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DOUBLE COVERING OF CURVES

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ABSTRACT. Let C be a smooth projective algebraic curve of genus q and g an integer with $g \geq 4q+5$. For all integers $d \geq g-2q+1$ we prove the existence of a double covering $f : X \to C$ with X a smooth curve of genus g and the existence of a degree d morphism $u : X \to \mathbb{P}^1$ that does not factor through f. By the Castelnuovo-Severi inequality, the result is sharp (except perhaps the bound $g \geq 4q+5$).

1. INTRODUCTION

The main aim of this paper is the proof of the following result.

Theorem 1.1. *Fix integers g, q and d with $q \geq 0$, $g \geq 4q+5$ and $d \geq g-2q+1$. Let C be a smooth projective curve of genus q. Then there exist a double covering $f : X \to C$ with X a smooth connected curve of genus g and $M \in \mathrm{Pic}^d(X)$ with M spanned and such that the morphism $X \to \mathbb{P}(H^0(X,M))$ induced by M does not factor through f.*

If $d \leq g-2q$, no such M may exist for any genus g double covering of a smooth genus q curve by the Castelnuovo-Severi inequality (see e.g. [Ka]). Hence Theorem 1.1 seems to be quite strong. We discuss here the main differences between Theorem 1.1 and [BK1, Theorem 0.1], which was proved using a quite different method. Theorem 1.1 here has the following strong advantages with respect to [BK1, Theorem 0.1]: the bound $d \geq g-2q+1$ is optimal, while in [BK1, Theorem 0.1], we only obtained $d \geq g-q$; the curve C is arbitrary; the algebraically closed base field may have arbitrary characteristic; the bound $g \geq 4q+5$ is usually better than the bound $g \geq 5q+1$ used in [BK1]. The weak point of Theorem 1.1 here with respect to [BK1, Theorem 0.1] is that X is not an arbitrary smooth genus g covering of C and that for any such X we allow here only one integer d. The bound in [BK2, Theorem 1.1 and Proposition 1.5] is $d \geq g-q+1$, but we considered there a much harder problem about the irreducibility of $W^1_d(X)$.

For the proof of Theorem 1.1 we were motivated by the study of reducible curves, which are defined in the following way. Let C be an integral projective curve. A reducible double covering of C is a connected reduced projective curve X with

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exactly two irreducible components, say $X = X_1 \cup X_2$, equipped with a morphism $f : X \to C$ such that $f_{|X_1} : X_1 \to C$ and $f_{|X_2} : X_2 \to C$ are isomorphisms. Alternatively, X is uniquely determined by taking two curves, say X_1 and X_2, isomorphic to C, fixing isomorphisms $j_1 : X_1 \to C$, $j_2 : X_2 \to C$, a zero-dimensional subscheme Z of C, $Z \neq \emptyset$, and fixing an isomorphism between $j_1^{-1}(Z)$ and $j_2^{-1}(Z)$. We have

$$p_a(X) = 2p_a(C) + \mathrm{length}(Z) - 1 = 2p_a(C) + \mathrm{length}(X_1 \cap X_2) - 1,$$

where $X_1 \cap X_2$ is just seen as a zero-dimensional scheme.

2. Reducible coverings

Let $f : X = X_1 \cup X_2 \to C$ be a reducible and connected double covering. If C is smooth, then at each point of $X_1 \cap X_2$ the double covering X has two smooth branches and hence a planar singularity with multiplicity two (a node or a tacnode, perhaps a non-ordinary one). For every $M \in \mathrm{Pic}(X)$ we have $\deg(M) = \chi(M) - \chi(\mathcal{O}_X)$. M uniquely determines $M_1 := M_{|X_1} \in \mathrm{Pic}(X_1)$ and $M_2 := M_{|X_2} \in \mathrm{Pic}(X_2)$ and an isomorphism, j, between $M_{1|X_1 \cap X_2}$ and $M_{2|X_2 \cap X_1}$. Vice versa, at least if X is nodal at each point of $(X_1 \cap X_2)_{\mathrm{red}}$, any such triple (M_1, M_2, j) uniquely determines a line bundle on X. We have $\deg(M) = \deg(M_1) + \deg(M_2)$. If C is smooth, it is easy to extend the classical Castelnuovo-Severi inequality to reducible connected double coverings of C.

Lemma 2.1. *Let C be a smooth curve and $f : X \to C$ a reducible and connected double covering. Set $g := p_a(X)$ and $q := p_a(C)$. Fix integers d_1, d_2 with $d_1 > 0$, $d_2 > 0$ and $d_1 + d_2 \leq g - 2q$. Let $M \in \mathrm{Pic}(X)$ be a spanned line bundle such that $\deg(M_{|X_1}) = d_1$ and $\deg(M_{|X_2}) = d_2$. Then $d_1 = d_2$, and there is an $R \in \mathrm{Pic}(C)$ with $\deg(R) = d_1$, $h^0(C, R) = h^0(X, M)$ and $M \cong f^*(R)$.*

Proof. Since $d_1 + d_2 \neq 0$ and M is spanned, we have $h^0(X, M) \geq 2$. Since $\dim(X) = 1$, there is a linear subspace V of $H^0(X, M)$ with $\dim(V) = 2$ and V spanning M. Fix any such V. V induces a morphism $u : X \to \mathbb{P}^1$ with $\deg(u_{|X_1}) = d_1$ and $\deg(u_{|X_2}) = d_2$. The pair (f, u) gives a morphism $h := (f, u) : X \to C \times \mathbb{P}^1$ such that $h(X_i)$ is an irreducible curve of type $(1, d_i)$ on the ruled surface $C \times \mathbb{P}^1$. Each curve $h(X_i)$ is isomorphic to C because $f_{|X_i}$ is an isomorphism. If $h(X_1) = h(X_2)$, we obtain $d_1 = d_2$, and, up to identification of X_1 with $h(X_1)$ given by h and of $h(X_1)$ with C obtained using the first projection of $C \times \mathbb{P}^1$, we may take as R the line bundle $M_{|X_1}$. Hence we may assume $h(X_1) \neq h(X_2)$. Since the scheme $h(X_1) \cap h(X_2)$ is zero-dimensional, it has length $d_1 + d_2$ by the intersection theory of the smooth ruled surface $C \times \mathbb{P}^1$. Hence $p_a(h(X)) = 2q + d_1 + d_2 - 1$. Since $p_a(h(X)) \geq p_a(X) = g$, we obtain $d_1 + d_2 \geq g - 2q + 1$, as wanted. □

Theorem 2.2. *Fix integers g, q, d_1 and d_2 with $q \geq 0$, $d_1 \geq q + 3$, $d_2 \geq q + 3$, $d_1 + d_2 \geq g - 2q + 1$ and $g \geq 2q$. Let C be a smooth curve with $p_a(C) = q$. Then there exists a reducible and connected double covering $f : X = X_1 \cup X_2 \to C$ with X a nodal curve, $p_a(X) = g$ and $M \in \mathrm{Pic}(X)$ with $\deg(M_{|X_1}) = d_1$, $\deg(M_{|X_2}) = d_2$, M spanned by its global sections and such that the morphism $X \to \mathbb{P}(H^0(X, M))$ induced by $H^0(X, M)$ does not factor through f.*

Proof. We have

$$\mathrm{Pic}(C \times \mathbb{P}^1) \cong \mathrm{Pic}(C) \oplus \mathbb{Z}[F] \cong \mathrm{Pic}(C) \oplus \mathbb{Z},$$

where F will denote the class of a fiber of the projection $C \times \mathbb{P}^1 \to \mathbb{P}^1$ onto the second factor. For any integer a, and any $M \in \mathrm{Pic}(C)$, $M + aF$ will denote the corresponding class in $\mathrm{Pic}(C \times \mathbb{P}^1)$. We will often use the same additive notation for line bundles and divisors on $C \times \mathbb{P}^1$. By the Künneth formula for every $a \in \mathbb{Z}$ and every $M \in \mathrm{Pic}(C)$ with $h^0(C, M) = 0$, we have $h^1(C \times \mathbb{P}^1, M + aF) = 0$. By the Künneth formula for every $a < 0$ and every $M \in \mathrm{Pic}(C)$, we have $h^0(C \times \mathbb{P}^1, M + aF) = 0$. By the Künneth formula for every $a \geq 0$ and every $M \in \mathrm{Pic}(C)$, we have $h^0(C \times \mathbb{P}^1, M + aF) = (a+1)h^0(C, M)$. Fix general $M_i \in \mathrm{Pic}(C)$ with $\deg(M_i) = d_i,\ i = 1, 2$. Hence $h^1(C, M_i) = 0,\ h^0(C, M_i) = d_i + 1 - q$ and M_i is very ample. Let C_i be a general element of $|M_i + F|$. The very ampleness of M_i implies that $|M_i + F|$ separates points and tangent vectors, and gives very easily, just by a dimension count (without even using any Bertini-type theorem), that each C_i has no fiber of the first projection of $C \times \mathbb{P}^1$ as an irreducible component. Since $C_i \in |M_i + F|$, this implies that the map $C_i \to C$ induced by the first projection is an isomorphism. In particular, C_1 and C_2 are smooth. We also may assume $C_1 \neq C_2$, and hence $\mathrm{length}(C_1 \cap C_2) = (M_1 + F) \cdot (M_2 + F) = d_1 + d_2$. We claim that for a general pair (C_1, C_2), the curve $C_1 \cup C_2$ has only ordinary nodes as singularities. To check the claim we fix C_1 (any smooth $C_1 \in |M_1 + F|$ will work) and consider the linear system $|M_2 + F|$. Since M_2 is very ample, it has no base point and it separates the tangent vectors. Hence by a characteristic free form of Bertini's theorem, the restriction of $|M_2 + F|$ to C_1 is a linear system whose general member is smooth, i.e., formed by $d_1 + d_2$ distinct points. Hence the claim, i.e., for general C_1 and C_2 the curve $C_1 \cup C_2$ is nodal. Now we fix a subset S of $C_1 \cap C_2$ with $\mathrm{card}(S) = d_1 + d_2 - g + 2q - 1$. By assumption we have $d_1 + d_2 - g + 2q - 1 \geq 0$; if $d_1 + d_2 - g + 2q - 1 = 0$ we take $S = \emptyset$. Let $\pi : X \to C_1 \cup C_2$ be the partial normalization of $C_1 \cup C_2$ in which we normalize exactly the nodes of $C_1 \cup C_2$ that lie in S. Since $g \geq 2q$ we have $d_1 + d_2 > \mathrm{card}(S)$, i.e., X is a connected nodal curve with exactly two irreducible components and $p_a(X) = g$. Let $f : X \to C$ be the composition of π with the morphism $C_1 \cup C_2 \to C$ induced by the projection $C \times \mathbb{P}^1 \to C$. The pair (X, f) is the connected reducible double covering of C we were looking for. □

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1.

Proof of Theorem 1.1. The result is obvious if $d \geq g + 1$; just take a general $M \in \mathrm{Pic}^d(X)$ (which is non-special). Hence from now on we assume $d \leq g$. We will use the notation concerning $\mathrm{Pic}(C \times \mathbb{P}^1)$ introduced in the proof of Theorem 2.2. Let $\mathcal{O}$ be the structure sheaf of $C \times \mathbb{P}^1$. For any $U \in \mathrm{Pic}(C \times \mathbb{P}^1)$ and all finite subsets $S,\ S'$ of $C \times \mathbb{P}^1$ with $S \cap S' = \emptyset$, set

$$|U|(-S - 2S') := \{D \in |U| : S \subset D \text{ and } D \text{ is singular at each point of } S'\}.$$

If $S = \{P_1, \cdots, P_s\}$ and $S' = \{Q_1, \cdots, Q_t\}$, set

$$|U|(-P_1 - \cdots - P_s - 2Q_1 - \cdots - 2Q_t) := |U|(-S - 2S').$$

The canonical line bundle of $C \times \mathbb{P}^1$ is $\omega_C - 2F$. Hence by the adjunction formula for every $A \in \mathrm{Pic}^x(C),\ x > 0$, and every integer $b \geq 0$ with $|A + bF| \neq \emptyset$ and every $D \in |A + bF|$ we have $\omega_D \cong (A + \omega_C + (b-2)F)_{|D}$ and hence $p_a(D) = 1 + bx - x + bq - b$. In particular, for every $D \in |A + 2F|$ with $\deg(A) = d$ we have $p_a(D) = -1 + d + 2q$.

For any such D we have $h^1(C \times \mathbb{P}^1, \mathcal{O}(-D)) = 0$ (Künneth formula); here we use $x \neq 0$. Hence by the exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0 \tag{1}$$

we obtain $h^0(D, \mathcal{O}_D) = 1$. In particular, D is connected. First we will do the case $d = g - 2q + 1$; by assumption we also have $d \geq 2q + 6$. Fix $A \in \mathrm{Pic}^d(C)$ and take a general $X \in |A + 2F|$.

Claim I. *X is a smooth connected curve of genus g.*

Proof of Claim I. We just saw that $p_a(X) = g$ and that X is connected. Since $d \geq 2q+6$, we may find integers d_1, d_2 with $d_1 \geq q+3,\ d_2 \geq q+3$ and $d_1 + d_2 = d$. Hence, taking a general $M_i \in \mathrm{Pic}(C)$ with $\deg(M_i) = d_i$ and a general $C_i \in |M_i + F|$, we obtain a nodal curve $C_1 \cup C_2 \in |M_1 + M_2 + 2F|$ (proof of Theorem 2.2). Hence, taking $A = M_1 \otimes M_2$, we know that X has at most nodal singularities. However, since each linear system $|M_i + F|$ has no base points and separates tangent vectors, Bertini's theorem gives the smoothness of X in arbitrary characteristic. □

Notice that Claim I proves Theorem 1.1 for $d = g - 2q + 1$, because every $X \in |A + 2F|$ comes with a morphism $X \to \mathbb{P}^1$ that does not factor through the double covering $X \to C$, which is the restriction to X of the projection map $C \times \mathbb{P}^1 \to C$. Now we assume $g \geq d > g - 2q + 1$ (and again $d \geq 2q + 6$). Fix a general $A \in \mathrm{Pic}^d(C)$.

Claim II. *There is $Y \in |A + 2F|$ with Y integral and with exactly $d - g + 2q - 1$ ordinary nodes as its only singularities.*

Proof of Claim II. By Claim I, there is $B \in |A_1 + 2F|$, $\deg(A_1) = g - 2q + 1$ and A_1 general, with B smooth. Since $\dim(|A_1 + 2F|) = 3(g - 3q + 2) - 1 \geq 3(q + 7) > 2q - 1 \geq d - g + 2q - 1$, for a general $S \subset C \times \mathbb{P}^1$ with $\mathrm{card}(S) = d - g + 2q - 1$ we may find a smooth $B \in |A_1 + 2F|$ with $S \subset B$. By the generality of S, the image of S by the projection $u_1 : C \times \mathbb{P}^1 \to C$ is a set S' with $\mathrm{card}(S') = d - g + 2q - 1$. Set $T := B \cup u_1^{-1}(S')$. Since S may be considered as a general subset of B, the curve T is nodal. First assume $d = g - 2q + 2$, i.e., $\mathrm{card}(S) = 1$, say $S = \{P\}$. T has two irreducible components and exactly two nodes. By the generality of P we have $\dim(|A + 2F|(-P)) = \dim(|A + 2F|) - 1$. Furthermore, the linear system $|A + 2F|(-2P)$ separates the tangent vectors at P. Here we may use that A is very ample and that $|A + 2F|(-P)$ contains reducible elements, since

$$\dim(|A(-u_1(P)) + 2F|) = \dim(|A + 2F|) - 3$$

and a general element E of $|A(-u_1(P)) + 2F|$ does not contain P, i.e.,

$$E \cup u_1^{-1}(u_1(P)) \notin |A + 2F|(-2P).$$

Hence a general element Y of $|A + 2F|(-2P)$ does not have $u_1^{-1}(u_1(P))$ as a component. Thus Y must be irreducible and with exactly one node, as wanted. Now assume $d > g - 2q + 2$, i.e., $\mathrm{card}(S) \geq 2$. Fix $P, Q \in S$ with $P \neq Q$ and set $T_Q := B \cup u_1^{-1}(S' \setminus \{u_1(Q_1)\})$ and $T_P := B \cup u_1^{-1}(S' \setminus \{u_1(P)\})$. By induction on d we may assume that a general element $T(Q)$ (resp. $T(P)$) of $|T_Q|(-2(S \setminus \{Q\}))$ (resp. $|T_P|(-2(S \setminus \{P\}))$ is irreducible, nodal and $\mathrm{Sing}(T(Q)) = S \setminus \{Q\}$ (resp. $\mathrm{Sing}(T(P)) = S \setminus \{P\}$). Since $T(Q) \cup u_1^{-1}(u_1(Q))$ and $T(P) \cup u_1^{-1}(u_1(P))$ are both elements of $|T|(-2S)$, we easily obtain that a general element of $|T|(-2S)$ is irreducible, nodal and singular only along S. Hence we obtain Claim II for the given integer d. □

Take $Y \in |A+2F|$ with Y integral and with exactly $d-g+2q-1$ ordinary nodes as its only singularities, and let $u : X \to Y$ be the normalization. There is a degree two morphism $f : X \to C$ obtained by composing u with the restriction to Y of the projection $C \times \mathbb{P}^1 \to C$. There is a degree d morphism $v : X \to \mathbb{P}^1$ obtained by composing u with the restriction to Y of the projection $C \times \mathbb{P}^1 \to \mathbb{P}^1$, proving Theorem 1.1 for the integer d. □

It should be possible to use [R] and [S, Theorem 7(i)] to give a proof of Theorem 1.1, and this is quite easy in the cases $d = g-2q+1$ and $d = g-2q+2$. We do not know if this approach may be used to obtain the same result for some genus g with $2q-1 \le g \le 4q+4$. On the other hand, if the genus of X is relatively higher with respect to the genus of C, say $g \ge 8q+4$, we can prove the existence of a double covering $f : X \to C$ such that for every $d \ge g-2q+1$, there is a spanned $M \in \mathrm{Pic}^d(X)$ such that the morphism $X \to \mathbb{P}(H^0(X,M))$ induced by M does not factor through f.

Theorem 3.1. *Fix integers $q \ge 0$ and g with $g \ge 8q-4$. Let C be a smooth projective curve of genus q defined over an algebraically closed field of characteristic zero. Then there exists a smooth connected curve of genus g that admits a surjective degree two morphism $f : X \to C$ having a spanned $M \in \mathrm{Pic}^d(X)$ for any $d \ge g-2q+1$ such that the morphism $X \to \mathbb{P}(H^0(X,M))$ induced by M does not factor through f.*

We need the following lemma, which one can prove easily by using elementary results in Brill-Noether theory; here we adopt the conventions and notation used in [ACGH].

Lemma 3.2. *Fix an integer $e \ge 1$. Let X be a smooth curve of genus $g \ge 4e-4$ defined over an algebraically closed field of characteristic zero. For an integer d, let Σ^1_d be the union of components of $W^1_d(X)$ whose general element is base-point-free and complete. If $\Sigma^1_{g-e+1} \neq \emptyset$, then every component of Σ^1_{g-e+1} has the expected dimension $\rho(g-e+1,g,1) = g-2e$. Furthermore, we have $\Sigma^1_{g-e+2} \neq \emptyset$.*

Proof. Since it is assumed that $\Sigma^1_{g-e+1} \neq \emptyset$, any component of Σ^1_{g-e+1} has dimension at least $\rho(g-e+1,g,1) = g-2e$. Suppose there exists a component $\Sigma \subset \Sigma^1_{g-e+1}$ such that $\dim \Sigma = n \ge g-2e+1$, and take a general $L \in \Sigma$. By the base-point-free pencil trick and the description of the tangent space to the scheme $W^r_d(X)$ in general, we have

$$\begin{aligned} h^0(X,L^2) &= 2(g-e+1)-g+1+h^1(X,L^2) \\ &= 2(g-e+1)-g+1+\ker \mu_0 \\ &\ge g-2e+3+n-\rho(g-e+1,g,1) = n+3, \end{aligned}$$

where $\mu_0 : H^0(X,L) \otimes H^0(X,KL^{-1}) \to H^0(X,K)$ is the natural map given by multiplication of sections; cf. [ACGH, Proposition 4.2; page 189]. Therefore, it follows that

$$g-2e+1 \le n \le \dim W^{n+2}_{2g-2e+2}(X) \le \dim W^{g-2e+3}_{2g-2e+2}(X) = 2e-4,$$

contrary to the assumption $g \geq 4e-4$. This completes the proof of the first assertion of our lemma. Suppose now that $\Sigma^1_{g-e+2} = \emptyset$. Then, we have

$$W^1_{g-e+2}(X) = [\Sigma^1_{g-e+1} + W_1(X)] \cup [W^1_{g-e}(X) + W_2(X)].$$

Since $\dim[\Sigma^1_{g-e+1} + W_1(X)] = \rho(g-e+1, g, 1) + 1 < \rho(g-e+2, g, 1)$, it follows that the closed locus $\Sigma^1_{g-e+1} + W_1(X)$ is contained in $W^1_{g-e}(X) + W_2(X)$. Note that a general element in the locus $\Sigma^1_{g-e+1} + W_1(X)$ is a complete pencil with only one base point, whereas a complete pencil in $W^1_{g-e}(X) + W_2(X)$ has at least two base points, which is an absurdity. □

Proof of Theorem 3.1. In the proof of Theorem 1.1, it was shown that there exists a double covering $f : X \to C$ that has a spanned $M \in \mathrm{Pic}^{g-2q+1}(X)$ such that the morphism $X \to \mathbb{P}(H^0(X, M))$ induced by M does not factor through f. We will argue that for this double covering $f : X \to C$ and for every $d \geq g-2q+1$, there also exists $M' \in \mathrm{Pic}^d(X)$ such that the morphism $X \to \mathbb{P}(H^0(X, M'))$ induced by M' does not factor through f. We now take $e = 2q$ in Lemma 3.2. By Theorem 1.1, we have $\Sigma^1_{g-2q+1} \neq \emptyset$ and hence $\Sigma^1_{g-2q+2} \neq \emptyset$ by Lemma 3.2. By taking $e' = 2q-1$ in Lemma 3.2, we again have $\Sigma^1_{g-e'+2} = \Sigma^1_{g-2q+3} \neq \emptyset$; note that $g \geq 8q-4 > 4e'-4$. We may continue this process by taking smaller e's until $e = 2$ and we stop. □

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