

MÖBIUS FUNCTION OF COORDINATE HYPERPLANES IN COMPLEX ELLIPSOIDS

WITOLD JARNICKI

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ABSTRACT. For $p_1, \dots, p_n > 0$, let $\mathbb{E} = \{z \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^{2p_j} < 1\}$ be a complex ellipsoid. We present effective formulas for the generalized Möbius and Green functions $m_{\mathbb{E}}(A, \cdot)$, $g_{\mathbb{E}}(A, \cdot)$ in the case where $A := \{z \in \mathbb{E} : z_1 \cdots z_k = 0\}$ ($1 \leq k \leq n$).

1. INTRODUCTION AND MAIN RESULTS

Let $G \subset \mathbb{C}^n$ be a domain, and let $A \subset G$. Define the *generalized Green function*

$$g_G(A, z) := \sup\{u(z) : u : G \rightarrow [0, 1), \log u \in \mathcal{PSH}(G), \\ \forall a \in A \exists C=C(u,a)>0 \forall w \in G : u(w) \leq C\|w-a\|\}, \quad z \in G,$$

and the *generalized Möbius function*

$$m_G(A, z) := \sup\{|f(z)| : f \in \mathcal{O}(G, \mathbb{D}), f|_A \equiv 0\}, \quad z \in G,$$

where \mathbb{D} denotes the unit disc (cf. [Jar-Jar-Pfl 2003]).

The generalized Möbius and Green functions were recently studied by many authors (e.g. [Lár-Sig 1998]). Despite various proven properties, effective formulas for $m_G(A, \cdot)$ and $g_G(A, \cdot)$ are known only in a few special cases. An effective formula for the generalized Green function in the case of the Euclidean ball $G = \mathbb{B}_2 \subset \mathbb{C}^2$ and the union of coordinate lines $A = \{(z, w) \in \mathbb{B}_2 : zw = 0\}$ was given in [Ngu 2003].

Let $n \geq 2$. Let $p_1, \dots, p_n > 0$. Define the *complex ellipsoid with exponents* p_1, \dots, p_n :

$$\mathbb{E} = \mathbb{E}_{p_1, \dots, p_n} := \left\{ z \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^{2p_j} < 1 \right\}.$$

Observe that $\mathbb{E}_{p_1, \dots, p_n}$ is convex if and only if $p_1, \dots, p_n \geq 1/2$ (cf. [Jar-Pfl 1993], §8.4).

For $G \subset \mathbb{C}^n$, $k \in \{1, \dots, n\}$, we consider the sets

$$A = A_{G,k} := \{z \in G : z_1 \cdots z_k = 0\}.$$

We give an effective formula for $g_{\mathbb{E}}(A_{\mathbb{E},k}, \cdot)$ for any complex ellipsoid $\mathbb{E} \subset \mathbb{C}^n$ and any $k \in \{1, \dots, n\}$ (Theorem 1.2(a)). This is a generalization of a result in [Ngu 2003] (with a much simpler proof). We prove that the $m_{\mathbb{E}}(A_{\mathbb{E},k}, \cdot) \equiv$

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$g_{\mathbb{E}}(A_{\mathbb{E},k}, \cdot)$ for the cases where \mathbb{E} is convex (Theorem 1.2(b)) or where $k = 1, n = 2, p_2 \geq 1/2$ (Theorem 1.2(c)). This is not true in the general nonconvex case; we prove that $m_{\mathbb{E}}(A_{\mathbb{E},k}, \cdot) \not\equiv g_{\mathbb{E}}(A_{\mathbb{E},k}, \cdot)$ if there exists a $j \in \{k + 1, \dots, n\}$ such that $p_j < 1/2$ (Theorem 1.2(d)). Partial results for the case $k = n = 2$ are given (Theorem 1.2(e)).

Definition 1.1. Let $p_1, \dots, p_n > 0, 1 \leq k \leq n$. Put $\mathbb{E} := \mathbb{E}_{p_1, \dots, p_n}, A := A_{\mathbb{E},k}$. Let $z \in \mathbb{E}$ be such that the sequence $(p_j |z_j|^{2p_j})_{j=1}^k$ is monotonically increasing. For $s \in \{1, \dots, k\}$ define

$$q_s := \sum_{j=1}^s (2p_j)^{-1}, \quad r_s(z) := 1 - \sum_{j=s+1}^n |z_j|^{2p_j}, \quad c_s(z) := r_s(z)/q_s.$$

Let $d := \max\{s \in \{1, \dots, k\} : 2p_s |z_s|^{2p_s} \leq c_s(z)\}$. Define

$$(*) \quad R_{\mathbb{E}}(A, z) = \prod_{j=1}^d |z_j| \left(\frac{2p_j}{c_d(z)} \right)^{\frac{1}{2p_j}}.$$

The main results of the paper are the following.

Theorem 1.2. *Under the above assumptions we have:*

- (a) $g_{\mathbb{E}}(A, \cdot) \equiv R_{\mathbb{E}}(A, \cdot)$,
- (b) $m_{\mathbb{E}}(A, \cdot) \equiv g_{\mathbb{E}}(A, \cdot) \equiv R_{\mathbb{E}}(A, \cdot)$, for $p_j \geq 1/2, j = 1, \dots, n$,
- (c) $m_{\mathbb{E}}(A, \cdot) \equiv g_{\mathbb{E}}(A, \cdot) \equiv R_{\mathbb{E}}(A, \cdot)$, for $k = 1, n = 2, p_2 \geq 1/2$,
- (d) $m_{\mathbb{E}}(A, \cdot) \not\equiv g_{\mathbb{E}}(A, \cdot)$ if there exists a $j \in \{k + 1, \dots, n\}$ with $p_j < 1/2$,
- (e) $m_{\mathbb{E}}(A, \cdot) \equiv g_{\mathbb{E}}(A, \cdot) \equiv R_{\mathbb{E}}(A, \cdot)$, for $k = n = 2, p_1 \leq p_2$, and either $p_2 \geq 1/2$ or $8p_1 + 4p_2(1 - p_2) > 1$.

Observe that one can obtain the monotonicity of $(p_j |z_j|^{2p_j})_{j=1}^k$ for arbitrary $z \in \mathbb{E}$ by permutation of coordinates. The final result is a subdivision of \mathbb{E} into $2^k - 1$ subsets, each with $R_{\mathbb{E}}(A, \cdot)$ given by a formula of type (*).

Remark 1.3. In the case where $p_1 = \dots = p_n = 1$, the domain \mathbb{E} is the Euclidean ball \mathbb{B}_n . Theorem 1.2 may then be formulated in the following simpler way.

Corollary 1.4. *Assume that $|z_1| \leq \dots \leq |z_k|$, and let*

$$d := \max \left\{ s \in \{1, \dots, k\} : s|z_s|^2 + \sum_{j=s+1}^n |z_j|^2 \leq 1 \right\}.$$

Then

$$m_{\mathbb{B}_n}(A, z) = g_{\mathbb{B}_n}(A, z) = \left(\frac{d}{1 - \sum_{j=d+1}^n |z_j|^2} \right)^{\frac{d}{2}} \prod_{j=1}^d |z_j|.$$

Remark 1.5. It is unknown to the author whether $m_{\mathbb{E}}(A, \cdot) \equiv g_{\mathbb{E}}(A, \cdot)$ for any $p_j > 0, j = 1, \dots, k, p_j \geq 1/2, j = k + 1, \dots, n$. However, Theorem 1.2(e) and the author's research (cf. Remark 4.4) indicate that this is true at least in the case $k = n = 2$.

Remark 1.6. Take $p_{j,\ell} \nearrow +\infty, j = 1, \dots, n$. Then

$$G_{\ell} := \mathbb{E}_{p_{1,\ell}, \dots, p_{n,\ell}} \nearrow \mathbb{D}^n.$$

Consequently,

$$m_{G_{\ell}}(A_{G_{\ell},k}, \cdot) = g_{G_{\ell}}(A_{G_{\ell},k}, \cdot) \searrow m_{\mathbb{D}^n}(A_{\mathbb{D}^n,k}, \cdot) = g_{\mathbb{D}^n}(A_{\mathbb{D}^n,k}, \cdot)$$

(cf. [Jar-Jar-Pfl 2003], Property 2.7) and hence

$$m_{\mathbb{D}^n}(A_{\mathbb{D}^n,k}, z) = |z_1| \cdots |z_k|, \quad z \in \mathbb{D}^n.$$

However, we need the formula for $g_{\mathbb{D}^n}(A_{\mathbb{D}^n,k}, \cdot)$ before we prove Theorem 1.2 (cf. Lemma 2.1).

2. PROOF OF THEOREM 1.2

Lemma 2.1. *Let $n \in \mathbb{N}$, $1 \leq k \leq n$. Then*

$$m_{\mathbb{D}^n}(A_{\mathbb{D}^n,k}, z) = g_{\mathbb{D}^n}(A_{\mathbb{D}^n,k}, z) = |z_1| \cdots |z_k|, \quad z \in \mathbb{D}^n.$$

Proof of Lemma 2.1 (due to P. Pflug). Obviously,

$$|z_1| \cdots |z_k| \leq m_{\mathbb{D}^n}(A_{\mathbb{D}^n,k}, z) \leq g_{\mathbb{D}^n}(A_{\mathbb{D}^n,k}, z).$$

It remains to prove that $g_{\mathbb{D}^n}(A_{\mathbb{D}^n,k}, z) \leq |z_1| \cdots |z_k|$.

It is enough to prove that for any $1 \leq k \leq n$ and for any function $u : \mathbb{D}^n \rightarrow [0, 1]$ such that $\log u \in \mathcal{PSH}(\mathbb{D}^n)$ and $u(z) \leq C(a)\|z - a\|$, $a \in A_{\mathbb{D}^n,k}$, $z \in \mathbb{D}^n$, we have $u(z) \leq |z_1| \cdots |z_k|$. We proceed by induction on k .

For $k = 1$ the inequality follows from the Schwarz-type lemma for logarithmically subharmonic functions $u(\cdot, z_2, \dots, z_n)$, $z_2, \dots, z_n \in \mathbb{D}$.

For $k > 1$ we first apply the case $k = 1$ and get $u(z_1, \dots, z_n) \leq |z_1|$, $z \in \mathbb{D}^n$. Applying the inductive assumption to $u(z_1, \cdot)/|z_1|$, $z_1 \in \mathbb{D}$, finishes the proof. \square

Take a $u : \mathbb{E} \rightarrow [0, 1]$ with $\log u \in \mathcal{PSH}(\mathbb{E})$ and $u(\zeta) \leq C(a)\|\zeta - a\|$, $a \in A$, $\zeta \in \mathbb{E}$. Consider the mapping

$$\iota_z : \mathbb{D}^d \ni (\zeta_1, \dots, \zeta_d) \mapsto \left(\zeta_1 \left(\frac{c_d(z)}{2p_1} \right)^{\frac{1}{2p_1}}, \dots, \zeta_d \left(\frac{c_d(z)}{2p_d} \right)^{\frac{1}{2p_d}}, z_{d+1}, \dots, z_n \right) \in \mathbb{E}.$$

Applying the holomorphic contractivity of the generalized Green function and Lemma 2.1 proves that $m_{\mathbb{E}}(A, z) \leq g_{\mathbb{E}}(A, z) \leq R_{\mathbb{E}}(A, z)$.

It remains to prove that $m_{\mathbb{E}}(A, z) \geq R_{\mathbb{E}}(A, z)$ (resp. $g_{\mathbb{E}}(A, z) \geq R_{\mathbb{E}}(A, z)$). In the case $d = k = n$ it suffices to take

$$f(\zeta) := \prod_{j=1}^d \zeta_j \left(\frac{2p_j}{c_d(z)} \right)^{\frac{1}{2p_j}}.$$

Then $f \in \mathcal{O}(\mathbb{E}, \mathbb{D})$ and $|f(z)| = R_{\mathbb{E}}(A, z)$.

It remains to prove Theorem 1.2 in the remaining cases. We may assume that $z_j \neq 0$, $j = 1, \dots, d$. Consider the following lemma.

Lemma 2.2. *Define $\mathbb{E}' := \mathbb{E}_{p_{d+1}, \dots, p_n}$.*

- (a) *Let $v : \mathbb{E}' \rightarrow [0, 1]$, $v \not\equiv 0$ be such that $\log v \in \mathcal{PSH}(\mathbb{E}')$ and $v(\zeta) \leq |\zeta_j|$, $\zeta \in \mathbb{E}'$, $j = d + 1, \dots, k$, and the mapping*

$$\mathbb{E}' \ni (\zeta_{d+1}, \dots, \zeta_n) \mapsto v(\zeta_{d+1}, \dots, \zeta_n) r_d^{q_d}(\zeta)$$

attains its maximum value M at (z_{d+1}, \dots, z_n) . Then $g_{\mathbb{E}}(A, z) = R_{\mathbb{E}}(A, z)$.

- (b) *The following conditions are equivalent:*

- *There exists an $h \in \mathcal{O}(\mathbb{E}')$, $h \not\equiv 0$ such that $h(\zeta) = 0$ for $\zeta_{d+1} \cdots \zeta_k = 0$ and the mapping*

$$\mathbb{E}' \ni (\zeta_{d+1}, \dots, \zeta_n) \mapsto |h(\zeta_{d+1}, \dots, \zeta_n)| r_d^{q_d}(\zeta)$$

attains its maximum value M at the point (z_{d+1}, \dots, z_n) ,

- *$m_{\mathbb{E}}(A, z) = R_{\mathbb{E}}(A, z)$.*

We present a proof in Section 3. The result above reduces the proof of Theorem 1.2 to the following propositions.

Proposition 2.3. *There exists a v as required in Lemma 2.2(a).*

Proposition 2.4. *Using the notation of Theorem 1.2, assume that $p_j \geq 1/2$, $j = d + 1, \dots, n$. Then there exists an $h \in \mathcal{O}(\mathbb{E}')$ as required in Lemma 2.2(b).*

Proposition 2.5. *Using the notation of Theorem 1.2, assume that $p_{k+1} < 1/2$. Then one cannot find an h as required in Lemma 2.2(b) for $|z_\ell| \neq 0$ small enough, $\ell = 1, \dots, k + 1$ and $z_\ell = 0$, $\ell = k + 2, \dots, n$.*

Proposition 2.6. *Using the notation of Theorem 1.2, assume that $n = k = 2$. Additionally, assume that $p_2 \geq 1/2$ or $8p_1 + 4p_2(1 - p_2) > 1$. Then there exists an $h \in \mathcal{O}(\mathbb{D})$ as required in Lemma 2.2(b).*

3. PROOF OF LEMMA 2.2

(a) Put

$$u(\zeta_1, \dots, \zeta_n) := M^{-1} \left(\prod_{j=1}^d (2p_j)^{\frac{1}{2p_j}} |\zeta_j| \right) q_d^{q_d} v(\zeta_{d+1}, \dots, \zeta_n).$$

Obviously $\log u \in \mathcal{PSH}(\mathbb{E})$ and $u(\zeta) \leq C|\zeta_j| \leq C\|\zeta - a\|$, $\zeta \in \mathbb{E}$, whenever $a_j = 0$ for some $j \in \{1, \dots, k\}$. For $\zeta \in \mathbb{E}$ we have

$$u(\zeta) \leq M^{-1} \left(\frac{\sum_{j=1}^d |\zeta_j|^{2p_j}}{q_d} \right)^{q_d} q_d^{q_d} v(\zeta_{d+1}, \dots, \zeta_n) < 1.^1$$

Consequently $u : \mathbb{E} \rightarrow [0, 1)$. On the other hand,

$$u(z) = M^{-1} R_{\mathbb{E}}(A, z) r_d^{q_d}(z) v(z_{d+1}, \dots, z_n) = R_{\mathbb{E}}(A, z).$$

(b) Assume that such an h exists. Put

$$f(\zeta_1, \dots, \zeta_n) := M^{-1} \left(\prod_{j=1}^d (2p_j)^{\frac{1}{2p_j}} \zeta_j \right) q_d^{q_d} h(\zeta_{d+1}, \dots, \zeta_n).$$

Observe that $f(\zeta) = 0$ for $\zeta \in A$. Similarly as in (a) we prove that $|f(\zeta)| < 1$, $\zeta \in \mathbb{E}$ and $f(z) = R_{\mathbb{E}}(A, z)$.

Assume now that $m_{\mathbb{E}}(A, z) = R_{\mathbb{E}}(A, z)$. Let $f \in \mathcal{O}(\mathbb{E}, \mathbb{D})$ be such that $f|_A \equiv 0$ and $|f(z)| = R_{\mathbb{E}}(A, z)$ (cf. [Jar-Jar-Pfl 2003], Property 2.5). Put

$$h(\zeta) := \frac{\partial^d f}{\partial z_1 \dots \partial z_d}(0, \zeta).$$

¹Let $a_1, \dots, a_d \geq 0$, $w_1, \dots, w_d > 0$. Then

$$\prod_{j=1}^d a_j^{w_j} \leq \left(\frac{\sum_{j=1}^d w_j a_j}{\sum_{j=1}^d w_j} \right)^{\sum_{j=1}^d w_j}.$$

By definition, we have $h(\zeta) = 0$ for $\zeta_{d+1} \cdots \zeta_k = 0$. Applying the Schwarz lemma to the mapping $f \circ \iota_\zeta$, $\zeta \in \mathbb{E}'$, we get

$$|h(\zeta_{d+1}, \dots, \zeta_n)|r_d^{q_d}(\zeta) \leq \left(\prod_{j=1}^d (2p_j)^{\frac{1}{2p_j}} \right) q_d^{q_d}, \quad \zeta \in \mathbb{E}',$$

$$|h(z_{d+1}, \dots, z_n)|r_d^{q_d}(z) = \left(\prod_{j=1}^d (2p_j)^{\frac{1}{2p_j}} \right) q_d^{q_d}.$$

4. PROOF OF PROPOSITIONS 2.3, 2.4, 2.5, AND 2.6

Proof of Proposition 2.3. We may assume that $z_{d+1}, \dots, z_n \geq 0$. Consider functions of the form

$$v_\alpha(\zeta_{d+1}, \dots, \zeta_n) = \left(\prod_{j=d+1}^k |\zeta_j|^{1+\alpha_j} \right) \left(\prod_{j=k+1}^n |\zeta_j|^{\alpha_j} \right),$$

where $\alpha = (\alpha_{d+1}, \dots, \alpha_n)$, $\alpha_{d+1}, \dots, \alpha_n \geq 0$. Obviously $v : \mathbb{E}' \rightarrow [0, 1)$, $\log v \in \mathcal{PSH}(\mathbb{E}')$, and $v(\zeta) \leq |\zeta_j| \leq \|\zeta - a\|$, $\zeta \in \mathbb{E}'$, whenever $\alpha_j = 0$ for some $j \in \{d+1, \dots, k\}$. Since $v_\alpha(\zeta_{d+1}, \dots, \zeta_n) = v_\alpha(|\zeta_{d+1}|, \dots, |\zeta_n|)$, it is enough to find an α such that the function

$$\mathbb{E}' \cap \mathbb{R}_+^{n-d} \ni (t_{d+1}, \dots, t_n) \mapsto v_\alpha(t_{d+1}, \dots, t_n)r_d^{q_d}(t)$$

attains its maximum at (z_{d+1}, \dots, z_n) . Considering the partial (logarithmic) derivatives results in the following equations:

$$0 = 1 + \alpha_j - 2p_j q_d \frac{t_j^{2p_j}}{r_d(t)}, \quad j = d+1, \dots, k,$$

$$0 = \alpha_j - 2p_j q_d \frac{t_j^{2p_j}}{r_d(t)}, \quad j = k+1, \dots, n.$$

These give formulas for $\alpha_{d+1}, \dots, \alpha_n$ such that (z_{d+1}, \dots, z_n) is the common zero of the derivatives. To prove that there are no other points like this, consider a reformulation of the above equations:

$$r_d(t) = \frac{2p_j q_d t_j^{2p_j}}{1 + \alpha_j}, \quad j = d+1, \dots, k,$$

$$r_d(t) = \frac{2p_j q_d t_j^{2p_j}}{\alpha_j}, \quad j = k+1, \dots, n.$$

The left side is decreasing in any of the variables, while the right sides are increasing. Thus, at most one common zero is allowed.

It remains to check whether $\alpha_j \geq 0$, $j = d+1, \dots, n$. Obviously, this is true for $j = k+1, \dots, n$ and in the remaining cases we have

$$\alpha_j = \frac{2p_j q_d t_j^{2p_j} - r_d(t)}{t_j r_d(t)} \geq 0,$$

since this is the way we have chosen d . □

Proof of Proposition 2.4. We may assume that $z_{d+1}, \dots, z_n \geq 0$. Consider functions of the form

$$h_\alpha(\zeta_{d+1}, \dots, \zeta_n) = \left(\prod_{j=d+1}^k \zeta_j e^{\alpha_j \zeta_j} \right) \left(\prod_{j=k+1}^n e^{\alpha_j \zeta_j} \right),$$

where $\alpha = (\alpha_{d+1}, \dots, \alpha_n)$, $\alpha_{d+1}, \dots, \alpha_n \geq 0$.

Since $|h_\alpha(\zeta_{d+1}, \dots, \zeta_n)| \leq h_\alpha(|\zeta_{d+1}|, \dots, |\zeta_n|)$, it is enough to find an α such that

$$\mathbb{E}' \cap \mathbb{R}_+^{n-d} \ni (t_{d+1}, \dots, t_n) \mapsto h_\alpha(t_{d+1}, \dots, t_n) r_d^{q_d}(t)$$

attains its maximum at (z_{d+1}, \dots, z_n) . Considering the partial (logarithmic) derivatives results in the following equations:

$$\begin{aligned} 0 &= \frac{1}{t_j} + \alpha_j - 2p_j q_d \frac{t_j^{2p_j-1}}{r_d(t)}, & j &= d+1, \dots, k, \\ 0 &= \alpha_j - 2p_j q_d \frac{t_j^{2p_j-1}}{r_d(t)}, & j &= k+1, \dots, n. \end{aligned}$$

We continue as in the proof of Proposition 2.3. □

Proof of Proposition 2.5. Consider the following two lemmas.

Lemma 4.1. *Assume that there exist $0 < c < b < 1$ such that the function*

$$\varphi : [0, b] \ni t \mapsto (1 - t^p)^{-q}$$

is strictly concave and

$$\varphi(0) + \frac{b}{c}(\varphi(c) - \varphi(0)) > \varphi(b) + 2.$$

Let $f \in \mathcal{O}(\mathbb{D})$, $f \not\equiv 0$ such that $|f(\zeta)|/\varphi(|\zeta|)$ attains its maximum at w_0 . Then $w_0 = 0$ or $|w_0| \geq c$.

Proof. Assume that $|w_0| \in (0, c)$. We may assume that $|f(w_0)| = \varphi(|w_0|)$. Consider the function

$$\psi : [0, b] \ni t \mapsto |f(0)| + \frac{t}{|w_0|} |f(w_0) - f(0)|.$$

From $\psi(0) \leq \varphi(0)$, $\psi(|w_0|) \geq \varphi(|w_0|)$, and the convexity condition we get

$$\begin{aligned} \psi(b) &= |f(0)| + \frac{b}{|w_0|} |f(w_0) - f(0)| \geq \varphi(0) + \frac{b}{|w_0|} |\varphi(|w_0|) - \varphi(0)| \\ &\geq \varphi(0) + \frac{b}{c} |\varphi(c) - \varphi(0)| > \varphi(b) + 2. \end{aligned}$$

The Schwarz lemma and the maximum principle imply that there exists a $w \in \mathbb{D}$ with $|w| = b$ and

$$\frac{|f(w) - f(0)|}{|w|} \geq \frac{|f(w_0) - f(0)|}{|w_0|}.$$

This means that

$$\begin{aligned} |f(w)| &\geq |f(w) - f(0)| - |f(0)| = |f(0)| + |f(w) - f(0)| - 2|f(0)| \\ &\geq \psi(b) - 2|f(0)| > \varphi(b) + 2 - 2|f(0)| \geq \varphi(b) = \varphi(|w|). \end{aligned}$$

This contradicts the maximality of $|f(\zeta)|/\varphi(|\zeta|)$ at w_0 . □

Lemma 4.2. *Assume that $p \in (0, 1)$, $q > 0$. Then there exist $b \in (0, 1)$, $c \in (0, b)$ as required in Lemma 4.1.*

Proof. We have

$$\varphi'(t) = \frac{pq}{t^{1-p}(1-t^p)^{q+1}},$$

which is decreasing for small t . To prove the existence of c observe that

$$\lim_{d \rightarrow 0^+} \frac{\varphi(d) - \varphi(0)}{d} = +\infty.$$

□

Observe that $d = k$ for $|z_\ell|$ small enough, $\ell = 1, \dots, k$. Let $h \in \mathcal{O}(\mathbb{E}')$ and consider $f(\zeta) := h(\zeta, 0, \dots, 0)$, $\zeta \in \mathbb{D}$. Put $p := 2p_{k+1}$, $q = q_d$. Let b, c be as in Lemma 4.2. It follows from Lemma 4.1 that $0 \neq |w_0| < c$ cannot be a maximum of $\mathbb{D} \ni \zeta \mapsto |f(\zeta)|/\varphi(|\zeta|)$. In particular, z cannot be a maximum of $\mathbb{E}' \ni \zeta \mapsto |h(\zeta)|r_d^{q_d}(\zeta)$ for $0 \neq |z_{k+1}| < c$. □

Proof of Proposition 2.6. Consider the following lemma.

Lemma 4.3. *Let $a, c > 0$, $t_0 \in (0, 1)$ be such that $c \geq 1$ or $4a + 2c > 1 + c^2$, $t_0^c > \tau := a/(a + c)$. Then there exist $b > 0$ and $r \geq 1$ such that*

$$h : [0, 1] \ni t \mapsto \frac{t^a}{(r-t)^b}(1-t^c) \in [0, 1]$$

admits its maximum at t_0 .

Proof. Comparing $\varphi := th'(t)/h(t)$ to zero we get the equation

$$\varphi(r, t) = a + \frac{bt}{r-t} - \frac{ct^c}{1-t^c} = 0.$$

This gives us a formula for r :

$$r(t) = \frac{t(b - bt^c - a + at^c + t^c c)}{-a + at^c + t^c c}.$$

Observe that

$$\lim_{t \rightarrow \tau^{1/c}} r(t) = +\infty, \quad \lim_{t \rightarrow 1} r(t) = 1.$$

In order to prove that $r(t) > 1$ and that $\varphi(r, \cdot)$ has only one zero it suffices to show that $r'(t) < 0$. We have

$$r'(t) = \frac{-(a+c)t^{2c} + (2a+c-c^2)t^c - a}{(at^c + t^c c - a)^2} b + \frac{(at^c + t^c c - a)^2}{(at^c + t^c c - a)^2}.$$

It remains to show that the coefficient $\alpha(t^c)$ next to b is negative. We have

$$\begin{aligned} \alpha(\tau) &= \frac{-ac^2}{a+c} < 0, \\ \alpha(1) &= -c^2 < 0, \\ \alpha'(u) &= -2(a+c)u + 2a + c - c^2. \end{aligned}$$

Let u_0 be the zero of $\alpha'(u)$. For $c \geq 1$ we have $u_0 \leq \tau$ and we are done. Otherwise $u_0 \in (\tau, 1)$ and $4(a+c)\alpha(u_0) = c^2(1+c^2-4a-2c) < 0$. □

We may assume that $z_0 > 0$. Put $a = 2p_1$, $c = 2p_2$, $t_0 = z_0$. Let r be as in Lemma 4.3. Putting

$$h(\zeta) = \frac{\zeta}{(r - \zeta)^{b/a}}$$

completes the proof. \square

Remark 4.4. The calculations in the proof of Proposition 2.6 can be performed using alternative function families, e.g.,

$$\begin{aligned} h_j(\zeta) &:= (\zeta + r)^k, & r \geq 0, k = 0, 1, 2, \dots, \\ h_j(\zeta) &:= (\zeta + 1)^\alpha, & \alpha > 0, \\ h_j(\zeta) &:= (r - \zeta)^\alpha \zeta^k, & \alpha < 0, r \geq 1, k = 0, 1, 2, \dots, \\ h_j(\zeta) &:= \left(\frac{\zeta + \delta}{1 + \delta\zeta} \right) \zeta^k, & \delta \in [0, 1], k = 0, 1, 2, \dots \end{aligned}$$

However, the author was unable to solve the general case using any of them.

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JAGIELLONIAN UNIVERSITY, INSTITUTE OF MATHEMATICS, REYMONTA 4, 30-059 KRAKÓW, POLAND

E-mail address: wmj@im.uj.edu.pl

Current address: Universität Osnabrück, Fachbereich Mathematik/Informatik, Albrechtstraße 28, 49069 Osnabrück, Germany