

HYPERBOLIC DERIVATIVES AND GENERALIZED SCHWARZ-PICK ESTIMATES

PRATIBHA GHATAGE AND DECHAO ZHENG

(Communicated by Joseph A. Ball)

ABSTRACT. In this paper we use the beautiful formula of Faa di Bruno for the n th derivative of composition of two functions to obtain the generalized Schwarz-Pick estimates. By means of those estimates we show that the hyperbolic derivative of an analytic self-map of the unit disk is Lipschitz with respect to the pseudohyperbolic metric.

1. INTRODUCTION

For each $z \in D$, let φ_z denote the Möbius transformation of D ,

$$\varphi_z = \frac{z - w}{1 - \bar{z}w},$$

for $w \in D$. The pseudo-hyperbolic distance on D is defined by

$$\rho(z, w) = |\varphi_z(w)|, \quad z, w \in D.$$

The pseudohyperbolic distance is Möbius invariant, that is,

$$\rho(gz, gw) = \rho(z, w),$$

for all $g \in \text{Aut}(D)$, the Möbius group of D , and all $z, w \in D$. It has the following useful property:

$$(1.1) \quad 1 - \rho(z, w)^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{z}w|^2}.$$

The Bergman metric on D is the hyperbolic metric whose element of length is

$$ds = \frac{|dz|}{1 - |z|^2}.$$

In this metric every rectifiable arc γ has a length

$$\int_{\gamma} \frac{|dz|}{1 - |z|^2}.$$

It is easy to show that the induced distance on D is given by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)},$$

for $z, w \in D$.

Received by the editors July 9, 2003 and, in revised form, August 12, 2003.

2000 *Mathematics Subject Classification*. Primary 30C80.

The second author was supported in part by the National Science Foundation.

Let φ be an analytic self-map of the unit disk. By the classical Schwarz-Pick Lemma [2], [5], φ decreases the hyperbolic distance between two points and the hyperbolic length of an arc. The explicit inequality is

$$\left| \frac{\varphi(z_1) - \varphi(z_2)}{1 - \overline{\varphi(z_1)}\varphi(z_2)} \right| \leq \left| \frac{z_1 - z_2}{1 - \overline{z_1}z_2} \right|$$

for any z_1, z_2 in D . In particular,

$$(1.2) \quad \frac{|\varphi'(z)|}{1 - |\varphi(z)|^2} \leq \frac{1}{1 - |z|^2}$$

for z in D . Let

$$\tau_\varphi(z) = \frac{(1 - |z|^2)\varphi'(z)}{1 - |\varphi(z)|^2}.$$

Then

$$|\tau_\varphi(z)| \leq 1,$$

for all $z \in D$. Nontrivial equality $|\tau_\varphi(z)| = 1$ holds for some $z \in D$ only when φ is a fractional linear transformation $e^{i\theta}\varphi_a(z)$. For each $z \in D$, the hyperbolic derivative of φ at z is defined by

$$\lim_{\beta(z,w) \rightarrow 0} \frac{\beta(\varphi(z), \varphi(w))}{\beta(z, w)}.$$

In Section 3 we will show that the hyperbolic derivative of φ equals $|\tau_\varphi(z)|$ and that $\tau_\varphi(z)$ is Lipschitz with respect to the pseudohyperbolic metric. Hyperbolic derivatives have been used in studying composition operators on the Bloch space [7], [9] and [10].

Recently, MacCluer, Stroethoff, and Zhao [8] used the formula of Faa di Bruno and the theory of the weighted composition operators [11] to obtain the generalized Schwarz-Pick estimates:

$$(1.3) \quad \sup_{z \in D} \frac{(1 - |z|^2)^n |\varphi^{(n)}(z)|}{(1 - |\varphi(z)|^2)} < \infty$$

for each analytic self-map φ of the unit disk. We obtain the following generalized Schwarz-Pick estimates: for each $0 < r < 1$ and each positive integer n , there is a positive constant $M_{n,r}$ such that for each analytic self-map φ of the unit disk:

$$(1.4) \quad \frac{(1 - |z|^2)^n |\varphi^{(n)}(z)|}{1 - |\varphi(z)|^2} \leq M_{n,r} \max_{\rho(w,z) < r} \frac{(1 - |w|^2) |\varphi'(w)|}{1 - |\varphi(w)|^2},$$

for z in D . Clearly, Combining (1.2) with (1.4) gives (1.3). Moreover, (1.4) directly leads to the result [8] that if φ is in the little Bloch class, then

$$\lim_{|z| \rightarrow 1} \frac{(1 - |z|^2)^n |\varphi^{(n)}(z)|}{1 - |\varphi(z)|^2} = 0$$

for each n . The main tool is the beautiful formula of Faa di Bruno [13] for the n th derivative of the composition of two functions.

Based on the generalized Schwarz-Pick estimates we will show in Section 3 that $\tau_{\varphi,n}(z) = \frac{(1 - |z|^2)^n |\varphi^{(n)}(z)|}{(1 - |\varphi(z)|^2)}$ is Lipschitz with respect to the pseudohyperbolic metric. Thus $\tau_{\varphi,n}(z)$ admits a continuous extension to the set of nontrivial Gleason parts of the maximal ideal space of H^∞ .

2. GENERALIZED SCHWARZ-PICK ESTIMATES

In this section, we will present a proof of the generalized Schwarz-Pick estimates. The generalized Schwarz-Pick estimates will be used in the proof of Theorem 6. The main tool is the beautiful formula of Faa di Bruno, which deals with the n th derivative of composition of an analytic function f on the unit disk with a self-map φ of the the unit disk [13].

Theorem 1 (The Formula of Faa di Bruno). *If φ is an analytic function from the unit disk to the unit disk and if f is an analytic function on the unit disk, then*

$$(f \circ \varphi)^{(n)}(z) = \sum \frac{n!}{k_1!k_2! \cdots k_n!} f^{(k)}(\varphi(z)) \prod_{j=1}^n \left(\frac{\varphi^{(j)}(z)}{j!}\right)^{k_j}$$

where $k = k_1 + \cdots + k_n$ and the sum is over all k_1, \dots, k_n for which $k_1 + 2k_2 + \cdots + nk_n = n$.

The following result is well known [12]. We include a proof to motivate our Theorem 2.

Proposition 1. *If φ is a univalent analytic self-map of D , then*

$$(1 - |z|^2)|\varphi''(z)| \leq 10|\varphi'(z)|$$

for all $z \in D$.

Proof. For a fixed z in D , let h be the Koebe transform of φ ,

$$h(w) = \frac{\varphi\left(\frac{w+z}{1+\bar{z}w}\right) - \varphi(z)}{(1 - |z|^2)\varphi'(z)}.$$

Then $h(0) = 0$ and $h'(0) = 1$. It follows from Bieberbach’s theorem ([12], page 8) that

$$|h''(0)| \leq 4.$$

On the other hand, an easy computation gives

$$h''(0) = \frac{1}{2}(1 - |z|^2)\frac{\varphi''(z)}{\varphi'(z)} - \bar{z}.$$

Hence

$$\left|\frac{1}{2}(1 - |z|^2)\frac{\varphi''(z)}{\varphi'(z)} - \bar{z}\right| \leq 4.$$

Since $|z| \leq 1$, we conclude that

$$|(1 - |z|^2)|\varphi''(z)| \leq 10|\varphi'(z)|.$$

This completes the proof. □

As a consequence of the proposition, we have

$$(2.1) \quad \frac{(1 - |z|^2)^2|\varphi''(z)|}{1 - |\varphi(z)|^2} \leq \frac{10(1 - |z|^2)|\varphi'(z)|}{1 - |\varphi(z)|^2}$$

for all $z \in D$ if φ is a univalent self-map of the unit disk.

Example. Let b be an interpolating Blaschke product with zeros $\{z_n\}$ in the unit disk and $\varphi = b^2$. Clearly, $\varphi'(z_n) = 0$ and $\varphi''(z_n) = 2[b'(z_n)]^2$. Let $\delta = \inf_{z_n} (1 - |z_n|^2)|b'(z_n)|$. Thus

$$\frac{(1 - |z_n|^2)|\varphi''(z_n)|}{1 - |\varphi(z_n)|^2} = 2(1 - |z_n|^2)[b'(z_n)]^2 \geq 2\delta|b'(z_n)| \geq \frac{2\delta^2}{1 - |z_n|^2}.$$

So the inequality (2.1) does not hold for some analytic self-maps of the unit disk. But by means of the formula of Faa di Bruno we still have the generalized Schwarz-Pick estimates:

Theorem 2. *For each positive integer n and each number $0 < r < 1$, there is a positive constant $M_{n,r}$ such that for each analytic self-map φ of the unit disk,*

$$\frac{(1 - |z|^2)^n |\varphi^{(n)}(z)|}{1 - |\varphi(z)|^2} \leq M_{n,r} \max_{\rho(w,z) < r} \frac{(1 - |w|^2) |\varphi'(w)|}{1 - |\varphi(w)|^2}$$

for z in D .

As we mentioned in the introduction, by the Schwarz-Pick estimates (1.2), we have

$$\frac{(1 - |w|^2) |\varphi'(w)|}{1 - |\varphi(w)|^2} \leq 1.$$

Thus Theorem 2 implies the following generalized Schwarz-Pick Estimates [8].

Theorem 3. *For each n , there is a positive constant M_n such that for each analytic self-map φ of the unit disk,*

$$\frac{(1 - |z|^2)^n |\varphi^{(n)}(z)|}{1 - |\varphi(z)|^2} \leq M_n,$$

for z in D .

If φ is in the little Bloch class, i.e.,

$$\frac{(1 - |z|^2) |\varphi'(z)|}{1 - |\varphi(z)|^2} \rightarrow 0$$

as $|z| \rightarrow 1$, then noting that for the given $0 < s < 1$, for every $w \in D$ with $\rho(w, z) < s$, $|w| \rightarrow 1$ as $|z| \rightarrow 1$, Theorem 2 gives the following result in [8].

Theorem 4. *Let φ be an analytic self-map of the unit disk. If*

$$\lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2) |\varphi'(z)|}{1 - |\varphi(z)|^2} = 0,$$

then for each n ,

$$\lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^n |\varphi^{(n)}(z)|}{1 - |\varphi(z)|^2} = 0.$$

Proof of Theorem 2. For a fixed z in D , let $g = \varphi \circ \varphi_z$. Clearly, $g(0) = \varphi(z)$. By the formula of Faa di Bruno, we have

$$g^{(n)}(w) = \sum \frac{n!}{k_1! k_2! \dots k_n!} \varphi^{(k)}(\varphi_z(w)) \prod_{j=1}^n \left(\frac{\varphi_z^{(j)}(w)}{j!}\right)^{k_j}$$

where $k = k_1 + \dots + k_n$ and the sum is over all k_1, \dots, k_n for which $k_1 + 2k_2 + \dots + nk_n = n$.

Evaluating the value of $g^{(n)}$ at 0 gives

$$g^{(n)}(0) = \sum \frac{n!}{k_1!k_2! \dots k_n!} \varphi^{(k)}(\varphi_z(0)) \prod_{j=1}^n \left(\frac{\varphi_z^{(j)}(0)}{j!}\right)^{k_j}.$$

Noting that $\varphi_z(0) = z$ and $\varphi_z^{(j)}(w) = -(1 - |z|^2)\bar{z}^{j-1}j!(1 - \bar{z}w)^{-j-1}$, we have

$$\begin{aligned} g^{(n)}(0) &= \sum \frac{n!}{k_1!k_2! \dots k_n!} \varphi^{(k)}(z) \prod_{j=1}^n (-1 - |z|^2)\bar{z}^{j-1}j!^{k_j} \\ &= \sum (-1)^k \frac{n!}{k_1!k_2! \dots k_n!} \varphi^{(k)}(z)(1 - |z|^2)^k \bar{z}^{n-k}. \end{aligned}$$

The last equality follows from $k_1 + \dots + k_n = k$ and $k_1 + 2k_2 + \dots + nk_n = n$.

Thus

$$(-1)^n(1 - |z|^2)^n \varphi^{(n)}(z) = g^{(n)}(0) - \sum_{k < n} (-1)^k \frac{n!}{k_1!k_2! \dots k_n!} \varphi^{(k)}(z)(1 - |z|^2)^k \bar{z}^{n-k}.$$

So

$$\begin{aligned} \frac{(1 - |z|^2)^n |\varphi^{(n)}(z)|}{1 - |\varphi(z)|^2} &\leq \frac{|g^{(n)}(0)|}{1 - |\varphi(z)|^2} \\ &+ \sum_{k < n} \frac{n!}{k_1!k_2! \dots k_n!} \frac{|\varphi^{(k)}(z)|(1 - |z|^2)^k |z|^{n-k}}{1 - |\varphi(z)|^2}. \end{aligned}$$

Let $M_k(z) = \frac{|\varphi^{(k)}(z)|(1 - |z|^2)^k}{1 - |\varphi(z)|^2}$. The above inequality gives

$$M_n(z) \leq \frac{|g^{(n)}(0)|}{1 - |\varphi(z)|^2} + \sum_{k < n} \frac{n!}{k_1!k_2! \dots k_n!} M_k(z).$$

We need to estimate $\frac{|g^{(n)}(0)|}{1 - |\varphi(z)|^2}$.

Let $\lambda = g(0)$, $h = \varphi_\lambda \circ g$. Then h is still an analytic self-map of the unit disk, $h(0) = 0$, and $\|h\|_\infty \leq 1$. Since $\varphi_\lambda \circ \varphi_\lambda(z) = z$, we obtain $g = \varphi_\lambda \circ h$. The formula of Faa di Bruno again gives

$$g^{(n)}(w) = \sum \frac{n!}{k_1!k_2! \dots k_n!} \varphi_\lambda^{(k)}(h(w)) \prod_{j=1}^n \left(\frac{h^{(j)}(w)}{j!}\right)^{k_j}$$

where $k = k_1 + \dots + k_n$ and the sum is over all k_1, \dots, k_n for which $k_1 + 2k_2 + \dots + nk_n = n$.

Evaluating $g^{(n)}$ at 0 gives

$$g^{(n)}(0) = \sum \frac{n!}{k_1!k_2! \dots k_n!} \varphi_\lambda^{(k)}(0) \prod_{j=1}^n \left(\frac{h^{(j)}(0)}{j!}\right)^{k_j}$$

since $h(0) = 0$. Noting $\varphi_\lambda^{(k)}(w) = -(1 - |\lambda|^2)\bar{\lambda}^{k-1}k!(1 - \bar{\lambda}w)^{-k-1}$, the above equality leads to

$$g^{(n)}(0) = \sum \frac{n!}{k_1!k_2! \dots k_n!} [-(1 - |\lambda|^2)\bar{\lambda}^{k-1}k!] \prod_{j=1}^n \left(\frac{h^{(j)}(0)}{j!}\right)^{k_j}.$$

Hence

$$\frac{|g^{(n)}(0)|}{1 - |g(0)|^2} \leq \sum \frac{n!}{k_1!k_2! \cdots k_n!} |\lambda|^{k-1} k! \prod_{j=1}^n \left(\frac{|h^{(j)}(0)|}{j!} \right)^{k_j}.$$

Let $a_n = \sum_{k < n} \frac{n!}{k_1!k_2! \cdots k_n!}$. So far we have shown

$$M_n(z) \leq a_n \max_{k < n} M_k(z) + \sum \frac{n!}{k_1!k_2! \cdots k_n!} |\lambda|^{k-1} k! \prod_{j=1}^n \left(\frac{|h^{(j)}(0)|}{j!} \right)^{k_j}.$$

Note that $h = \varphi_\lambda \circ g$, $g = \varphi \circ \varphi_z$, and $\lambda = g(0) = \varphi(z)$. Then

$$h'(w) = \frac{(1 - |\lambda|^2)g'(w)}{(1 - \bar{\lambda}g(w))^2}$$

and

$$h'(w) = \sum_{j=1}^{\infty} \frac{h^{(j)}(0)}{(j-1)!} w^{j-1}.$$

Let $0 < r < 1$. Thus

$$h^{(j)}(0) = r^{-(j-1)}(j-1)! \frac{1}{2\pi} \int_0^{2\pi} h'(re^{i\theta}) e^{-i(j-1)\theta} d\theta.$$

So

$$\begin{aligned} |h^{(j)}(0)| &\leq r^{-(j-1)}(j-1)! \frac{1}{2\pi} \int_0^{2\pi} |h'(re^{i\theta})| d\theta \\ &\leq r^{-(j-1)}(j-1)! \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - |\lambda|^2)|g'(re^{i\theta})|}{|1 - \bar{\lambda}g(re^{i\theta})|^2} d\theta \\ &\leq r^{-(j-1)}(j-1)! \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - |\varphi_\lambda(g(re^{i\theta}))|^2)|g'(re^{i\theta})|}{1 - |g(re^{i\theta})|^2} d\theta \\ &\leq r^{-(j-1)}(1 - r^2)^{-1}(j-1)! \max_{|w| \leq r} \frac{(1 - |w|^2)|g'(w)|}{1 - |g(w)|^2} \\ &\leq C_r \frac{1}{r^{(j-1)}}(j-1)! \max_{\rho(z,u) \leq r} \frac{(1 - |u|^2)|\varphi'(u)|}{1 - |\varphi(u)|^2} \end{aligned}$$

for some constant $C_r > 0$. The third inequality follows from

$$\frac{(1 - |\lambda|^2)(1 - |g(re^{i\theta})|^2)}{|1 - \bar{\lambda}g(re^{i\theta})|^2} = 1 - |\varphi_\lambda(g(re^{i\theta}))|^2.$$

The last inequality follows from making the change of variable $u = \varphi_z(w)$ and the fact that

$$\begin{aligned} (1 - |w|^2)|g'(w)| &= (1 - |w|^2)|\varphi'(\varphi_z(w))\varphi'_z(w)| \\ &= \frac{(1 - |w|^2)(1 - |z|^2)}{|1 - \bar{z}w|^2} |\varphi'(\varphi_z(w))| = (1 - |\varphi_z(w)|^2)|\varphi'(\varphi_z(w))|. \end{aligned}$$

Hence

$$\frac{|h^{(j)}(0)|}{j!} \leq [jr^{(j-1)}(1 - r^2)]^{-1} \max_{\rho(z,u) \leq r} \frac{(1 - |u|^2)|\varphi'(u)|}{1 - |\varphi(u)|^2}.$$

The Schwarz-Pick estimate gives

$$\frac{(1 - |u|^2)|\varphi'(u)|}{1 - |\varphi(u)|^2} \leq 1$$

for each $u \in D$. Thus

$$\begin{aligned} & \sum \frac{n!}{k_1!k_2! \cdots k_n!} |\lambda|^{k-1} k! \prod_{j=1}^n \left(\frac{|h^{(j)}(0)|}{j!} \right)^{k_j} \\ & \leq \sum \frac{n!}{k_1!k_2! \cdots k_n!} |\lambda|^{k-1} k! r^{k-n} (1-r^2)^{-k} \max_{\rho(z,u) \leq r} \frac{(1-|u|^2)|\varphi'(u)|}{1-|\varphi(u)|^2}. \end{aligned}$$

Let $b_{n,r} = \sum \frac{n!}{k_1!k_2! \cdots k_n!} k! r^{k-n} (1-r^2)^{-k}$. The above inequality gives

$$M_n(z) \leq a_n \max_{k < n} M_k(z) + b_{n,r} \max_{\rho(z,u) \leq r} \frac{(1-|u|^2)|\varphi'(u)|}{1-|\varphi(u)|^2}.$$

By the induction, we conclude that

$$M_n(z) \leq M_{n,r} \max_{\rho(z,u) \leq r} \frac{(1-|u|^2)|\varphi'(u)|}{1-|\varphi(u)|^2}$$

to complete the proof. □

3. HYPERBOLIC DERIVATIVES

In this section we will first show that the hyperbolic derivative of an analytic self-map φ of the unit disk equals $|\tau_\varphi(z)|$. Then we will show that $\tau_\varphi(z)$ is Lipschitz with respect to the pseudo-hyperbolic metric.

Theorem 5. *Let $\varphi : D \rightarrow D$ be an analytic self-map. Then, for each point $z \in D$, the hyperbolic derivative of φ is equal to*

$$\lim_{\beta(z,w) \rightarrow 0} \frac{\beta(\varphi(z), \varphi(w))}{\beta(z, w)} = |\tau_\varphi(z)|.$$

Proof. Assume that φ is not constant. For each fixed $z \in D$, $\rho(\varphi(z), \varphi(w))$ converges to zero as $\beta(w, z)$ converges to zero because φ is continuous in D and $|\varphi(z)| < 1$. Thus

$$\lim_{\beta(z,w) \rightarrow 0} \frac{\beta(\varphi(z), \varphi(w))}{\beta(z, w)} = \lim_{\beta(z,w) \rightarrow 0} \frac{\beta(\varphi(z), \varphi(w))}{\rho(\varphi(z), \varphi(w))} \frac{\rho(\varphi(z), \varphi(w))}{\rho(z, w)} \frac{\rho(z, w)}{\beta(z, w)}.$$

Both the first and third factors of the product on the right converge to one. Now the second factor is

$$\frac{\rho(\varphi(z), \varphi(w))}{\rho(z, w)} = \frac{|\varphi(z) - \varphi(w)|}{|z - w|} \frac{|1 - \bar{z}w|}{|1 - \overline{\varphi(z)\varphi(w)}|}.$$

Thus

$$\lim_{\beta(z,w) \rightarrow 0} \frac{\rho(\varphi(z), \varphi(w))}{\rho(z, w)} = \frac{|\varphi'(z)|(1-|z|^2)}{1-|\varphi(z)|^2}.$$

So

$$\lim_{\beta(z,w) \rightarrow 0} \frac{\beta(\varphi(z), \varphi(w))}{\beta(z, w)} = \frac{|\varphi'(z)|(1-|z|^2)}{1-|\varphi(z)|^2}.$$

This completes the proof. □

For each n , define

$$\tau_{\varphi,n}(z) = \frac{(1-|z|^2)^n \varphi^{(n)}(z)}{1-|\varphi(z)|^2}.$$

Theorem 6. Let φ be an analytic self-map of the unit disk D . Then for each n , $\tau_{\varphi,n}(z)$ is Lipschitz with respect to the pseudohyperbolic metric. More precisely,

$$|\tau_{\varphi,n}(z) - \tau_{\varphi,n}(w)| \leq C_n \rho(z, w)$$

for any $z, w \in D$. Here C_n is a positive constant only depending on n .

Proof. Suppose that f is a differentiable function on the unit disk. Let $\partial_z f = \frac{\partial f}{\partial z}$ and $\partial_{\bar{z}} f = \frac{\partial f}{\partial \bar{z}}$. Note that $\tau_{\varphi,n}(z)$ is differentiable on the unit disk. Easy calculations give

$$\partial_{\bar{z}} \tau_{\varphi,n}(z) = \frac{-zn(1-|z|^2)^{n-1} \varphi^{(n)}(z)(1-|\varphi(z)|^2) + (1-|z|^2)^n \varphi^{(n)}(z) \overline{\varphi'(z)} \varphi(z)}{(1-|\varphi(z)|^2)^2}$$

and

$$\begin{aligned} \partial_z \tau_{\varphi,n}(z) &= \frac{1}{(1-|\varphi(z)|^2)^2} \{ [(1-|z|^2)^n \varphi^{(n+1)}(z) - \bar{z}n(1-|z|^2)^{n-1} \varphi^{(n)}(z)] \\ &\quad \times (1-|\varphi(z)|^2) + (1-|z|^2)^n \varphi^{(n)}(z) \varphi'(z) \overline{\varphi(z)} \}. \end{aligned}$$

Thus

$$\begin{aligned} |\partial_{\bar{z}} \tau_{\varphi,n}(z)| &\leq \frac{1}{1-|z|^2} \left[\frac{n(1-|z|^2)^n |\varphi^{(n)}(z)|}{1-|\varphi(z)|^2} \right. \\ &\quad \left. + |\varphi(z)| \left[\frac{(1-|z|^2) |\varphi'(z)|}{1-|\varphi(z)|^2} \right] \left[\frac{(1-|z|^2)^n |\varphi^{(n)}(z)|}{1-|\varphi(z)|^2} \right] \right] \leq \frac{(n+1)M_n}{1-|z|^2}, \end{aligned}$$

where the last inequality follows from Theorem 3, and

$$\begin{aligned} |\partial_z \tau_{\varphi,n}(z)| &\leq \frac{1}{1-|z|^2} \left\{ \frac{(1-|z|^2)^{n+1} |\varphi^{(n+1)}(z)|}{1-|\varphi(z)|^2} + n \frac{(1-|z|^2)^n |\varphi^{(n)}(z)|}{1-|\varphi(z)|^2} \right. \\ &\quad \left. + |\varphi(z)| \left[\frac{(1-|z|^2)^n |\varphi^{(n)}(z)|}{1-|\varphi(z)|^2} \right] \left[\frac{(1-|z|^2) |\varphi'(z)|}{1-|\varphi(z)|^2} \right] \right\} \leq \frac{M_{n+1} + (n+1)M_n}{1-|z|^2}, \end{aligned}$$

where the last inequality follows from Theorem 3. Given z and w in D , let $\gamma(t) : [0, 1] \rightarrow D$ be a smooth curve to connect z and w , i.e.,

$$\begin{aligned} |\tau_{\varphi,n}(z) - \tau_{\varphi,n}(w)| &= \left| \int_0^1 \frac{d\tau_{\varphi,n}(\gamma(t))}{dt} dt \right| \\ &\leq \int_0^1 \left| \frac{d}{dt} \tau_{\varphi,n}(\gamma(t)) \right| dt \\ &\leq \int_0^1 \left[|\partial_z \tau_{\varphi,n}(\gamma(t))| \left| \frac{d\gamma(t)}{dt} \right| + |\partial_{\bar{z}} \tau_{\varphi,n}(\gamma(t))| \left| \frac{d\bar{\gamma}(t)}{dt} \right| \right] dt, \end{aligned}$$

where the last inequality follows from the first chain rule:

$$\frac{d}{dt} \tau_{\varphi,n}(\gamma(t)) = \partial_z \tau_{\varphi,n}(\gamma(t)) \frac{d\gamma(t)}{dt} + \partial_{\bar{z}} \tau_{\varphi,n}(\gamma(t)) \frac{d\bar{\gamma}(t)}{dt}.$$

Combining the above estimates gives

$$|\tau_{\varphi,n}(z) - \tau_{\varphi,n}(w)| \leq \int_{\gamma} \frac{M_{n+1} + 2(n+1)M_n}{1-|\gamma(t)|^2} d|\gamma(t)|.$$

If we choose γ to be a geodesic to connect z and w , then the above inequality gives

$$\begin{aligned} |\tau_{\varphi,n}(z) - \tau_{\varphi,n}(w)| &\leq (M_{n+1} + 2(n + 1)M_n)\beta(z, w) \\ &\leq \frac{(M_{n+1} + 2(n + 1)M_n)\rho(z, w)}{1 - \rho(z, w)^2}. \end{aligned}$$

The last inequality comes from the fact that for all $0 < x < 1$,

$$\frac{1}{2} \ln \frac{1+x}{1-x} \leq \frac{x}{1-x^2}.$$

If $|\rho(z, w)| < 1/8$, the above inequality gives

$$|\tau_{\varphi,n}(z) - \tau_{\varphi,n}(w)| \leq 2(M_{n+1} + 2(n + 1)M_n)\rho(z, w).$$

If $|\rho(z, w)| \geq 1/8$, we have $8|\rho(z, w)| \geq 1$, and

$$|\tau_{\varphi,n}(z) - \tau_{\varphi,n}(w)| \leq \max\{|\tau_{\varphi,n}(z)|, |\tau_{\varphi,n}(w)|\} \leq M_n \leq 8M_n\rho(z, w).$$

Choosing $C_n = \max\{2(M_{n+1} + 2(n + 1)M_n), 8M_n\}$, we have

$$|\tau_{\varphi,n}(z) - \tau_{\varphi,n}(w)| \leq C_n\rho(z, w)$$

to complete the proof. □

Theorem 6 has an application to closed-range composition operators on the Bloch space.

Hoffman [6] showed that $(1 - |z|^2)^n \varphi^{(n)}(z)$ continuously extends to the maximal ideal space of H^∞ . Let \mathcal{G} be the subset of the maximal ideal space of H^∞ consisting of nontrivial Gleason parts. As a corollary of a result in [1] and Theorem 1.2 [4], we have the following result.

Corollary 1. *Suppose that φ is an analytic self-mapping of the unit disk. Then $\tau_{\varphi,n}(z)$ admits a continuous extension to \mathcal{G} .*

ADDENDUM

After we finished this paper, we obtained K. Stroethoff's paper [14], which showed that

$$\rho(|\tau_\varphi(z)|, |\tau_\varphi(w)|) \leq 2\rho(z, w),$$

for $z, w \in D$. This generalizes Beardon's result [3]: If $\varphi(0) = 0$, then

$$\rho(\tau_\varphi(0), \tau_\varphi(w)) < 2\rho(0, w)$$

for $w \in D$. We thank K. Stroethoff.

REFERENCES

[1] S. Axler and K. Zhu, *Boundary behavior of derivatives of analytic functions*, Michigan Math. J. 39 (1992), 129–143. MR 93e:30073
 [2] L. V. Ahlfors, *Conformal invariants: topics in geometric function theory*, McGraw-Hill, New York, 1973. MR 50:10211
 [3] A. Beardon, *The Schwarz-Pick Lemma for derivatives*, Proc. Amer. Math. Soc. 125 (1997), 3255–3256. MR 97m:30062
 [4] A. Brudnyi, *Topology of the maximal ideal space of H^∞* , J. Funct. Analysis 189 (2002), 21–52. MR 2003c:46066
 [5] J. Garnett *Bounded analytic functions*, Academic Press, New York, 1981. MR 83g:30037
 [6] K. Hoffman, *Bounded analytic functions and Gleason parts*, Ann. Math. 86 (1967), 74–111. MR 35:5945
 [7] P. Ghatage, J. Yan and D. Zheng, *Composition operators with closed range on the Bloch space*, Proc. Amer. Math. Soc. 129 (2001), 2039–2044. MR 2002a:47034

- [8] B. MacCluer, K. Stroethoff, and R. Zhao, *Generalized Schwarz-Pick estimates*, Proc. Amer. Math. Soc. 131 (2003), 593–599. MR 2003g:30038
- [9] K. Madigan and A. Matheson, *Compact composition operators on the Bloch space*, Trans. Amer. Math. Soc. 347 (1995), 2679–2687. MR 95i:47061
- [10] A. Montes-Rodríguez, *The essential norm of a composition operator on Bloch spaces*, Pacific J. Math. 188 (1999), 339–351. MR 2000d:47044
- [11] S. Ohno, K. Stroethoff and R. Zhao, *Weighted composition operators between Bloch-type space*, Rocky Mountain J. Math. 33 (2003), 191–215. MR 2004d:47058
- [12] Ch. Pommerenke, *Boundary behaviour of conformal maps*, Springer-Verlag 299, New York, 1991. MR 95b:30008
- [13] S. Roman, *The Formula of Faa di Bruno*, Amer. Math. Monthly 87 (1980), 805–809. MR 82d:26003
- [14] K. Stroethoff, *Lecture notes on The Schwarz-Pick Lemma for derivatives*, Preprint.

DEPARTMENT OF MATHEMATICS, CLEVELAND STATE UNIVERSITY, CLEVELAND, OHIO 44115
E-mail address: `p.ghatage@csuohio.edu`

DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY, NASHVILLE, TENNESSEE 37240
E-mail address: `zheng@math.vanderbilt.edu`