an isomorphism has codimension 1, a contradiction. \(\square\)

**Problem 1.** For which actions does \(C_{0,m}\) remain a unique factorization domain upon completion? In particular, does this hold for proper actions on \(C^4\)?

### 4. Remarks

Related to Lemma 3.2 we have the following.

**Proposition 4.1.** Let \(k\) be a field of characteristic 0 and \(A\) an affine \(k\) algebra satisfying the following conditions:

1. \(A\) is a unique factorization domain;
2. with \(T\) denoting an indeterminate, \(A[[T]]\) is a unique factorization domain (i.e., \(A\) has discrete divisor class group, e.g. \(A\) is a regular UFD);
3. \(G_a = G_a(k)\) acts on \(A\) via the locally nilpotent derivation \(d\) with kernel \(A^d\).

Then \(A^d\) satisfies Serre’s \(S_3\) condition.

**Proof.** Consider the extension \(D\) of the derivation to \(A[[T]]\), defined by \(D\sum_{i=0}^\infty a_iT^i = \sum_{i=0}^\infty d(a_i)T^i\), and the extension of the \(G_a\) action by \(\sigma_t\sum_{i=0}^\infty a_iT^i = \sum_{i=0}^\infty \exp(td)(a_i)T^i\) for each element \(t \in k\). It is straightforward to check that
Theorem 4.3. Let $A^A = A^T = A^G$. Moreover, $A^A$ is factorial in $A^T$. Indeed, suppose that $a \in A^A$ has the factorization $a = a_1a_2 \ldots a_k$ in $A^T$. Then $G_a$ permutes the ideals $(a_i)$ inducing a homomorphism from $G_a$ to the symmetric group on $k$ letters. However, only the trivial homomorphism exists, so that $\sigma_i(a_i) = \lambda(t)a_i$ where $\lambda : G_a \rightarrow A^T$. Comparing coefficients of $T_i$, we find that $(\lambda(t) - 1)a_i = \sum_{j=1}^N \frac{t_j}{j!}d^j(a_i)$, where $N + 1$ is the least power of $\lambda(t) = 1$, we obtain a contradiction. Thus each $a_i \in A^A$, and therefore this ring is a unique factorization domain. Thus $A^d$ has discrete divisor class group and consequently satisfies Serre’s condition $S_3$. □

Theorem 4.2. Let $X$ be a smooth factorial quasifinite variety. Suppose that $G_a$ acts algebraically on $X$ and that $O(X)^G_a$ is finitely generated over $\mathbb{C}$. If $\dim X \leq 5$, then $O(X)^G_a$ is Gorenstein.

Proof. Since $O(X)^G_a$ has dimension at most 4 and satisfies $S_3$, Corollary 1.8 shows that all of its localizations are Gorenstein. □

The proposition also enables a strengthening of [11, Theorem 3.1] by removing the Cohen-Macaulay hypothesis on the ring of invariants.

Theorem 4.3. Let $X$ be a smooth factorial complex affine variety of dimension $n \geq 4$ with a GICO $G_a$ action generated by the locally nilpotent derivation $\delta$ of $O(X)$. If $O(X)^G_a$ is finitely generated and the height of the ideal image($\delta$) of $O(X)^G_a$ is at least 3, then the action is equivariantly trivial.

Proof. Let $P$ be a prime ideal of $O(X)^G_a$ minimal over image($\delta$) of $O(X)^G_a$. Set $R = O(X)^G_a$, denote the closed point of $\text{Spec} R$ by $M$, and let $U = \text{Spec} R - \{M\}$. The Cohen-Macaulay hypothesis was used to show that $\text{Ext}^1(O_U, O_U) \cong H^2_M(W, O_W) = 0$. But this follows from the $S_3$ condition.

Greuel and Pfister have conjectured [8] that any proper action of a unipotent group on an affine scheme $X$ lifts to locally trivial action on some étale covering of $X$. If by étale covering one means a finite étale morphism, then the conjecture fails for $X = \mathbb{C}^n$ and the connected unipotent group by the simple connectivity of $\mathbb{C}^n$. Indeed, suppose $X = \bigcup_{i=1}^n X_i$, with $q_i : X_i \cong X$ for each $i$. Connectivity implies that each orbit will lie in exactly one $X_i$ so that the action is locally trivial on $X_i$ and $q_i$ is $G_a$ equivariant (i.e., the action was already locally trivial on $X_i$). On the other hand, if one drops the finiteness requirement, then section 3 indicates why the conjecture does hold for $X = \mathbb{C}^n$ and proper $G_a$ actions.

References


