

## AN EXTREMAL PROBLEM OF QUASICONFORMAL MAPPINGS

ZHONG LI, SHENGJIAN WU, AND ZEMIN ZHOU

(Communicated by Juha M. Heinonen)

ABSTRACT. In this paper, the following problem is studied. Let  $\Omega_1$  and  $\Omega_2$  be two domains in the complex plane with  $\Omega_1 \cap \Omega_2 \neq \emptyset$ . Suppose that  $f_j : \Omega_j \rightarrow f_j(\Omega_j)$  ( $j = 1, 2$ ) are two quasiconformal mappings satisfying  $f_1|_{\Omega_1 \cap \Omega_2} = f_2|_{\Omega_1 \cap \Omega_2}$ . Let  $F$  be the mapping in  $\Omega_1 \cup \Omega_2$  defined by  $F|_{\Omega_j} = f_j$  ( $j = 1, 2$ ). If both  $f_1$  and  $f_2$  are uniquely extremal, is  $F$  always uniquely extremal? It is shown in this paper that the answer to this problem is no.

### §1. INTRODUCTION

Let  $\Omega$  and  $D$  be two domains in the complex plane  $\mathbb{C}$  and let  $f : \Omega \rightarrow D$  be a quasiconformal mapping from  $\Omega$  onto  $D$ . This means that  $f$  is an orientation-preserving homeomorphism of  $\Omega$  onto  $D$  with locally  $L^2$ -generalized derivatives  $\partial_z f$  and  $\partial_{\bar{z}} f$  which satisfy the Beltrami equation

$$\partial_{\bar{z}} f(z) = \mu(z) \partial_z f(z), \quad z \in \Omega,$$

where  $\mu$  is a bounded measurable function with  $\|\mu\|_\infty < 1$ . The function  $\mu$  is called the Beltrami coefficient of  $f$ , and

$$K[f] = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}$$

is called the maximal dilatation of  $f$ . It is well known that a quasiconformal mapping can be continuously extended to the accessible boundary points of  $\Omega$ . So the boundary values of a quasiconformal mapping between two domains whose boundaries consist of Jordan arcs and isolated points are well defined. In what follows we always assume that the domains under consideration have such boundaries.

A quasiconformal mapping  $f : \Omega \rightarrow D$  is said to be extremal if  $K[f] \leq K[g]$  for any quasiconformal mapping  $g : \Omega \rightarrow D$  with  $g|_{\partial\Omega} = f|_{\partial\Omega}$ . If  $f$  is an extremal quasiconformal mapping and  $K[f] < K[g]$  for any quasiconformal mapping  $g : \Omega \rightarrow D$  with  $g|_{\partial\Omega} = f|_{\partial\Omega}$  and  $g \neq f$ , then  $f$  is said to be uniquely extremal.

A basic problem in the theory of quasiconformal mappings is to determine whether a given quasiconformal mapping is extremal or uniquely extremal, and to characterize the uniquely extremal mapping (see [Ah], [BLM], [LV], [RS1], [RS2]

---

Received by the editors December 3, 2002 and, in revised form, July 15, 2003.

2000 *Mathematics Subject Classification*. Primary 30C75, 30C62.

The first author was supported by the 973-Project Foundation of China (Grant TG199075105) and the second author was supported by the NNSF of China (Grants 10171003 and 10231040).

and so on). There has been important progress in characterizing uniquely extremality in recent years (see [BLM] and [Re]).

In his paper [Re1], E. Reich studied the following problem. Let  $\Omega_1$  and  $\Omega_2$  be two domains with  $\Omega_1 \cap \Omega_2 = \emptyset$  and  $\partial\Omega_1 \cap \partial\Omega_2 = \gamma$ , where  $\gamma$  is a Jordan arc. Let  $f_j$  be a quasiconformal mapping of  $\Omega_j$  ( $j = 1, 2$ ) with  $f_1|_\gamma = f_2|_\gamma$  and let  $F$  be the quasiconformal mapping in  $\Omega = \Omega_1 \cup \Omega_2 \cup \gamma$  defined by  $F|_{\Omega_j} = f_j$  ( $j = 1, 2$ ). If both  $f_1$  and  $f_2$  are uniquely extremal, is  $F$  uniquely extremal? Reich provided a counterexample to this problem in [Re1].

In this paper, we shall study the following problem, posed by Chen Jixiu and Shen Yuliang [CS], which is an improvement of the above problem. Let  $\Omega_1$  and  $\Omega_2$  be two domains with  $\Omega_1 \cap \Omega_2 \neq \emptyset$ . Suppose that  $f_1 : \Omega_1 \rightarrow D_1$  and  $f_2 : \Omega_2 \rightarrow D_2$  are two quasiconformal mappings satisfying  $f_1|_{\Omega_1 \cap \Omega_2} = f_2|_{\Omega_1 \cap \Omega_2}$ . Let  $F$  be the mapping in  $\Omega_1 \cup \Omega_2$  defined by  $F|_{\Omega_j} = f_j$  ( $j = 1, 2$ ). If both  $f_1$  and  $f_2$  are uniquely extremal, is  $F$  always uniquely extremal?

This problem is also connected with many other studies in characterizing unique extremality of quasiconformal mappings (see Theorems 2.3, 3.1, 4.1 and Example 5.3.1 in [Re]).

In this paper, we will construct some counterexamples where  $F$  is not uniquely extremal or even not extremal.

## §2. SOME COUNTEREXAMPLES

We look at the quadratic differential

$$\varphi(z)dz^2 = \frac{dz^2}{(z^2 - 1)^2}.$$

Then  $\varphi dz^2$  is a holomorphic quadratic differential in  $\overline{\mathbb{C}} \setminus \{1, -1\}$  and has poles of order two at  $z = 1$  and  $z = -1$ . It is easy to see that

$$\varphi(x)dx^2 = \frac{dx^2}{(x^2 - 1)^2} > 0, \quad \text{for all } x \in \mathbb{R} \setminus \{1, -1\},$$

and

$$\varphi(iy)d(iy)^2 = \frac{-dy^2}{(y^2 + 1)^2} < 0, \quad \text{for all } y \in \mathbb{R}.$$

So the intervals  $(-\infty, -1)$ ,  $(-1, 1)$  and  $(1, +\infty)$  are horizontal trajectories of  $\varphi dz^2$  and the imaginary axis is a vertical trajectory of  $\varphi dz^2$ .

To study the trajectory structure of the quadratic differential  $\varphi dz^2$ , we look at the function

$$w = \Phi(z) := \frac{1}{2} \log \frac{z-1}{z+1} - i\frac{1}{2}\pi,$$

which is a single-valued holomorphic function in  $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$  with  $\Phi(i) = -i\pi/4$ . The function  $w = \Phi(z)$  maps  $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$  to the strip

$$\Sigma := \{u + iv \mid -\infty < u < +\infty; -\frac{1}{2}\pi < v < \frac{1}{2}\pi\}.$$

On the other hand,

$$\{\Phi'(z)\}^2 = \varphi(z).$$

So each horizontal line in  $\Sigma$  corresponds to a horizontal trajectory of  $\varphi dz^2$  and each vertical segment in  $\Sigma$  corresponds to a vertical trajectory of  $\varphi dz^2$ .

Let

$$u(z) := \operatorname{Re} \Phi(z) = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right|$$

and let

$$v(z) := \operatorname{Im} \Phi(z) = \frac{1}{2} [\arg(z-1) - \arg(z+1)] - \frac{1}{2} \pi.$$

Then  $u(z) = C$  ( $C \in \mathbb{R}$ ) is a horizontal trajectory of  $\varphi dz^2$  and  $v(z) = C$  ( $C \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ ) is a vertical trajectory of  $\varphi dz^2$ .

It is easy to check that  $u(z) = C$  is a circle with the radius

$$r = \left( \frac{1+e^C}{1-e^C} \right)^2 - 1$$

and the center  $z = (1+e^C)/(1-e^C)$ . One can easily see that, for each  $C \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ ,  $v(z) = C$  is a circle (or a straight line) that passes through 1 and  $-1$ , for each point at which, the circumference angle with respect to 1 and  $-1$  is  $C + \pi/2$  or  $\pi/2 - C$ .

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be the quasiconformal mapping of  $\mathbb{C}$  onto  $\mathbb{C}$  with the Beltrami coefficient

$$\mu = k \frac{\bar{\varphi}}{|\varphi|}, \quad \text{where } k \in (0, 1),$$

keeping the points 0 and  $i$  fixed.

The quasiconformal mapping  $w = f(z)$  can also be got by the following construction.

Let  $\zeta = g(w)$  be a stretch mapping of  $\Sigma$  onto itself defined by

$$w = u + iv \mapsto \zeta = Ku + iv,$$

where  $K = (1+k)/(1-k)$ . Then the mapping  $\tilde{f} : z \mapsto \Phi^{-1} \circ g \circ \Phi(z)$  is a quasiconformal mappings of  $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$  onto itself with the Beltrami coefficient  $\mu = k\bar{\varphi}/|\varphi|$ . Obviously,  $\tilde{f}$  can be extended to the slits  $(-\infty, -1]$  and  $[1, +\infty)$  so that  $\tilde{f}$  is a quasiconformal mapping of the whole plane. By the above construction we see that every point on the imaginary axis is fixed by  $\tilde{f}$ . From the uniqueness theorem of quasiconformal mappings, we see that  $\tilde{f} \equiv f$ .

Now we give our first counterexample, as follows.

**Example 1.** Let

$$\Omega_1 = \{z = x + iy \mid x < 0, x^2 + (y-1)^2 < 2; \text{ or } x < 0, x^2 + (y+1)^2 < 2\}$$

and let

$$\Omega_2 = \{z = x + iy \mid (x+2)^2 + y^2 < 1\}.$$

Let  $f_j$  be the restriction of  $f$  to  $\Omega_j$  ( $j = 1, 2$ ). Note that  $\Phi \circ f_1 \circ \Phi^{-1}$  is a stretch mapping of the trip

$$\{w = u + iv \mid u > 0; -3\pi/8 < v < 3\pi/8\}.$$

By a result of Strebel [St],  $\Phi \circ f_1 \circ \Phi^{-1}$  is uniquely extremal, and hence so is  $f_1$ .

Since  $\varphi$  is meromorphic on  $\bar{\Omega}_2$  and has only one pole of order 2 at  $z = -1$ , it follows from a result of Sethares [Se] that  $f_2$  is also uniquely extremal.

Let  $F := f|_{\Omega_1 \cup \Omega_2}$ . Then  $F$  is not uniquely extremal. In fact,  $\Omega = \Omega_1 \cup \Omega_2$  is bounded by the imaginary axis and 3 circles with a puncture  $z = -1$ , and the mapping  $F$  is a Teichmüller mapping associated with a quadratic differential  $\varphi dz^2$  that has a pole at  $z = -1$ . Taking a sufficiently small disk  $D$  around  $z = -1$ ,

then  $F$  is smooth on  $\partial D$  and hence the boundary dilatation of  $F|_{D \setminus \{-1\}}$  is one. Then there is a Teichmüller mapping  $\tilde{F}$  of  $D \setminus \{-1\}$ , associated with a holomorphic quadratic differential of finite norm, such that

$$\tilde{F}|_{\partial D \cup \{-1\}} = F|_{\partial D \cup \{-1\}}.$$

As the quadratic differential associated with  $\tilde{F}$  is regular on  $D$  or has a pole of order one at  $z = -1$ , we see that  $\tilde{F}$  is different from  $F|_{D \setminus \{-1\}}$ . Therefore  $F = f|_{\Omega}$  is not uniquely extremal.

This example shows that the union of two uniquely extremal quasiconformal mappings is still extremal, but it is not uniquely extremal. We shall give another example which shows that the union of two uniquely extremal quasiconformal mappings need not even be extremal.

**Example 2.** Let  $K > 1$  be a real number and let  $f_K(z) := |z|^{K-1}z$ . Then  $f_K$  is a quasiconformal mapping of  $\mathbb{C}$  onto itself with 0 and 1 fixed and with the Beltrami coefficient  $\mu = kz/\bar{z}$ , where  $k = (K-1)/(K+1)$ . The Beltrami coefficient  $\mu$  can be expressed as  $\mu = k\bar{\phi}/|\phi|$ , where

$$\phi(z) = \frac{1}{z^2}.$$

We consider

$$\Omega_1 := \{x + iy \mid x^2 + y^2 < 1 \text{ and } y > x, \text{ or } y < -x\}$$

and

$$\Omega_2 := \{x + iy \mid x^2 + y^2 < 1 \text{ and } x > 0\}.$$

Let  $f_j = f_K|_{\Omega_j}$  ( $j = 1, 2$ ) and let  $F = f_K|_{\Omega}$ , where  $\Omega = \Omega_1 \cup \Omega_2$ .

Both  $f_1$  and  $f_2$  are uniquely extremal. In fact,  $f_1$  can be expressed as

$$f_1 = \Psi^{-1} \circ g \circ \Psi(z), \quad z \in \Omega_1,$$

where  $\Psi$  is a single-valued branch of  $-\log z$  on  $\mathbb{C}$  with a slit  $[0, +\infty)$  and  $g$  is a stretch mapping  $u \mapsto Ku$ ,  $v \mapsto v$  of a strip. Making use of the result in [St] again, we see that  $f_1$  is uniquely extremal. Similarly, we can prove that  $f_2$  is also uniquely extremal.

On the other hand, it is easy to see that  $\Omega = \Delta \setminus \{0\}$ , where  $\Delta$  is the unit disc, and the boundary correspondence of  $F : \Omega \rightarrow \Omega$  is the identity. Obviously,  $F$  is not extremal.

In the above examples, the union of domains  $\Omega$  is doubly connected and one component of its boundary is an isolated point. Now we are going to give other examples in which the union of domains is a simply connected domain and its boundary consists of Jordan arcs.

To construct such examples, we need a new result [Ma] obtained by V. Marković. It says that an affine stretch of the plane  $\mathbb{C}$  punctured at integer lattices is uniquely extremal.

**Example 3.** Let  $\mathbb{Z}$  be the set of integer numbers. Define

$$\Lambda_1 := \{m + ni \in \mathbb{C} \mid m \in \mathbb{Z}; n \in \mathbb{Z}\}$$

and

$$\Lambda_2 := \{m + (n + \frac{1}{2})i \in \mathbb{C} \mid m \in \mathbb{Z}; n \in \mathbb{Z}\}.$$

Let

$$\Omega = \{x + yi \mid y > \max\{C, x^2\}\},$$

where  $C > 0$  is a constant. Now we consider the stretch mapping

$$g_K : x \mapsto Kx; y \mapsto y,$$

where  $K > 1$ . It is known that  $g_K|_\Omega$  is extremal but not uniquely extremal (see [AH] or [RS2]).

Let

$$\Omega_1 := \Omega \setminus \{x + ni \in \Omega \mid n \in \mathbb{Z}; x \geq x'_n\},$$

where  $x'_n$  the smallest number of the set  $A_n := \{m \mid m \in \mathbb{Z}, m + ni \in \Omega\}$ , and let

$$\Omega_2 := \Omega \setminus \{x + (n + \frac{1}{2})i \mid n \in \mathbb{Z}; x \geq x''_n\},$$

where  $x''_n$  is the smallest number of the set  $B_n := \{m \mid m \in \mathbb{Z}, m + (n + \frac{1}{2})i \in \Omega\}$ .

Now define  $f_j = g_K|_{\Omega_j}$  ( $j = 1, 2$ ). Then both  $f_1$  and  $f_2$  are uniquely extremal. In fact, any quasiconformal mapping  $g$  of  $\Omega_1$  with  $g|_{\partial\Omega_1} = f_1|_{\partial\Omega_1}$  can be extended to the whole plane by defining  $g = g_K$  outside of  $\Omega$ . The resulting mapping  $g$  has the same values on  $\Lambda_1$  as  $g_K$ . It follows from the result in [Ma] that  $g_K$  is extremal with respect to the boundary correspondence

$$\Lambda_1 \rightarrow g_K(\Lambda_1) : m + ni \mapsto Km + ni, \quad m, n \in \mathbb{Z}.$$

So  $K[g] \geq K = K[f_1]$ , and hence  $f_1 = g_K|_{\Omega_1}$  is extremal with respect to its boundary correspondence. The uniqueness of the extremal mapping  $g_K$  with respect to the stretch of  $\Lambda_1$  also implies the uniqueness of the extremal mapping  $f_1$  with respect to its boundary correspondence. Thus  $f_1$  is uniquely extremal. Similarly, we can also conclude that  $f_2$  is uniquely extremal.

However, it is known that  $F = g_K|_\Omega$  is extremal but not uniquely extremal.

If we consider the domain

$$\{x + yi \mid y > \max\{C, |x|\}\}$$

instead of the parabolic domain in Example 3, and use a result of Reich and Strebel (cf. [AH], [Re2] and [RS3]), then we can get another example which shows that the union of two uniquely extremal mappings is not extremal ([RS3] or [AH]).

#### ACKNOWLEDGEMENT

The authors would like to thank the referee for his or her helpful suggestions.

#### REFERENCES

- [Ah] L. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Nostrand, New York, 1966. MR **34**:336
- [AH] M. Anderson and A. Hinkkanen, *Quadrilaterals and extremal quasiconformal extensions*, Comment. Math. Helv. **70** (1995), 455-474. MR **96g**:30042
- [BLM] V. Božin, N. Lakic, V. Marković and M. Mateljić, *Unique extremality*, J. Anal. Math. **75** (1998), 299-338. MR **2000a**:30045
- [CS] J. Chen and Y. Shen, *Oral communication*.
- [LV] O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, Springer-Verlag, 1973. MR **49**:9202
- [Ma] V. Marković, *Extremal problems for quasiconformal mappings of punctured plane domains*, Trans. Amer. Math. Soc. **354** (2002.) 1631-1650. MR **2002j**:30074
- [Re] E. Reich, *Extremal Quasiconformal mapping of the Disk*, in the book "Handbook of Complex Analysis: Geometric function theory, Volume 1", Edited by R.Kühnau, Elsevier Science B.V., 2002, pp. 75-135. MR **2004c**:30036
- [Re1] E. Reich, *An extremum problem for analytic functions with area norm*, Ann. Acad. Sci. Fenn. Ser. A. I. Math. **2** (1976), 429-445. MR **58**:17102

- [Re2] E. Reich, *Uniqueness of Hahn-Banach extensions from certain spaces of analytic functions*, Math. Z. **167** (1979), 81-89. MR **80j**:30074
- [RS1] E. Reich and K. Strebel, *On quasiconformal mappings which keep the boundary points fixed*, Trans. Amer. Math. Soc. **138** (1969), 211-222. MR **38**:6059
- [RS2] E. Reich and K. Strebel, *Extremal quasiconformal mappings with given boundary values, in the book "Contributions to Analysis (a collection of papers dedicated to Lipman Bers)"*, Academic Press, 1974, pp. 375-392. MR **50**:13511
- [RS3] E. Reich and K. Strebel, *On the extremality of certain Teichmüller mappings*, Comment. Math. Helv. **45** (1970), 353-362. MR **43**:514
- [Se] G. C. Sethares, *The extremal property of certain Teichmüller mappings*, Comment. Math. Helv. **43** (1968), 98-119. MR **37**:4253
- [St] K. Strebel, *On the extremality and unique extremality of quasiconformal mappings of a parallel strip*, Rev. Roumaine Math. Pures Appl. **32** (1987), 923-928. MR **89f**:30042

SCHOOL OF MATHEMATICAL SCIENCES, LMAM, PEKING UNIVERSITY, BEIJING 100871,  
PEOPLE'S REPUBLIC OF CHINA

*E-mail address*: [lizhong@math.pku.edu.cn](mailto:lizhong@math.pku.edu.cn)

SCHOOL OF MATHEMATICAL SCIENCES, LMAM, PEKING UNIVERSITY, BEIJING 100871,  
PEOPLE'S REPUBLIC OF CHINA

*E-mail address*: [wusj@math.pku.edu.cn](mailto:wusj@math.pku.edu.cn)

SCHOOL OF MATHEMATICAL SCIENCES, LMAM, PEKING UNIVERSITY, BEIJING 100871,  
PEOPLE'S REPUBLIC OF CHINA

*E-mail address*: [zeminzhou2000@163.com](mailto:zeminzhou2000@163.com)