

## ON TWO PROBLEMS OF ERDŐS AND HECHLER: NEW METHODS IN SINGULAR MADNESS

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ABSTRACT. For an infinite cardinal  $\mu$ ,  $\text{MAD}(\mu)$  denotes the set of all cardinalities of *nontrivial maximal almost disjoint families* over  $\mu$ .

Erdős and Hechler proved in 1973 the consistency of  $\mu \in \text{MAD}(\mu)$  for a singular cardinal  $\mu$  and asked if it was ever possible for a singular  $\mu$  that  $\mu \notin \text{MAD}(\mu)$ , and also whether  $2^{\text{cf } \mu} < \mu \implies \mu \in \text{MAD}(\mu)$  for every singular cardinal  $\mu$ .

We introduce a new method for controlling  $\text{MAD}(\mu)$  for a singular  $\mu$  and, among other new results about the structure of  $\text{MAD}(\mu)$  for singular  $\mu$ , settle both problems affirmatively.

### 1. INTRODUCTION

**1.1. Background.** Let  $\mu$  be an infinite cardinal. A family of sets  $\mathcal{A}$  is  $\mu$ -almost disjoint ( $\mu$ -ad for short) if  $|A| = \mu = |\bigcup \mathcal{A}|$  for every  $A \in \mathcal{A}$  and  $|A \cap B| < \mu$  for every distinct  $A, B \in \mathcal{A}$ .  $\mathcal{A}$  is *maximal  $\mu$ -almost disjoint* ( $\mu$ -mad) if there is no  $C \subseteq \bigcup \mathcal{A}$  such that  $\mathcal{A} \cup \{C\}$  is  $\mu$ -almost disjoint; in this case we also say that  $\mathcal{A}$  is *mad in  $\mu$* . It is clear that every  $\mu$ -almost disjoint family consisting of fewer than  $\text{cf } \mu$  sets is mad in  $\mu$ ; such a family will be called *trivial*. We denote by  $\text{MAD}(\mu)$  the set of all cardinalities of nontrivial mad families in  $\mu$ . A standard diagonalization argument shows that  $\text{cf } \mu \notin \text{MAD}(\mu)$ . Therefore,  $\text{MAD}(\mu)$  is contained in the interval of cardinals  $[\text{cf}(\mu)^+, 2^\mu]$ .

W. W. Comfort asked (see [7]) under what conditions it follows that  $\mu \in \text{MAD}(\mu)$  for a singular cardinal  $\mu$ . P. Erdős and S. Hechler [7] proved that  $\mu \in \text{MAD}(\mu)$  if  $\lambda^{\text{cf } \mu} < \mu$  for every  $\lambda < \mu$ . Thus, if  $2^{\aleph_0} < \aleph_\omega$ , then the interval  $[2^{\aleph_0}, \aleph_\omega]$  of cardinals is contained in  $\text{MAD}(\aleph_\omega)$ .

Erdős and Hechler asked in [7] whether it is consistent that  $\mu \notin \text{MAD}(\mu)$  for some singular cardinal  $\mu$  and, more concretely, whether Martin's axiom together with  $2^{\aleph_0} > \aleph_\omega$  implies that  $\aleph_\omega \notin \text{MAD}(\aleph_\omega)$ . They also asked whether  $2^{\text{cf } \mu} < \mu$  implies  $\mu \in \text{MAD}(\mu)$  for singular cardinals  $\mu$  other than  $\aleph_\omega$ .

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Both problems are settled affirmatively by the general results below on  $\text{MAD}(\mu)$  for a singular  $\mu$ .

**1.2. Notation.** Let  $\mathfrak{a}_\mu = \min \text{MAD}(\mu)$ , and let  $\mathfrak{a} = \mathfrak{a}_{\aleph_0}$ . For a singular  $\mu$  it follows that  $\text{MAD}(\text{cf } \mu) \subseteq \text{MAD}(\mu)$ ; therefore  $\mathfrak{a}_\mu \leq \mathfrak{a}_{\text{cf } \mu}$ .

A crucial role in the results is played by two *bounding numbers*:  $\mathfrak{b}_\mu$  and  $\mathfrak{b}_{\text{cf } \mu}$ .

For every quasi-ordering  $(P, \leq)$  with no maximum, the *bounding number*  $\mathfrak{b}(P, \leq)$  is the least cardinality of a subset of  $P$  with no upper bound. For a regular cardinal  $\kappa$ , let  $\mathfrak{b}_\kappa$  denote the bounding number of  $(\kappa^\kappa, \leq^*)$ , where  $f \leq^* g$  means that  $|\{i < \kappa : f(i) > g(i)\}| < \kappa$ ; let  $\mathfrak{b} = \mathfrak{b}_{\aleph_0}$ . It is well known that  $\kappa < \mathfrak{b}_\kappa \leq \mathfrak{a}_\kappa$  for a regular cardinal  $\kappa$  (for  $\kappa = \aleph_0$  see [6]; the general case is similar) and that under Martin's axiom,  $\mathfrak{b} = 2^{\aleph_0}$ .

Suppose that  $\mu$  is a singular cardinal of cofinality  $\kappa$  and that  $\langle \mu_i : i < \kappa \rangle$  is a strictly increasing sequence of regular cardinals with supremum  $\mu$ . Standard diagonalization shows that  $\mathfrak{b}(\prod_{i < \kappa} \mu_i, \leq^*) > \mu$ . Denote by  $\mathfrak{b}_\mu$  the *supremum of  $\mathfrak{b}(\prod \mu_i, \leq^*)$  over all strictly increasing sequences of regular cardinals  $\langle \mu_i : i < \kappa \rangle$  with supremum  $\mu$* .

Each of the following three relations is consistent with ZFC:  $\mathfrak{b} < \mathfrak{b}_{\aleph_\omega}$ ,  $\mathfrak{b} = \mathfrak{b}_{\aleph_\omega}$  and  $\mathfrak{b} > \mathfrak{b}_{\aleph_\omega}$ .

**1.3. The results.** We prove that for every singular cardinal  $\mu$ :

- (1)  $\mathfrak{a}_\mu \geq \min\{\mathfrak{b}_\mu, \mathfrak{b}_{\text{cf } \mu}\}$ ;
- (2)  $\mathfrak{a}_\mu \leq \lambda < \mathfrak{b}_\mu \implies \lambda \in \text{MAD}(\mu)$ .

Thus, if  $\mathfrak{b}_{\text{cf } \mu} > \mu$ , it follows from (1) that  $\mathfrak{a}_\mu > \mu$ , and hence  $\mu \notin \text{MAD}(\mu)$ ; and if  $\mathfrak{a}_{\text{cf } \mu} < \mu$ , it follows from (2) that  $\mu \in \text{MAD}(\mu)$ . In particular,

- (b)  $MA + 2^{\aleph_0} > \aleph_\omega \implies \aleph_\omega \notin \text{MAD}(\aleph_\omega)$ ,
- (a)  $2^{\text{cf } \mu} < \mu \implies \mu \in \text{MAD}(\mu)$  for every singular  $\mu$ ,

which, respectively, settle in the affirmative both problems of Erdős and Hechler from [7].

If one assumes the consistency of large cardinals,  $\mathfrak{b}_{\aleph_\omega}$  can be shifted up arbitrarily high below  $\aleph_{\omega_1}$ . Following this with a ccc forcing for controlling  $\mathfrak{b}$  proves the following:

- (3) for every regular  $\lambda \in (\aleph_\omega, \aleph_{\omega_1})$  and regular uncountable  $\theta \leq \lambda^+$ , it is consistent that

$$\text{MAD}(\aleph_\omega) = [\theta, \lambda^+].$$

So, e.g., the following are consistent:

- $\text{MAD}(\aleph_\omega) = \{\aleph_1, \aleph_2, \dots, \aleph_{\omega+\beta+2} = 2^{\aleph_\omega}\}$  for an arbitrary  $\beta < \omega_1$ ,
- $\text{MAD}(\aleph_\omega) = \{\aleph_{\omega+\beta+2}\}$  for an arbitrary  $\beta < \omega_1$ ,
- $\text{MAD}(\aleph_\omega) = [\aleph_{\omega+\alpha+1}, \aleph_{\omega+\beta+2}]$  for arbitrary  $\alpha \leq \beta < \omega_1$ ,

and so on.

We refer the reader to the comprehensive list of references in D. Monk's recent paper [12], in which maximal almost disjoint families are viewed as partitions of unity in the Boolean algebra  $\mathcal{P}(\mu)/_{[\mu] < \mu}$ .

**1.4. Preliminary facts.** We will use the following facts from [7]:

- (1)  $\text{MAD}(\text{cf } \mu) \subseteq \text{MAD}(\mu)$ , and
- (2)  $\text{MAD}(\mu)$  is closed under singular suprema.

The latter fact is stated in [7] in a less general form. So we give a proof here.

**Lemma 1.1.** *Assume that  $\lambda = \sup_{i < \theta} \lambda_i$ , where  $\{\lambda_i : i < \theta\} \subseteq \text{MAD}(\mu)$  and  $\theta < \lambda$ . Then  $\lambda \in \text{MAD}(\mu)$ .*

*Proof.* We may assume that  $\theta \leq \lambda_0$ . Let  $\mathcal{A}$  be a mad family in  $\mu$  with  $|\mathcal{A}| = \lambda_0$ . Write  $\mathcal{A} = \{A_i : i < \lambda_0\}$  and for each  $i < \theta$  choose a mad family  $\mathcal{B}_i$  with  $\bigcup \mathcal{B}_i = A_i$  and  $|\mathcal{B}_i| = \lambda_i$ . Set

$$\mathcal{C} = \bigcup_{i < \theta} \mathcal{B}_i \cup \{A_j : \theta \leq j < \lambda_0\}.$$

Then  $|\mathcal{C}| = \lambda$  and  $\mathcal{C}$  is mad in  $\mu$ .  $\square$

The following fact will also be used in some proofs.

**Lemma 1.2.** *Let  $\kappa = \text{cf } \mu$ , and let  $\mathcal{A}$  be a  $\mu$ -almost disjoint family of size  $\kappa$ . Then there exists a mad family  $\mathcal{A}' \supseteq \mathcal{A}$  such that  $|\mathcal{A}'| = \mathfrak{a}_\mu$  and  $\bigcup \mathcal{A}' = \bigcup \mathcal{A}$ .*

*Proof.* Fix a  $\mu$ -mad family  $\mathcal{B}$  with  $|\mathcal{B}| = \mathfrak{a}_\mu$ . Choose  $\mathcal{B}_0 = \{B_i : i < \kappa\} \subseteq \mathcal{B}$ . Let  $X = \bigcup_{i < \kappa} B_i$  and define

$$\mathcal{B}' = \{B \cap X : X \in \mathcal{B} \setminus \mathcal{B}_0 \text{ \& } |X \cap B| = \mu\}.$$

Let  $\langle A_i : i < \kappa \rangle$  be a one-to-one enumeration of  $\mathcal{A}$ . Define a bijection  $f : \bigcup \mathcal{A} \rightarrow X$  so that  $f[A_i \setminus \bigcup_{j < i} A_j] = B_i \setminus \bigcup_{j < i} B_j$ . Finally, set  $\mathcal{A}' = \mathcal{A} \cup \{f^{-1}[B] : B \in \mathcal{B}'\}$ . Observe that  $\mathcal{A}'$  is mad and  $|\mathcal{A}'| = \mathfrak{a}_\mu$ .  $\square$

## 2. INEQUALITIES

From now on,  $\mu$  will always denote a singular cardinal whose cofinality is denoted by  $\kappa$ .

### 2.1. Bounding numbers and madness in singular cardinals.

**Theorem 2.1.** *For every singular cardinal  $\mu$ ,*

$$(1) \quad \mathfrak{a}_\mu \geq \min\{\mathfrak{b}_\mu, \mathfrak{b}_{\text{cf } \mu}\}.$$

*Proof.* Let  $\kappa = \text{cf } \mu$ . Suppose to the contrary that  $\mathfrak{a}_\mu < \min\{\mathfrak{b}_\mu, \mathfrak{b}_\kappa\}$ , and fix a strictly increasing sequence of regular cardinals  $\langle \mu_i : i < \kappa \rangle$  with supremum  $\mu$  such that  $\mathfrak{b}(\prod_{i < \kappa} \mu_i, \leq^*) > \mathfrak{a}_\mu$ .

Let  $\mathcal{A} = \{\{i\} \times \mu : i < \kappa\}$ . By Lemma 1.2, there exists a family  $\mathcal{B} \subseteq [\kappa \times \mu]^\mu$  such that  $\mathcal{B} \cup \mathcal{A}$  is mad in  $\mu$ ,  $\mathcal{B} \cap \mathcal{A} = \emptyset$  and  $|\mathcal{B}| = \mathfrak{a}_\mu$ .

For each  $B \in \mathcal{B}$ , define a function  $f_B : \kappa \rightarrow \kappa$  by  $f_B(i) = \min\{j < \kappa : |B \cap (\{i\} \times \mu)| < \mu_j\}$ . This function is well defined, since  $|B \cap (\{i\} \times \mu)| < \mu$  for each  $i < \kappa$ .

Since  $|\mathcal{B}| = \mathfrak{a}_\mu < \mathfrak{b}_\kappa$ , there exists a function  $f : \kappa \rightarrow \kappa$  so that  $f_B <^* f$  for all  $B \in \mathcal{B}$ . Without loss of generality, we may assume that  $f$  is strictly increasing.

For each  $B \in \mathcal{B}$ , for all but boundedly many  $i < \kappa$ , it follows that  $\sup\{\alpha < \mu_{f(i)} : (i, \alpha) \in B\} < \mu_{f(i)}$ . Let  $g_B(i)$  be defined by

$$g_B(i) = \begin{cases} 0 & \text{if } \sup\{\alpha < \mu_{f(i)} : (i, \alpha) \in B\} = \mu_{f(i)}, \\ \sup\{\alpha < \mu_{f(i)} : (i, \alpha) \in B\} & \text{otherwise.} \end{cases}$$

For each  $B \in \mathcal{B}$  the function  $g_B$  belongs to  $\prod_{i < \kappa} \mu_{f(i)}$ . Since

$$\mathfrak{b}(\prod_{i < \kappa} \mu_{f(i)}, \leq^*) \geq \mathfrak{b}(\prod_{i < \kappa} \mu_i) > \mathfrak{a}_\mu,$$

we can fix a function  $g \in \prod_{i < \kappa} \mu_{f(i)}$  so that  $g_B <^* g$  for all  $B \in \mathcal{B}$ .

Define

$$C = \bigcup_{i < \kappa} \{i\} \times [g(i), \mu_{f(i)}).$$

Clearly,  $|C| = \mu$ . For each  $B \in \mathcal{B}$  there exists  $j_B < \kappa$  such that  $g_B(i) < g(i)$  for all  $i > j_B$ . This implies that  $\{i\} \times [g(i), \mu_{f(i)})$  is disjoint from  $B$  for all  $i > j_B$ . Hence  $|B \cap C| \leq \mu_{f(j_B)} < \mu$ . Clearly,  $|C \cap (\{i\} \times \mu)| \leq \mu_{f(i)} < \mu$  for all  $i < \kappa$ ; so  $\mathcal{A} \cup \mathcal{B} \cup \{C\}$  is  $\mu$ -almost disjoint, contrary to the maximality of  $\mathcal{A} \cup \mathcal{B}$ .  $\square$

A positive answer to the first question of Comfort, Erdős and Hechler follows now as a corollary:

**Corollary 2.2.** *If Martin's Axiom holds and  $2^{\aleph_0} > \mu > \text{cf } \mu = \aleph_0$ , then  $\mu \notin \text{MAD}(\mu)$ .*

**2.2. Between  $\mathfrak{a}_\mu$  and  $\mathfrak{b}_\mu$ .** In this section we shall show that  $\text{MAD}(\mu)$  contains the interval of cardinals  $[\mathfrak{a}_\mu, \mathfrak{b}_\mu)$  and even  $[\mathfrak{a}_\mu, \mathfrak{b}_\mu]$  in the case that  $\mathfrak{b}_\mu$  is a successor of a regular cardinal.

**Theorem 2.3.** *For every singular cardinal  $\mu$  and every cardinal  $\lambda$ ,*

$$(2) \quad \mathfrak{a}_\mu \leq \lambda < \mathfrak{b}_\mu \implies \lambda \in \text{MAD}(\mu).$$

*If  $\mathfrak{b}_\mu$  is a successor of a regular cardinal, then  $\mathfrak{a}_\mu \leq \mathfrak{b}_\mu \implies \mathfrak{b}_\mu \in \text{MAD}(\mu)$ .*

To prove the theorem it suffices, by Lemma 1.1, to show that every regular  $\lambda \in [\mathfrak{a}_\mu, \mathfrak{b}_\mu)$  belongs to  $\text{MAD}(\mu)$ .

The proof of this will now be divided into two cases. First we prove that every regular  $\mathfrak{a}_\mu < \lambda < \mu$  belongs to  $\text{MAD}(\mu)$ . The proof in this case does not require any specialized techniques. Then we prove the same for regular  $\mu < \lambda < \mathfrak{b}_\mu$  and for  $\mathfrak{b}_\mu$  itself when it is the successor of a regular cardinal. In this case the proof requires some machinery from pcf theory.

Despite the technical differences between both proofs, they are similar, and could, in fact, be combined to a single proof. Both follow the same scheme of gluing together  $\lambda$  different  $\mu$ -mad families, each of size  $\mathfrak{a}_\mu$ , to a single  $\mu$ -mad family of size  $\lambda$ . In the case  $\lambda < \mu$ , a simple presentation of  $\mu$  as a disjoint union of  $\lambda$  parts works; in the second part we need to rely on smooth pcf scales to get a presentation of  $\mu$  as an *almost increasing* and *continuous* union of length  $\lambda$  of sets of size  $\mu$ .

**2.2.1. The case  $\lambda < \mu$ .**

**Lemma 2.4.** *Suppose  $\mu > \text{cf } \mu = \kappa$ . Then for every regular cardinal  $\lambda$ ,*

$$\mathfrak{a}_\mu \leq \lambda < \mu \implies \lambda \in \text{MAD}(\mu).$$

*Proof.* Suppose  $\lambda$  is regular and  $\mathfrak{a}_\mu \leq \lambda < \mu$ . Since  $\mathfrak{a}_\mu > \kappa = \text{cf } \mu$ ,  $\lambda > \kappa$ .

Fix a strictly increasing sequence of regular cardinals  $\langle \mu_i : i < \kappa \rangle$  such that  $\sup_{i < \kappa} \mu_i = \mu$  and  $\lambda < \mu_0$ . We will work in  $\mu \times \lambda$  instead of  $\mu$ . Let  $S = \{\delta < \lambda : \text{cf } \delta = \kappa\}$ . For each  $\delta \in S$  fix a strictly increasing, continuous sequence  $D_\delta = \langle \gamma_i^\delta : i < \kappa \rangle$  with limit  $\delta$  such that  $\gamma_0^\delta = 0$ . Define

$$F_j^\delta = \bigcup \{ \mu \times \{ \beta \} : \gamma_j^\delta \leq \beta < \gamma_{j+1}^\delta \}.$$

Thus  $\mathcal{F}_\delta = \{F_j^\delta : j < \kappa\}$  is a disjoint family of sets, each set of size  $\mu$ , which covers  $\mu \times \delta$ . Let  $\mathcal{A}_\delta \subseteq [\mu \times \delta]^\mu$  be such that  $\mathcal{A}_\delta \cup \mathcal{F}_\delta$  is mad in  $\mu \times \delta$ ,  $\mathcal{A}_\delta \cap \mathcal{F}_\delta = \emptyset$  and  $|\mathcal{A}_\delta| = \mathfrak{a}_\mu$ .

Define

$$\mathcal{B} = \{\mu \times \{\alpha\} : \alpha < \lambda\} \cup \bigcup_{\delta \in S} \mathcal{A}_\delta.$$

Then  $|\mathcal{B}| = \lambda$  and  $\mathcal{B} \subseteq [\mu \times \lambda]^\mu$ . We will show that  $\mathcal{B}$  is  $\mu$ -mad.

First, observe that  $\mathcal{B}$  is almost disjoint: clearly each element of  $\mathcal{A}_\delta$  is almost disjoint from any set of the form  $\mu \times \{\alpha\}$ , because if  $\alpha < \delta$ , then  $\mu \times \{\alpha\} \subseteq F_j^\delta$  for  $j < \kappa$  such that  $\gamma_j^\delta \leq \alpha < \gamma_{j+1}^\delta$ . Finally, consider  $A_i \in \mathcal{A}_{\delta_i}$ ,  $i < 2$ , with  $\delta_0 < \delta_1$ . Then  $A_0 \subseteq \bigcup_{j < j_0} F_j^{\delta_1}$ , where  $j_0 < \kappa$  is such that  $\delta_0 < \gamma_{j_0}^{\delta_1}$ . Thus  $|A_0 \cap A_1| < \mu$ .

To see that  $\mathcal{B}$  is mad, fix an arbitrary  $Z \in [\mu \times \lambda]^\mu$ . There exists a sequence  $\langle \alpha_i : i < \kappa \rangle$  in  $\lambda$  such that

$$|Z \cap (\mu \times \{\alpha_i\})| \geq \mu_i.$$

If  $|\{\alpha_i : i < \kappa\}| < \kappa$ , then  $|Z \cap (\mu \times \{\alpha\})| = \mu$  for some  $\alpha$ . So suppose that  $|Z \cap (\mu \times \{\alpha_i\})| < \mu$  for every  $i < \kappa$ . Taking a subsequence, we may assume that  $\langle \alpha_i : i < \kappa \rangle$  is strictly increasing. Let  $\delta$  be its supremum. By regularity of  $\lambda$ ,  $\delta \in S$  and therefore  $Z \in [\mu \times \delta]^\mu$ . Shrinking  $Z$  if necessary, assume that  $Z \subseteq \bigcup_{i < \kappa} \mu \times \{\alpha_i\}$ . Then  $|Z \cap F_j^\delta| < \mu$  for every  $j < \kappa$ . Thus,  $|Z \cap A| = \mu$  for some  $A \in \mathcal{A}_\delta$ . This completes the proof.  $\square$

**Corollary 2.5.** *Let  $\mu > \text{cf } \mu = \kappa$ . If  $\mathfrak{a}_\kappa \leq \mu$ , then  $[\mathfrak{a}_\kappa, \mu] \subseteq \text{MAD}(\mu)$ . In particular, if  $2^\kappa < \mu$ , then  $\mu \in \text{MAD}(\mu)$ .*

Corollary 2.5 answers affirmatively the second question of Erdős and Hechler in [7].

**2.2.2. The case  $\lambda > \mu$ .** A  $(\mu, \lambda)$ -scale, for a regular cardinal  $\lambda > \mu$ , is a sequence  $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle \subseteq \prod_{i < \kappa} \mu_i$  such that  $\langle \mu_i : i < \kappa \rangle$  is a strictly increasing sequence of regular cardinals with limit  $\mu$ , and so that  $\alpha < \beta < \lambda \implies f_\alpha <^* f_\beta$  and for every  $g \in \prod_{i < \kappa} \mu_i$  there is  $\alpha < \lambda$  with  $g <^* f_\alpha$ . The relation  $f <^* g$  means that the set  $\{i < \kappa : f(i) \geq g(i)\}$  is bounded in  $\kappa$ . When  $\mu$  is fixed, “ $(\mu, \lambda)$ -scale” will be abbreviated by “ $\lambda$ -scale”. A  $\lambda$ -scale  $\bar{f}$  is *smooth* if for every  $\delta < \lambda$  with  $\text{cf } \delta > \kappa$ , the sequence  $\bar{f} \upharpoonright \delta = \langle f_\alpha : \alpha < \delta \rangle$  is cofinal in  $(\prod_{i < \kappa} f_\delta(i), <^*)$ . In this case we say that  $f_\delta$  is an *exact upper bound* of  $\bar{f} \upharpoonright \delta$ . We will denote by  $[f, g]$  the set  $\{(i, \alpha) : i < \kappa \wedge f(i) \leq \alpha < g(i)\}$ .

The proof in the present case goes through two steps. First, it is shown that whenever a smooth  $(\mu, \lambda)$ -scale exists and  $\mathfrak{a}_\mu < \lambda$ , it follows that  $\lambda \in \text{MAD}(\mu)$ . Then it is shown that for every  $\mu < \lambda < \mathfrak{b}_\mu$  there is a smooth  $(\mu, \lambda)$ -scale and that in case  $\mathfrak{b}_\mu$  is a successor of a regular cardinal there is also a smooth  $(\mu, \mathfrak{b}_\mu)$ -scale.

**Lemma 2.6.** *Assume  $\lambda > \mu > \text{cf } \mu = \kappa$  and there exists a smooth  $(\mu, \lambda)$ -scale. If  $\mathfrak{a}_\mu \leq \lambda$ , then  $\lambda \in \text{MAD}(\mu)$ .*

*Proof.* Suppose there exists a smooth  $\lambda$ -scale  $\langle g_\xi : \xi < \lambda \rangle \subseteq \prod_{i < \kappa} \mu_i$ . Let  $S = \{\delta < \lambda : \text{cf } \delta = \kappa\}$ , and for each  $\delta \in S$  fix a strictly increasing, continuous, sequence  $\langle \gamma_i^\delta : i < \kappa \rangle$  with limit  $\delta$  such that  $\gamma_0^\delta = 0$  and put  $D_\delta = \{\gamma_i^\delta : i < \kappa\}$ .

By induction on  $\xi < \lambda$  we construct a smooth  $\lambda$ -scale  $\bar{f} = \langle f_\xi : \xi < \lambda \rangle \subseteq \prod_{i < \kappa} \mu_i$  that satisfies the following two conditions:

- (1) If  $\delta < \lambda$  is a limit and  $\text{cf } \delta \leq \kappa$ , then  $f_\delta(i) = \sup_{\xi \in D_\delta} f_\xi(i)$ .
- (2) For each  $\xi < \lambda$  the set  $[f_\xi, f_{\xi+1}] = \{(i, \alpha) : f_\xi(i) \leq \alpha < f_{\xi+1}(i)\}$  has cardinality  $\mu$ .

By induction on  $\xi < \lambda$  we define an increasing and continuous sequence of ordinals  $\zeta(\xi) < \lambda$  and a  $<^*$ -increasing sequence of functions  $f_\xi \in \prod_{i < \kappa} \mu_i$  so that  $f_\xi = g_{\zeta(\xi)}$  for all  $\xi < \lambda$  *except* when  $\xi$  is limit of cofinality  $\leq \kappa$ . Then  $\bar{f} := \langle f_\xi : \xi < \lambda \rangle$  will be a smooth  $\lambda$ -scale as required.

At a limit stage  $\xi$  of cofinality  $\leq \kappa$ , let  $\zeta(\xi) = \bigcup_{\xi' < \xi} \zeta(\xi')$  and use condition (1) to define  $f_\xi$ ; at successor  $\xi + 1$  choose  $\zeta(\xi + 1)$  so that  $\max\{f_\xi, g_{\zeta(\xi)}\} <^* g_{\zeta(\xi+1)}$  and (2) holds, and let  $f_{\xi+1} = g_{\zeta(\xi+1)}$ . Suppose now that  $\xi$  is a limit of cofinality  $> \kappa$ . By the smoothness of  $\bar{g}$ , and since  $\langle g_{\zeta(\xi')} : \xi' < \xi \rangle$  is  $<^*$ -increasing, after defining  $\zeta(\xi) = \bigcup_{\xi' < \xi} \zeta(\xi')$  we get that  $g_{\zeta(\xi)}$  is an exact upper bound of  $\langle g_{\zeta(\xi')} : \xi' < \xi \rangle$ . But then  $g_{\zeta(\xi)}$  is also an exact upper bound of  $\langle f_{\zeta(\xi')} : \xi' < \xi \rangle$ , and we let  $f_\xi = g_{\zeta(\xi)}$ .

Let  $f_\lambda$  be defined on  $\kappa$  by  $f_\lambda(i) = \mu_i$ .

*Claim 2.7.* Suppose  $\delta \leq \lambda$  and  $A \subseteq [0, f_\delta)$  has cardinality  $\mu$ . If  $\text{cf } \delta > \kappa$ , there is some  $\delta' < \delta$  so that  $|A \cap [0, f_{\delta'})| = \mu$ .

*Proof.* Find  $g < f_\delta$  so that  $\sum_{i < \kappa} |A \cap (i \times g(i))| = \mu$ . By smoothness there exists some  $\delta' < \delta$  so that  $g <^* g_{\delta'}$ . □

For every  $\xi < \lambda$ , let  $A_\xi = [f_\xi, f_{\xi+1})$  and let  $\mathcal{A} = \{A_\xi : \xi < \lambda\}$ . Then  $\mathcal{A} \subseteq \mathcal{P}([0, f_\lambda))$  is  $\mu$ -almost disjoint and  $|\mathcal{A}| = \lambda$ .

For each  $\delta \in S$  and  $i < \kappa$ , let  $F_i^\delta = [f_{\gamma_i^\delta}, f_{\gamma_{i+1}^\delta})$ . Then  $\mathcal{F}_\delta = \{F_i^\delta : i < \kappa\}$  is a  $\mu$ -almost disjoint family whose union is, by condition (1) on  $\bar{f}$ , equal to  $[0, f_\delta)$ . Fix a  $\mu$ -ad family  $\mathcal{B}_\delta \subseteq \mathcal{P}([0, f_\delta))$  such that  $|\mathcal{B}_\delta| = \mathfrak{a}_\mu$ ,  $\mathcal{B}_\delta \cup \mathcal{F}_\delta$  is  $\mu$ -mad and  $\mathcal{B}_\delta \cap \mathcal{F}_\delta = \emptyset$  (by Lemma 1.2).

*Claim 2.8.* If  $\delta \in S$  and  $B \in \mathcal{B}_\delta$ , then for all  $i < \kappa$ , it follows that  $|B \cap [0, f_{\gamma_i^\delta})| < \mu$ .

*Proof.* If not so, let  $i_0 < \kappa$  be the largest value so that  $|B \cap [0, f_{\gamma_{i_0}^\delta})| < \mu$ ;  $i_0$  exists because  $D_\delta$  is closed. Now  $|B \cap F_{i_0}^\delta| = \mu$ , a contradiction. □

Let  $\mathcal{B} = \bigcup_{\delta \in S} \mathcal{B}_\delta$ . Then  $|\mathcal{B}| = \mathfrak{a}_\mu \cdot \lambda = \lambda$ , and therefore  $|\mathcal{A} \cup \mathcal{B}| = \lambda$ . We will show now that  $\mathcal{A} \cup \mathcal{B}$  is  $\mu$ -mad.

Suppose that  $A = A_\xi \in \mathcal{A}$  and  $B \in \mathcal{B}_\delta$  for some  $\delta \in S$ . If  $\xi \geq \delta$ , then clearly  $|A \cap B| < \mu$ , and if  $\xi < \delta$ , there is some  $i < \kappa$  so that  $A_\xi \subseteq^* F_i^\delta$  and  $|A \cap B| < \mu$  follows from Claim 2.8.

If  $B_1 \in \mathcal{B}_{\delta_1}$  and  $B_2 \in \mathcal{B}_{\delta_2}$  with  $\delta_1 < \delta_2$  in  $S$ , then there is some  $i < \kappa$  so that  $f_{\delta_1} <^* f_{\gamma_i^{\delta_2}}$  and Claim 2.8 gives  $|B_1 \cap B_2| < \mu$ .

This establishes that  $\mathcal{A} \cup \mathcal{B}$  is  $\mu$ -mad. To verify maximality, let  $Z \subseteq [0, f_\lambda)$  be arbitrary of size  $\mu$ . By Claim 2.7 the first  $\xi \leq \lambda$  for which  $|Z \cap [0, f_\xi)| = \mu$  is either a successor or of cofinality  $\leq \kappa$ . Cofinality  $< \kappa$  is ruled out by condition (1) on  $\bar{f}$ . The case  $\xi$  successor implies that  $|Z \cap A_\xi| = \mu$ . Finally, in the remaining case  $\xi = \delta \in S$ , there is some  $B \in \mathcal{B}_\delta$  so that  $|Z \cap B| = \mu$ . □

Now the proof of Theorem 2.3 will be completed by the following lemma, whose proof is actually found implicitly in [15]. We shall sketch a proof here too.

**Lemma 2.9.** *Suppose  $\mu$  is singular and  $\mu < \lambda < \mathfrak{b}_\mu$ ,  $\lambda$  regular. Then there is a smooth  $(\mu, \lambda)$ -scale. If  $\mathfrak{b}_\mu$  is a successor of a regular cardinal, there is also a smooth  $(\mu, \mathfrak{b}_\mu)$ -scale.*

*Proof.* Since  $\lambda < \mathfrak{b}_\mu$ , there exists a product  $\prod_{i < \kappa} \mu_i$ , where  $\kappa = \text{cf } \mu$ , so that  $\mathfrak{b}(\prod_{i < \kappa} \mu_i, <^*) > \lambda$ .

By Claim 1.3 in [15] there exists a  $\lambda$ -scale  $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$  in some  $\prod_{i < \kappa} \mu'_i$  such that for all regular  $\theta \in (\kappa, \mu)$ , every  $\alpha < \lambda$  with  $\text{cf } \alpha = \theta$  satisfies that  $\bar{f} \upharpoonright \alpha$  is *flat*, that is, is equivalent modulo the bounded ideal on  $\kappa$  to a strictly increasing sequence of ordinal functions on  $\kappa$ .

By Lemma 15 in [10], every  $\alpha < \lambda$  with  $\text{cf } \alpha > \kappa$  satisfies that  $\bar{f} \upharpoonright \alpha$  has an exact upper bound. Now it is clear how to replace  $\bar{f}$  by a smooth  $\lambda$ -scale.

Suppose now that  $\mathfrak{b}_\mu = \lambda^+$ ,  $\lambda = \text{cf } \lambda$ . By [14], 4.1, the set  $S_{<\lambda}^{\lambda^+} := \{\alpha : \alpha < \lambda^+ \wedge \text{cf } \alpha < \lambda\}$  is a union of  $\lambda$  sets, each of which carries a square sequence. Therefore,  $S_{<\lambda}^{\lambda^+} \in I[\lambda]$ . By 2.5 in chapter 1 of [15], there exists a  $(\mu, \mathfrak{b}_\mu)$ -scale in which all points of cofinality  $< \mu$  are flat and therefore a smooth  $(\mu, \mathfrak{b}_\mu)$ -scale.  $\square$

In contrast to the case of singular  $\mu$ , let us mention the following result of A. Blass [4], which generalizes Hechler's result [8]: it is consistent that  $\text{MAD}(\aleph_0) = C$ , for any prescribed closed set of uncountable cardinals  $C$  that satisfies that  $[\aleph_1, \aleph_1 + |C|] \subseteq C$  and  $\lambda^+ \in C$  whenever  $\lambda \in C$  has countable cofinality. For example, by Blass' or by Hechler's results there are universes of set theory in which  $\text{MAD}(\aleph_0) = \{\aleph_1, \aleph_{\omega+1}\}$ . By Corollary 2.5, in any universe that satisfies this, it follows that  $[\aleph_1, \aleph_{\omega+1}] \subseteq \text{MAD}(\aleph_\omega)$ .

Recently Brendle [5], using techniques from [16], proved the consistency of  $\mathfrak{a} = \aleph_\omega$ .

**Problem 2.10.** Is it consistent that  $\mathfrak{a}_{\aleph_\omega} = \aleph_\omega$ ?

### 3. CONSISTENCY RESULTS ON $\text{MAD}(\aleph_\omega)$ FROM LARGE CARDINAL AXIOMS

The inequality (1) can be used to control  $\text{MAD}(\aleph_\omega)$  by first increasing  $\mathfrak{b}_{\aleph_\omega}$  and then increasing  $\mathfrak{b}$ . PCF theory implies that whenever the SCH fails at a singular cardinal  $\mu$ , it follows that  $\mathfrak{b}_\mu > \mu^+$ . On the other hand,  $\mathfrak{b}_\mu$  cannot be changed by a ccc forcing.

Before we state the result, let us recall some pcf terminology:

$$\text{pcf}\{\aleph_n : n < \omega\} = \left\{ \mathfrak{b} \left( \prod_n \aleph_n, \leq_I \right) : I \subseteq \mathcal{P}(\omega) \text{ is a proper ideal} \right\}.$$

The relation  $<_I$  is defined by  $f <_I g \Leftrightarrow \{n : f(n) \geq g(n)\} \in I$ .

$\text{pcf}\{\aleph_n : n < \omega\}$  is an interval of regular cardinals and has a maximum. For every  $\lambda \in \text{pcf}\{\aleph_n : n < \omega\}$  there exists a *pcf generator*  $B_\lambda \subseteq \omega$  so that the following holds: denote by  $J_{<\lambda}$  the ideal that is generated by  $\{B_\theta : \theta \in \text{pcf}\{\aleph_n : n < \omega\} \wedge \theta < \lambda\}$ ; then

$$\lambda = \mathfrak{b} \left( \prod_n \aleph_n, \leq_{J_{<\lambda}} \right).$$

Finally,  $(\aleph_\omega)^{\aleph_0} = \max \text{pcf}\{\aleph_n : n < \omega\} \times 2^{\aleph_0}$ . Therefore, if  $\aleph_\omega$  is a strong limit,  $2^{\aleph_\omega} = \max \text{pcf}\{\aleph_n : n < \omega\}$ .

*Fact 3.1.* For every  $\beta < \omega_1$  it is consistent (from large cardinal axioms) that  $2^{\aleph_\omega} = \mathfrak{b}_\mu = \aleph_{\omega+\beta+1}$ .

*Proof.* Let  $V$  be any universe of set theory in which  $\aleph_\omega$  is a strong limit cardinal and  $2^{\aleph_\omega} = \max \text{pcf}\{\aleph_n : n < \omega\} = \aleph_{\omega+\beta+1}$  [13], [9].

In  $V$ , the ideal  $J_{<\max \text{pcf}\{\aleph_n : n < \omega\}}$  is proper and is generated by countably many sets. Therefore, by simple diagonalization there exists an infinite  $B \subseteq \omega$  so that

$J_{\langle \max \text{pcf}\{\aleph_n : n < \omega\} \upharpoonright B}$  is contained in the ideal of finite subsets of  $B$ . Since  $\mathfrak{b}(\prod_n \aleph_n, \leq_{J_{\langle \max \text{pcf}\{\aleph_n : n < \omega\}}}) = \aleph_{\omega+\beta+1}$ , it follows that  $\mathfrak{b}(\prod_{n \in B} \aleph_n, \leq^*) = \aleph_{\omega+\beta+1}$ ; hence  $\mathfrak{b}_{\aleph_\omega} = \aleph_{\omega+\beta+1}$ .  $\square$

**Theorem 3.2.** *For every  $\beta < \omega_1$  and  $\alpha \leq \omega + \beta + 2$ , it is consistent (from large cardinals) that  $2^{\aleph_\omega} = \aleph_{\omega+\beta+2}$  and  $\text{MAD}(\aleph_\omega) = [\aleph_\alpha, \aleph_{\omega+\beta+2}]$ .*

*Proof.* Start from a model  $V$  in which  $2^{\aleph_0} = \aleph_1$ ,  $\aleph_\omega$  is strong limit and  $2^{\aleph_\omega} = \aleph_{\omega+\beta+2}$ . Such a model exists by the previous Fact.

For every regular  $\aleph_\omega < \lambda \leq \aleph_{\omega+\beta+2}$ , there is a smooth  $\lambda$ -scale by Lemma 2.9. Consequently, there is also a smooth  $\aleph_{\omega+\beta+2}$ -scale.

Now apply Theorem 2.3 to finish the proof.  $\square$

By Theorem 5.4(b) in [3], after adding many Cohen subsets to  $\omega_1$ ,  $\max \text{MAD}(\aleph_\omega)$  does not increase by much. Therefore, it is consistent to have  $\text{MAD}(\aleph_\omega) = [\aleph_1, \aleph_{\omega+\beta+2}]$  as above, and to have  $2^{\aleph_\omega}$  arbitrarily large.

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