THE IDENTITY IS ISOLATED
AMONG COMPOSITION OPERATORS

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ABSTRACT. Let $H^\infty(B)$ be the Banach algebra of bounded holomorphic functions on the open unit ball $B$ of a Banach space. We show that the identity operator is an isolated point in the space of composition operators on $H^\infty(B)$. This answers a conjecture of Aron, Galindo and Lindström.

1. INTRODUCTION

Let $B$ be the open unit ball of a complex Banach space $E$, and let $H^\infty(B)$ be the uniform algebra of bounded complex-valued holomorphic functions on $B$, with the supremum norm $\|f\| = \sup_{x \in B} |f(x)|$. Given any holomorphic self-map $\phi$ of $B$, we define the composition operator $C_\phi : H^\infty(B) \to H^\infty(B)$ by

$$C_\phi(f) = f \circ \phi \quad (f \in H^\infty(B)).$$

The collection of such operators with the operator norm topology is denoted by $C(H^\infty(B))$. This space has been widely studied, and recently, Aron, Galindo and Lindström [1] have determined its path connected components for some special Banach spaces $E$, thereby extending results of MacCluer, Ohno and Zhao [6] for the case when $B$ is the open unit disc $\Delta$ in the complex plane. A main result in [1] is

**Theorem 1.1** ([1, Theorem 16]). If $E = C_0(X)$ or $E$ is a Hilbert space, then the composition operators $C_\phi$ and $C_\psi$ lie in the same path connected component in $C(H^\infty(B))$ if and only if $\|C_\phi - C_\psi\| < 2$.

Furthermore, using techniques involving w-strong peak points and determining sets for $H^\infty(B)$ when $B$ belongs to some special Banach spaces, the following result is established.

**Theorem 1.2** ([1, Corollary 12]). The identity operator is an isolated point in $C(H^\infty(B))$ when $E$ is $C_0(X)$ or $\ell_1$ or any strictly convex reflexive Banach space.

Two open questions were raised in [1]. First, does Theorem 1.1 hold when $E$ is a $JB^*$-triple? A positive answer to this question has been given in [7]. The second conjecture is that Theorem 1.2 holds for every Banach space $E$. We give a positive
answer and a simple proof in this paper. We use only the hyperbolic metric, but
do not require w-strong peak points.

2. Proof of the conjecture

The space of all complex-valued homomorphisms on $H^\infty(B)$ forms the maximal
ideal space of $H^\infty(B)$ and contains, in particular, the point evaluation functionals
$\{\delta_x : x \in B\}$. The pseudo-hyperbolic distance on the maximal ideal space is defined by

$$\beta(m, n) = \text{sup}\{|\hat{f}(n)| : f \in H^\infty(B), \|f\| \leq 1, \hat{f}(m) = 0\}$$

where $\hat{f}$ is the Gelfand transform of $f$. We note from [11] Remark 2 that

$$(1) \quad ||C_\phi - C_\psi|| < 2 \quad \text{if and only if} \quad \text{sup } \beta(\delta_\phi(x), \delta_\psi(y)) < 1.$$

The Carathéodory distance on $B$ is given by

$$C_B(x, y) = \text{sup}\{\gamma(f(x), f(y)) : f \in H(B, \Delta)\}$$

for $x, y \in B$, where $\gamma$ is the Poincaré metric on the disc $\Delta$ and $H(B, \Delta)$ the space
of holomorphic maps from $B$ to $\Delta$. Both $C_B$ and $\beta$ are contracted by holomorphic
functions and preserved by biholomorphic functions.

The metric

$$d_B(x, y) := \text{sup}\{|f(x) - f(y)| : f \in H^\infty(B), \|f\| \leq 1\} \quad (x, y \in B)$$

and its relation to the Carathéodory distance $C_B$ are examined in [7] where it is shown that

$$(2) \quad d_B(x, y) = \frac{2 - 2\sqrt{1 - (\tanh C_B(x, y))^2}}{\tanh C_B(x, y)} \quad (x, y \in B).$$

We have $d_B(x, y) \geq d_\Delta(h(x), h(y))$ for any $h \in H(B, \Delta)$. Also,

$$(3) \quad d_B(x, y) \leq 2\text{sup}\{|f(x)| : f \in H^\infty(B), \|f\| \leq 1, f(y) = 0\}.$$

To see this, it suffices to note that for any $f \in H^\infty(B)$ with $\|f\| \leq 1$, the function
$f^y$ defined by $f^y(x) = \frac{1}{y}(f(x) - f(y))$ is also in the closed unit ball of $H^\infty(B)$.

Let $E^*$ be the dual of a complex Banach space $E$. We denote the unit spheres
of $E$ and $E^*$ by $S(E)$ and $S(E^*)$ respectively. Given $x \in S(E)$, we denote the set
of support functionals of $x$ by

$$\text{supp} \{x\} = \{f \in E^* : \|f\| = 1, f(x) = 1\}.$$

Let $T : E \rightarrow E$ be a bounded complex linear operator. We recall that the spatial numerical range of $T$ is defined by

$$V(T) = \{f(Tx) : x \in S(E), f \in \text{supp} \{x\}\}$$

(cf. [3]). By [3] Theorem 3.9.4, the numerical radius $v(T)$ of $T$ is given by

$$v(T) = \text{sup}\{|\lambda| : \lambda \in V(T)\}.$$

We also have, by [3] Theorem 1.4.1,

$$(4) \quad \|T\| \geq v(T) \geq \frac{1}{e}\|T\|.$$

We now prove the conjecture that subsumes Theorem [12].

**Theorem 2.1.** Let $E$ be a Banach space with open unit ball $B$. Then the identity
operator is an isolated point in the space of composition operators on $H^\infty(B)$. 

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Proof. Let \( I : H^\infty(B) \to H^\infty(B) \) be the identity operator, and suppose that \( C_\phi \) is in the component of \( I \) for some holomorphic self-map \( \phi \) of \( B \). We show that \( \phi \) is the identity map on \( B \). We have \( \| C_\phi - I \| < 2 \) by \([5]\) and \( \sup_{x \in B} \beta(\delta_{\phi(x)}, \delta_x) < 1 \) by \([3]\). Since

\[
\beta(\delta_{\phi(x)}, \delta_x) = \sup \{ \| \hat{f}(\delta_{\phi(x)}) \| : f \in H^\infty(B), \| f \| \leq 1, \hat{f}(\delta_x) = 0 \}
\]

\[
= \sup \{ \| f(\phi(x)) \| : f \in H^\infty(B), \| f \| \leq 1, f(x) = 0 \}
\]

\[
\geq \frac{1}{2} \sup \{ |f(\phi(x)) - f(x)| : f \in H^\infty(B), \| f \| \leq 1 \}
\]

we see that

\[
\sup_{x \in B} d_B(\phi(x), x) < 2
\]

and hence \([2]\) gives

\[
\sup_{x \in B} C_B(\phi(x), x) < \infty
\]

or that

\[
\sup_{x \in B, f \in H(B, \Delta)} \gamma(f(x), f(\phi(x))) < \infty.
\]

Let \( \lambda \in S(E^*) \) be norm-attaining; that is, there exists \( x_\lambda \in S(E) \) with \( \lambda(x_\lambda) = 1 \). Define \( \psi : \Delta \to \Delta \) by \( \psi(\zeta) = \lambda(\phi(\zeta x_\lambda)) \). Then \( \psi \) is holomorphic, and we have

\[
\sup_{\zeta \in \Delta} \gamma(\zeta, \psi(\zeta)) = \sup_{\zeta \in \Delta} \gamma(\lambda(\zeta x_\lambda), \lambda(\phi(\zeta x_\lambda)))
\]

\[
\leq \sup_{x \in B} \gamma(\lambda(x), \lambda(\phi(x)))
\]

\[
\leq \sup_{x \in B, f \in H(B, \Delta)} \gamma(f(x), f(\phi(x))) < \infty.
\]

Since \( \gamma(\zeta, \psi(\zeta)) = \tanh^{-1} \beta(\zeta, \psi(\zeta)) \) on \( \Delta \), we have \( \sup_{\Delta} \beta(\zeta, \psi(\zeta)) < 1 \) and it follows from the one-dimensional result that \( \psi = id_\Delta \). Hence we have

\[
(5) \quad \zeta = \lambda(\phi(\zeta x_\lambda)) \quad (\zeta \in \Delta).
\]

In particular, we have \( \lambda(\phi(0)) = 0 \). By the Bishop-Phelps theorem \([2]\), the norm-attaining functionals in \( E^* \) are norm-dense in \( E^* \). Therefore \( \phi(0) = 0 \). Let us write \([4]\) in the form

\[
id_\Delta = \lambda \circ \phi \circ i_{x_\lambda}
\]

where \( i_{x_\lambda} : \Delta \to B \) is the map \( i_{x_\lambda}(\zeta) = \zeta x_\lambda \). Taking the derivative at \( \zeta \in \Delta \) of both sides we obtain

\[
1 = \lambda(\phi'(\zeta x_\lambda)(x_\lambda))
\]

which gives

\[
1 = \lambda(\phi'(0)x_\lambda).
\]

The above arguments imply that, for any \( x \in S(E) \) and \( f \in \text{supp}(x) \), we have

\[
1 = \lambda(\phi'(0)x_\lambda).
\]

The above arguments imply that, for any \( x \in S(E) \) and \( f \in \text{supp}(x) \), we have

\[
V(T) = \{ f(Tx) : x \in S(E), f \in \text{supp}(x) \}
\]

\[
= \{ f(\phi'(0)x) - f(x) : x \in S(E), f \in \text{supp}(x) \}
\]

\[
= \{0\}.
\]

It follows from \([4]\) that \( \|T\| = 0 \). Hence \( \phi'(0) = I \). Since we have already established that \( \phi(0) = 0 \), Cartan’s uniqueness theorem asserts that \( \phi \) itself is the identity map on \( B \) as required (see \([4]\) Proposition 6.6)). \( \Box \)
Corollary 2.2. Let $E$ be a Banach space and $\psi$ a biholomorphic self-map of the open unit ball $B$ of $E$. Then $C_\psi$ is isolated in $C(H^\infty(B))$.

Proof. The result is true for $\psi = \text{id}$ from above. Now observe that $C_\psi$ is a homeomorphism of $C(H^\infty(B))$ that takes the identity to the composition operator $C_\psi$. □

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