

DESCRIPTIVE PROPERTIES
OF THE SET OF EXPOSED POINTS
OF COMPACT CONVEX SETS IN \mathbb{R}^3

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ABSTRACT. We construct a compact convex subset of \mathbb{R}^3 such that the set of its exposed points is not the intersection of an F_σ set and a G_δ set. The existence of such a set answers a question posed by G. Choquet, H.H. Corson and V.L. Klee.

Let us recall that $x \in C$ is called an *exposed point* of the convex set $C \subset \mathbb{R}^n$ if there is a real linear functional f on \mathbb{R}^n such that $f(y) < f(x)$ for all $y \in C$, $y \neq x$. Clearly, x is an exposed point of C if and only if $\{x\} = H \cap C$ for some supporting hyperplane H of C . The set of all exposed points of C is denoted by $\text{exp } C$. We use $\text{conv } A$ to denote the convex hull of A , and $\overline{\text{conv } A}$ to denote its closure.

It was shown in [1, Theorem 1.1] that if K is a compact convex set in \mathbb{R}^3 , then $\text{exp } K$ is the union of a G_δ set and a set which is the intersection of a G_δ set and an F_σ set.

V.L. Klee [3] constructed a compact convex set K in \mathbb{R}^3 such that $\text{exp } K$ is not a G_δ set. H.H. Corson [2] gave an example of a compact convex set K in \mathbb{R}^3 such that $\text{exp } K$ is not even the union of a G_δ set and an F_σ set.

The aim of our remark is to answer in the negative the following question posed by G. Choquet, H. Corson and V. Klee [1, Problem 1.2]: if K is a three-dimensional compact convex set, must $\text{exp } K$ be the intersection of a G_δ set and an F_σ set?

Theorem 1. *Given a set $E = F \cup G \subset S = \{x \in \mathbb{R}^2; \|x\| = 1\}$ such that F is an F_σ subset of S and G is a G_δ subset of S , then there is a compact convex set $C \subset \mathbb{R}^3$ with $\{x \in \mathbb{R}^2; (x, 0) \in \text{exp } C\} = F \cup G$.*

The following lemma is probably known; however, we were not able to find a reference. We shall provide the simple proof.

Lemma 2. *There are an F_σ subset F of S and a G_δ subset G of S such that their union $F \cup G$ is not the intersection of an F_σ set and a G_δ set.*

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If we apply Theorem 1 with the set constructed in Lemma 2 we obtain a negative answer to the problem of G. Choquet, H. Corson and V. Klee.

Proof of Lemma 2. As S contains an open subset homeomorphic to \mathbb{R} , it is sufficient to look for the required sets in \mathbb{R} . Also, as \mathbb{R} contains a homeomorphic copy of the Cantor set $C = \{0, 1\}^{\mathbb{N}}$, we may work in C . Finally, as the product $C^2 = C \times C$ is homeomorphic to C , it is enough to find the required sets in C^2 .

Let Q be a countable dense subset of C . Then Q^2 is an F_σ set and $(C \setminus Q)^2$ is a G_δ set. We claim that $Q^2 \cup (C \setminus Q)^2$ is not the intersection of an F_σ set and a G_δ set.

Suppose that $Q^2 \cup (C \setminus Q)^2 = E \cap H$, where E is F_σ and H is G_δ in C^2 . As E contains the second category set $(C \setminus Q)^2$, it follows that E itself is of second category in C^2 and, being an F_σ set, its interior is not empty. Suppose $U \times V \subset E$, where U and V are nonempty open subsets of C . Let $x \in U \cap Q$ be fixed. Then we have $(\{x\} \times Q) \cap (\{x\} \times V) = H \cap (\{x\} \times V)$, and thus $Q \cap V$ is a G_δ set, which is not the case. \square

Proof of Theorem 1. Let $F = \bigcup_{n \in \mathbb{N}} F_n$ and $S \setminus G = \bigcup_{n \in \mathbb{N}} E_n$, where the sets F_n, E_n ($n \in \mathbb{N}$) are closed.

For $x \in S$ and $n \in \mathbb{N}$, let $f_{x,n}$ be the real linear functional on \mathbb{R}^3 such that $f_{x,n}(0, 0, n) = 1$, $f_{x,n}(x, 0) = 1$ and $f_{x,n} \leq 1$ on S . Put $H_{x,n} = f_{x,n}^{-1}((-\infty, 1])$. Then $H_{x,n}$ is a half-space such that its boundary $\partial H_{x,n}$ touches the circle S .

Using the notation $B_+ = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3; \|(x, y)\| \leq 1, y \geq 0\}$ and $B_- = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}; (x, -y) \in B_+\}$, we define $C = C_+ \cup C_-$, where

$$C_+ = B_+ \cap \bigcap_{n \in \mathbb{N}} \bigcap_{x \in F_n} H_{x,n}$$

and

$$C_- = \overline{\text{conv}} \left(B_- \cup \left(\bigcup_{n=1}^{\infty} E_n \times \left[-\frac{1}{n}, 0 \right] \right) \right).$$

Clearly, C is compact, and it is convex since the convex sets C_+ and C_- satisfy

$$C_+ \subset \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}; \|x\| \leq 1, y \geq 0\},$$

$$C_- \subset \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}; \|x\| \leq 1, y \leq 0\},$$

and

$$C_+ \cap C_- \supset \{(x, 0) \in \mathbb{R}^2 \times \mathbb{R}; \|x\| \leq 1\}.$$

It remains to show that $\text{exp } C \cap (S \times \{0\}) = (F \cup G) \times \{0\}$.

To show that $F \times \{0\} \subset \text{exp } C$, suppose that $z_0 = (x_0, 0) \in F_n \times \{0\}$ for some $n \in \mathbb{N}$. Then the supporting plane $\{(x, 2y) \in \mathbb{R}^2 \times \mathbb{R}; f_{x_0,n}(x, y) = 1\}$ meets C in the single point z_0 , thus $z_0 \in \text{exp } C$.

Let $x_0 \in G$ and $z = (x_0, y) \in C$. We shall show that necessarily $y = 0$, and so the point $z_0 = (x_0, 0)$ is an exposed point of C (it is even an exposed point of the set $\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}; \|x\| \leq 1, x \neq x_0\} \cup \{(x_0, 0)\}$ which contains C). Clearly $y \leq 0$, as $C_+ \subset B_+$. In what follows we are using the fact that the convex hull of a compact subset of \mathbb{R}^3 is compact (and thus closed) [4, Theorem 17.2]. For an arbitrary $n \in \mathbb{N}$ we have

$$C_- = \text{conv} \left(\text{conv} \left(B_- \cup \bigcup_{k=1}^n (E_k \times [-\frac{1}{k}, 0]) \right) \cup \overline{\text{conv}} \left(\bigcup_{k=n}^{\infty} (E_k \times [-\frac{1}{k}, 0]) \right) \right).$$

Thus $z = (x_0, y) = \alpha_1 \cdot (x_1, y_1) + \alpha_2 \cdot (x_2, y_2)$, where α_1, α_2 are nonnegative reals with $\alpha_1 + \alpha_2 = 1$, $(x_1, y_1) \in \text{conv}(B_- \cup \bigcup_{k=1}^n (E_k \times [-\frac{1}{k}, 0]))$ and $(x_2, y_2) \in \overline{\text{conv}} \bigcup_{k=n}^\infty (E_k \times [-\frac{1}{k}, 0])$. Since $\|x_0\| = 1$ and $\|x_i\| \leq 1$ ($i = 1, 2$), it follows that $x_0 = x_1 = x_2$. Thus $y_1 = 0$, as $B_- \cap (S \times (-\infty, 0)) = \emptyset$, and $x_1 = x_0 \in G \subset S \setminus \bigcup_{k=1}^n E_k$. Therefore $y = \alpha_2 y_2 \geq \alpha_2 \cdot (-\frac{1}{n}) \geq -1/n$, and, as $n \in \mathbb{N}$ was arbitrary, we have $y = 0$.

To prove the other inclusion, we suppose that $x \in S \setminus (F \cup G)$ is such that $(x, 0) \in \text{exp } C$, and we proceed to a contradiction. As $x \notin G$, it belongs to E_{n_0} for some $n_0 \in \mathbb{N}$. Thus $\{x\} \times [-\frac{1}{n_0}, 0] \subset C$, and necessarily $\partial H_{x, n_1} \cap C = f_{x, n_1}^{-1}(1) \cap C = \{(x, 0)\}$ for some sufficiently large $n_1 \in \mathbb{N}$.

On the other hand, we shall prove that for any $n \in \mathbb{N}$ (we shall use it for $n = n_1$) there is an $\varepsilon > 0$ such that

$$(1) \quad B_+ \cap \text{conv} \left((S \times \{0\}) \cup \{(0, 0, n)\} \right) \cap \text{conv} \left\{ (x, 0), (x, \varepsilon), (0, 0, 1) \right\} \subset C.$$

Suppose for a while that (1) is proved. Then we make the observation that there is an element distinct from $(x, 0)$ in the set

$$\partial H_{x, n_1} \cap B_+ \cap \text{conv} \left((S \times \{0\}) \cup \{(0, 0, n_1)\} \right) \cap \text{conv} \left\{ (x, 0), (x, \varepsilon), (0, 0, 1) \right\} \cap C,$$

which contradicts the fact that $f_{x, n_1}^{-1}(1) \cap C = \{(x, 0)\}$.

It remains to prove (1). As $\text{dist}(x, \bigcup_{k=1}^n F_k) > 0$, there is an $\varepsilon > 0$ such that

$$\bigcap_{k=1}^n \bigcap_{x' \in F_k} H_{x', k} \supset \bigcap_{k=1}^n \bigcap_{x' \in F_k} H_{x', 1} \supset \text{conv} \left\{ (x, 0), (x, \varepsilon), (0, 0, 1) \right\}.$$

Also

$$\bigcap_{k=n}^\infty \bigcap_{x' \in F_k} H_{x', k} \supset \bigcap_{k=n}^\infty \bigcap_{x' \in F_k} H_{x', n} \supset \text{conv} \left((S \times \{0\}) \cup \{(0, 0, n)\} \right).$$

Now the inclusion (1) follows from the definition of C_+ . □

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