

CLASS NUMBERS OF QUADRATIC FIELDS $\mathbb{Q}(\sqrt{D})$ AND $\mathbb{Q}(\sqrt{tD})$

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ABSTRACT. Let t be a square free integer. We shall show that there exist infinitely many positive fundamental discriminants $D > 0$ with a positive density such that the class numbers of quadratic fields $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt{tD})$ are both not divisible by 3.

1. INTRODUCTION

Let r and s be the 3-rank of the ideal class group of a real quadratic field $\mathbb{Q}(\sqrt{D})$ and an imaginary quadratic field $\mathbb{Q}(\sqrt{-3D})$. Scholz [7] showed that

$$r \leq s \leq r + 1.$$

This is a classical case of Leopoldt's reflection theorem. On the other hand, the Davenport-Heilbronn theorem [3] and a subsequent refinement by Nakagawa and Horie [6] state that there exist infinitely many positive fundamental discriminants $D > 0$ with a positive density such that the class numbers of the imaginary quadratic fields $\mathbb{Q}(\sqrt{-3D})$ are not divisible by 3. Thus we can make the following observation:

There exist infinitely many positive fundamental discriminants $D > 0$ with a positive density such that the class numbers of quadratic fields $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt{-3D})$ are both not divisible by 3.

Recently, combining this observation and the Gross-Zagier theorem [1], [2] on the Heegner points and derivatives of L -series, Vatsal [8] obtained a positive proportion of rank-one quadratic twists of the modular elliptic curve $X_0(19)$. The aim of this paper is to extend the above observation to the pair of fields $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt{tD})$ with any square free integer t .

Theorem 1.1. *Let t be a square free integer. Then there exist infinitely many positive fundamental discriminants $D > 0$ with a positive density such that the class numbers of quadratic fields $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt{tD})$ are both not divisible by 3.*

From this theorem and the class number product formula of bicyclic biquadratic fields, due to Kubota [5], we can easily obtain the following corollary.

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Corollary 1.2. *Let t be a square free integer such that the class number of the quadratic field $\mathbb{Q}(\sqrt{t})$ is not divisible by 3. Then there exist infinitely many bicyclic biquadratic fields $\mathbb{Q}(\sqrt{t}, \sqrt{D})$ whose class number is not divisible by 3.*

Finally, as an application, we shall use Theorem 1.1 to get another positive proportion of rank-one twists of the modular elliptic curve $X_0(19)$.

Remark. For the complementary question, Komatsu [4] explicitly constructed a family of infinite pairs of quadratic fields $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt{tD})$ whose class numbers are both divisible by 3, for any square free integer t .

2. PRELIMINARIES

We recall the result of Nakagawa and Horie in [6], which is a refinement of the result of Davenport and Heilbronn in [3]. Let m and N be two positive integers satisfying the following condition:

- (*) If an odd prime number p is a common divisor of m and N , then p^2 divides N but not m . Further if N is even, then (i) 4 divides N and $m \equiv 1 \pmod{4}$, or (ii) 16 divides N and $m \equiv 8$ or $12 \pmod{16}$.

For any positive real number $X > 0$, we denote by $S_+(X)$ the set of positive fundamental discriminants $D < X$ and by $S_-(X)$ the set of negative fundamental discriminants $D > -X$, and put

$$S_+(X, m, N) := \{D \in S_+(X) \mid D \equiv m \pmod{N}\},$$

$$S_-(X, m, N) := \{D \in S_-(X) \mid D \equiv m \pmod{N}\}.$$

Theorem 2.1 (Nakagawa and Horie). *Let D be a fundamental discriminant and $r_3(D)$ the 3-rank of the quadratic field $\mathbb{Q}(\sqrt{D})$. Then for any two positive integers m, N satisfying (*),*

$$\lim_{X \rightarrow \infty} \frac{\sum_{D \in S_+(X, m, N)} 3^{r_3(D)}}{\sum_{D \in S_+(X, m, N)} 1} = \frac{4}{3}$$

and

$$\lim_{X \rightarrow \infty} \frac{\sum_{D \in S_-(X, m, N)} 3^{r_3(D)}}{\sum_{D \in S_-(X, m, N)} 1} = 2.$$

From Theorem 2.1 and the fact that

$$\begin{aligned} & \sum_{\substack{D \in S_{\pm}(X, m, N) \\ r_3(D)=0}} 3^{r_3(D)} + 3 \left(\sum_{D \in S_{\pm}(X, m, N)} 1 - \sum_{\substack{D \in S_{\pm}(X, m, N) \\ r_3(D)=0}} 3^{r_3(D)} \right) \\ & \leq \sum_{D \in S_{\pm}(X, m, N)} 3^{r_3(D)}, \end{aligned}$$

we can easily obtain the following lemma.

Lemma 2.2. *Let D be a fundamental discriminant and $h(D)$ the class number of the quadratic field $\mathbb{Q}(\sqrt{D})$. Then for any two positive integers m, N satisfying (*),*

$$\liminf_{X \rightarrow \infty} \frac{\#\{D \in S_+(X, m, N) \mid h(D) \not\equiv 0 \pmod{3}\}}{\#S_+(X, m, N)} \geq \frac{5}{6}$$

and

$$\liminf_{X \rightarrow \infty} \frac{\#\{D \in S_-(X, m, N) \mid h(D) \not\equiv 0 \pmod{3}\}}{\#S_-(X, m, N)} \geq \frac{1}{2}.$$

3. PROOF OF THEOREM 1.1

Theorem 1.1 follows from the following proposition.

Proposition 3.1. *Let t be a square free integer and let m, N be two positive integers satisfying $(*)$ and $(m, t) = 1$. Then there exist infinitely many positive fundamental discriminants $D \equiv m \pmod{N}$ with a positive density such that the class numbers of quadratic fields $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt{tD})$ are both not divisible by 3.*

Proof. We shall give the details of the case $t \equiv 1 \pmod{16}$ and $(mN, t) = 1$, because the other cases are routine modifications of this case. Let

$$S'_+(X, m, tN) := \{tD \mid D \in S_+(X, m, tN)\}.$$

Since t is relatively prime to any $D \in S_+(X, m, tN)$, we have $\#S'_+(X, m, tN) = \#S_+(X, m, tN)$ and

$$\begin{aligned} S'_+(X, m, tN) &= S_+(tX, tm, t^2N) \quad \text{if } t \text{ is positive,} \\ S'_+(X, m, tN) &= S_-(-tX, tm, t^2N) \quad \text{if } t \text{ is negative.} \end{aligned}$$

Note that two positive integers m, tN satisfy the condition $(*)$. Then from Lemma 2.2, we have

$$(1) \quad \liminf_{X \rightarrow \infty} \frac{\#\{D \in S_+(X, m, tN) \mid h(D) \not\equiv 0 \pmod{3}\}}{\#S_+(X, m, tN)} \geq \frac{5}{6}.$$

Assume t is positive. Since tm, t^2N also satisfy $(*)$, we know that

$$\begin{aligned} (2) \quad & \liminf_{X \rightarrow \infty} \frac{\#\{D \in S'_+(X, m, tN) \mid h(D) \not\equiv 0 \pmod{3}\}}{\#S'_+(X, m, tN)} \\ (3) \quad &= \liminf_{X \rightarrow \infty} \frac{\#\{D \in S_+(X, tm, t^2N) \mid h(D) \not\equiv 0 \pmod{3}\}}{\#S_+(X, tm, t^2N)} \geq \frac{5}{6}. \end{aligned}$$

Suppose that

$$(4) \quad \liminf_{X \rightarrow \infty} \frac{\#\{D \in S_+(X, m, tN) \mid h(D) \not\equiv 0 \text{ and } h(tD) \not\equiv 0 \pmod{3}\}}{\#S_+(X, m, tN)} < \frac{2}{3}.$$

Then (4) contradicts (3) and we get

$$\liminf_{X \rightarrow \infty} \frac{\#\{D \in S_+(X, m, tN) \mid h(D) \not\equiv 0 \text{ and } h(tD) \not\equiv 0 \pmod{3}\}}{\#S_+(X, m, tN)} \geq \frac{2}{3}.$$

If we assume that n is negative, then we have

$$\liminf_{X \rightarrow \infty} \frac{\#\{D \in S'_+(X, m, tN) \mid h(D) \not\equiv 0 \pmod{3}\}}{\#S'_+(X, m, tN)} \geq \frac{1}{2}.$$

By a similar argument in the case t is positive, we can obtain

$$\liminf_{X \rightarrow \infty} \frac{\#\{D \in S_+(X, m, tN) \mid h(D) \not\equiv 0 \text{ and } h(tD) \not\equiv 0 \pmod{3}\}}{\#S_+(X, m, tN)} \geq \frac{1}{3}.$$

The proof is completed since $S_+(X, m, tN)$ has a positive density in $S_+(X, m, N)$. □

4. APPLICATION TO RANK-ONE TWISTS

We shall follow Vatsal's paper [8]. Let $t < 0$ be a negative square free integer such that 19 splits in the imaginary quadratic field $K = \mathbb{Q}(\sqrt{t})$ and $t \equiv 1 \pmod{4}$. Let $c > 0$ be a positive square free integer satisfying the following conditions:

- (i) $c \equiv 1 \pmod{4}$ and $(c, t \cdot 19) = 1$,
- (ii) 19 is inert in $K = \mathbb{Q}(\sqrt{c})$,
- (iii) the class numbers of quadratic fields $\mathbb{Q}(\sqrt{c})$ and $\mathbb{Q}(\sqrt{t \cdot c})$ are both not divisible by 3.

Let E be the modular elliptic curve $X_0(19)$ and $L(s, E)$ the corresponding L -series. We denote ψ and ψ' the quadratic Galois characters associated to $k = \mathbb{Q}(\sqrt{c})$ and $k' = \mathbb{Q}(\sqrt{t \cdot c})$. Then the Gross-Zagier theorem [1], [2] on the Heegner points and derivatives of L -series implies that if c satisfies the above conditions (i)–(iii), then $L(s, E \otimes \psi)$ has a simple zero and $L(s, E \otimes \psi')$ is non-zero at $s = 1$.

On the other hand, we can find two positive integers m, N , which depend on $\{4, 19, t\}$ and satisfy $(*)$, such that if c satisfies the congruence $c \equiv m \pmod{N}$, then c satisfies conditions (i) and (ii). From Proposition 3.1, we know that a positive proportion of $c \equiv m \pmod{N}$ satisfies condition (iii). Finally, if we add the condition $3|c$, which is different from Vatsal's $(3, c) = 1$, then we have another positive proportion of rank-one twists of $X_0(19)$.

REFERENCES

- [1] B. Gross, Heegner points on $X_0(N)$, Modular forms (R. Rankin, ed.), Chichester, Ellis Horwood Company, 1984. MR 86f:11003
- [2] B. Gross and D. Zagier, Heegner points and derivatives of L -series, *Invent. Math.* **84** (1986), 225–320. MR 87j:11057
- [3] H. Davenport and H. Heilbronn, On the density of discriminants of cubic fields II, *Proc. Roy. Soc. London A*, **322** (1971), 405–420. MR 58:10816
- [4] T. Komatsu, An infinite family of pairs of quadratic fields $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt{mD})$ whose class numbers are both divisible by 3, *Acta Arith.*, **104** (2002), 129–136. MR 2003f:11166
- [5] T. Kubota, Über die Beziehung der Klassenzahlen der Unterkörper des Bizyklischen Biquadratischen Zahlkörpers, *Nagoya Math. J.* **6** (1953), 119–127. MR 15:605e
- [6] J. Nakagawa and K. Horie, Elliptic curves with no torsion points, *Proc. A.M.S.* **104** (1988), 20–25. MR 89k:11113
- [7] A. Scholz, Über die Beziehung der Klassenzahlen quadratischer Körper zueinander, *J. Reine Angew. Math.*, **166** (1932), 201–203.
- [8] V. Vatsal, Rank-one twists of a certain elliptic curve, *Mathematische Annalen*, **311** (1998), 791–794. MR 99i:11050

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