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SIMPLE CORONA C^*-ALGEBRAS

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Abstract. Let A be a non-unital and σ-unital simple C^*-algebra. We show that if $M(A)/A$ is simple, then $M(A)/A$ is purely infinite. We also show that $M(A)/A$ is simple if and only if A has a continuous scale provided that A is not isomorphic to $\mathcal{K}$, the compact operators.

Introduction

Let $\mathcal{K}$ be the C^*-algebra of all compact operators on an infinite-dimensional separable Hilbert space H and $B(H)$ the C^*-algebra of all bounded operators on H. It is well known that $\mathcal{K}$ is a non-unital simple C^*-algebra and $\mathcal{K}$ is the only closed ideal of $B(H)$. Consequently the Calkin algebra, $B(H)/\mathcal{K}$ is simple. It is also known that the multiplier algebra $M(\mathcal{K})$ of $\mathcal{K}$ is isomorphic to $B(H)$.

Let A be a non-unital and σ-unital simple C^*-algebra. The ideal structure of $M(A)/A$ was first studied by G. A. Elliott. He showed in [Ell] that for a non-unital matroid C^*-algebra A, $M(A)/A$ is simple if and only if A is finite. It was later proved in [Ln1] that, for a non-unital separable simple C^*-algebra $A \ncong \mathcal{K}$, $M(A)/A$ is simple if and only if A has a continuous scale (see Definition 1.1). Furthermore, it is proved (in [Ln1]) that if $A \ncong \mathcal{K}$ is a non-unital and σ-unital simple C^*-algebra with a continuous scale, then $M(A)/A$ is always simple. In this short note we first show that in fact the converse also holds: if $A \ncong \mathcal{K}$ is a non-unital and σ-unital simple C^*-algebra such that $M(A)/A$ is simple, then A has a continuous scale. The renewed interests in the corona algebras $M(A)/A$ are related to the classification of nuclear C^*-algebras and the study of essential extensions by simple C^*-algebras (see for example, [Ln2] and [Ln4]).

S. Zhang showed (in [Zh1]) that if A has real rank zero and $M(A)/A$ is simple, then $M(A)/A$ is a purely infinite simple C^*-algebra. Seemingly $M(A)/A$ is infinite whenever A is simple. Recently M. Rørdam ([Ro3]) showed that there exist separable nuclear simple C^*-algebras that are infinite but not purely infinite. Such non-unital simple C^*-algebras can have continuous scales (see 2.1 below). The main purpose of this note is to prove that $M(A)/A$ is always a purely infinite simple C^*-algebra if it is simple. Thus if $A \ncong \mathcal{K}$ and has a continuous scale, then $M(A)/A$ is in fact a purely infinite simple C^*-algebra even though A may be infinite but not purely infinite. It was proved by S. Zhang (see [Zh2]) that every purely infinite

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simple C^*-algebra has real rank zero. Therefore, $M(A)/A$ has real rank zero if A has a continuous scale regardless of whether A contains any projections or not. An immediate consequence is that, for any non-unital separable simple C^*-algebra A, $M(A)/A$ always contains infinite projections and every projection is infinite.

1. Preliminaries

Definition 1.1. Let A be a C^*-algebra and $a, b \in A$ be two positive elements. We write $a \overset{\sim}{<} b$ if there is $x \in A$ such that $x^*x = a$ and $xx^* \in \overline{bAb}$.

We write $a \lesssim b$ if there is a sequence of elements $r_n \in A$ such that $r_n^* b r_n \to a$ as $n \to \infty$. If $a \overset{\sim}{<} b$, then $a \lesssim b$. If $a \le b$, then $a \lesssim b$ and $a \overset{\sim}{<} b$. If $a \lesssim b$ and $b \lesssim c$, then $a \lesssim c$. Also, if $a \overset{\sim}{<} b$ and $b \overset{\sim}{<} c$, then $a \overset{\sim}{<} c$. If $p, q \in A$ are projections, then $p \overset{\sim}{<} q$ (or $p \lesssim q$) if and only if there is $v \in A$ such that $v^*v = p$ and $vv^* \le q$.

For more details about these two relations, readers are referred to [Cu1], [Cu2], [Cu3] and [Ro1].

Definition 1.2. Let $\varepsilon > 0$. We define a function $f_\varepsilon \in C_0((0,1])$ as follows:

$$f_\varepsilon(t) = \begin{cases} 1, & \text{if } t > \varepsilon; \\ 2\varepsilon^{-1}(t - \varepsilon/2) & \text{if } \varepsilon/2 < t \le \varepsilon; \\ 0 & \text{if } 0 < t \le \varepsilon/2. \end{cases}$$

Lemma 1.3. *Let A be a C^*-algebra and $a, b, c, d \in A$ be four positive elements in A. Suppose that $a \overset{\sim}{<} c$, $b \overset{\sim}{<} d$ and $cd = dc = 0$. Then*

$$a + b \overset{\sim}{<} c + d.$$

Proof. Suppose that $x_1, x_2 \in A$ such that $x_1^*x_1 = a$, $x_1x_1^* \in \overline{cAc}$, $x_2^*x_2 = b$ and $x_2x_2^* \in \overline{dAd}$. Since $cd = dc = 0$, we have $x_1^*x_2 = x_2^*x_1 = 0$. Let $z = x_1 + x_2$, $c_1 = x_1x_1^*$ and $c_2 = x_2x_2^*$. Then

$$z^*z = (x_1 + x_2)^*(x_1 + x_2) = x_1^*x_1 + x_2^*x_2 = a + b \quad \text{and}$$

$$zz^* = x_1x_1^* + x_1x_2^* + x_2x_1^* + x_2x_2^* = c_1 + d_1 + (x_1x_2^* + x_2x_1^*).$$

For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|f_\delta(|x_1^*|)|x_1^*| - |x_1^*|\| < \varepsilon/(2\|x_2\| + 2) \quad \text{and} \quad \|f_\delta(|x_2^*|)|x_2^*| - |x_2^*|\| < \varepsilon/(2\|x_1\| + 2).$$

Write $x_1 = |x_1^*|u_1$, $x_2 = |x_2^*|u_2$, $x_1^* = u_3|x_1^*|$ and $x_2^* = u_3|x_2^*|$ in A'' and put $e = |x_1^*| + |x_2^*|$. Note that $f_\delta(e) = f_\delta(|x_1^*|) + f_\delta(|x_2^*|)$. We estimate that

$$\|f_\delta(e)(x_1x_2^*)f_\delta(e) - x_1x_2^*\| < \varepsilon.$$

Note that $f_\delta(e) \in \overline{(c+d)A(c+d)}$. Hence $f_\delta(e)(x_1x_2^*)f_\delta(e) \in \overline{(c+d)A(c+d)}$ for all $\delta > 0$. It follows that $x_1x_2^* \in \overline{(c+d)A(c+d)}$. Exactly the same argument shows that $x_2x_1^* \in \overline{(c+d)A(c+d)}$. Hence

$$zz^* \in \overline{(c+d)A(c+d)}.$$

□

Lemma 1.4. *Let A be a σ-unital C^*-algebra and $\{e_n\}$ be an approximate identity such that*

$$e_{n+1}e_n = e_ne_{n+1} = e_n, \;\; n = 1, 2, \ldots.$$

Then, for any $1 > \delta > 0$ and integers $n_0 < n_1 < n_2 < n$,

$$f_\delta(e_n - e_{n_0})(e_{n_2} - e_{n_1}) = (e_{n_2} - e_{n_1}).$$

In particular,

$$(e_{n_2} - e_{n_1}) \overset{\sim}{<} f_\delta(e_n - e_{n_0}).$$

Proof. Suppose that $a, b \in A$ are positive elements with $0 \le a, b \le 1$ such that $ab = ba = b$. Then $f_\varepsilon(a)b = b$ if $0 < \varepsilon < 1$. This follows from the fact that we may assume that a and b are functions on a compact subset of the plane since a commutes with b. We have

$$\begin{aligned}(e_n - e_{n_0})(e_{n_2} - e_{n_1}) &= e_n e_{n_2} - e_n e_{n_1} - e_{n_0} e_{n_2} + e_{n_0} e_{n_1} \\ &= e_{n_2} - e_{n_1} - e_{n_0} + e_{n_0} = e_{n_2} - e_{n_1}.\end{aligned}$$

Thus the lemma follows. □

2. Simple C^*-algebras with continuous scales

Definition 2.1. Let $A \not\cong \mathcal{K}$ be a non-unital and σ-unital simple C^*-algebra. Let $\{e_n\}$ be an approximate identity for A such that $e_{n+1}e_n = e_n e_{n+1} = e_n$, $n = 1, 2, \ldots$. We say A has a continuous scale if for any nonzero element $a \in A_+$, there exists an integer $n_0 > 0$ such that

$$e_m - e_n \lesssim a \text{ for } m > n \ge n_0.$$

It is proved in [LZ] that in a purely infinite simple C^*-algebra A, $a \overset{\sim}{<} b$ for any two nonzero positive elements in A. It follows that every non-unital but σ-unital purely infinite simple C^*-algebra has a continuous scale.

The following proposition was known in the case that A is an AF-algebra. It justifies the terminology "continuous scale".

Suppose that A is a non-unital and σ-unital simple C^*-algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$. Fix any nonzero projection $e \in A$. Denote by T the set of those quasi-traces τ on A such that $\tau(e) = 1$. Note that T is a (weak $*$-) compact convex set. Let $a \in M(A)_+$. Define $\hat{a}(\tau) = \tau(a)$ for $\tau \in T$. Then $\hat{a}$ is a lower semi-continuous affine function on T. If $a \in A$, then $\hat{a}$ is continuous.

Proposition 2.2. *Let A be a non-unital but σ-unital simple C^*-algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$. Let 1 be the identity of $M(A)$. Then A has a continuous scale if and only if $\hat{1}(\tau) = \tau(1)$ for $\tau \in T$ is a continuous function on T.*

Proof. Suppose that

$$T = \{\tau : \tau(e) = 1, \tau \text{ quasi-traces defined on } A\}.$$

Let $\{e_n\}$ be an approximate identity consisting of projections.

Suppose that A has a continuous scale. For any $\varepsilon > 0$, it is known (see Lemma 3.5.7 in [Ln3] for example) that there exists a projection $p_0 \in A$ such that $\tau(p_0) < \varepsilon$ for all $\tau \in T$. Since A has a continuous scale, there exists $N > 0$ such that

$$(e_m - e_n) \lesssim p_0 \text{ for all } m > n \ge N.$$

This implies that $\tau(e_m - e_n) < \varepsilon$ for all $\tau \in T$ whenever $m > n \geq N$. This implies that $\widehat{e_n}$ converges to $\hat{1}$ uniformly on T. Therefore $\hat{1}$ is continuous on T.

Now suppose that $\hat{1}$ is continuous. For any $a \in A$, since A has real rank zero, there is a nonzero projection $q_0 \in \overline{aAa}$. Let $d = \inf\{\tau(q_0) : \tau \in T\}$. Since A is simple and T is compact, we see that $d > 0$. Since $\hat{1}$ is continuous, there is $N > 0$ such that

$$\tau(e_m - e_n) < d \ \text{ for all } \tau \in T \quad \text{and} \quad \text{for } m > n \geq N.$$

It follows from III2.2 and III 2.3 in [BH] that $e_m - e_n \lesssim q_0 \lesssim a$. Therefore A has a continuous scale. □

To see more simple C^*-algebras with continuous scales, we offer the following.

Proposition 2.3. *For any separable simple C^*-algebra $A \not\cong \mathcal{K}$ there exists a non-unital hereditary C^*-subalgebra $B \subset A$ such that B has a continuous scale.*

Proof. Let $\{e_n\}$ be an approximate identity for A such that $e_{n+1}e_n = e_ne_{n+1} = e_n$, $n = 1, 2, ...$ Let I_0 be as in Lemma 2.1 in [Ln1] and I be the closure of I_0. It follows from Lemma 2.4 in [Ln1] that I is a closed ideal containing A properly. In particular, there is a nonzero positive element $b \in I_0$ that is not in A. In fact, by the proof of Lemma 2.4 in [Ln1], one may write $b = \sum_{n=1}^\infty b_n$ with each $b_n \in \overline{(e_{n+1} - e_n)A(e_{n+1} - e_n)}$. Let $b_1 = \sum_{k=1}^\infty b_{2k}$ and $b_2 = \sum_{n=1}^\infty b_{2k+1}$. Then one of them is not in A, say $b_1 \notin A$. For any $a \in A_+ \setminus \{0\}$, there exists n_0 such that

$$\sum_{k=n_0}^{m} b_{2k} \lesssim a \ \text{ for all } m > n_0.$$

Put $h = \sum_{k=1}^\infty (1/k)b_{2k}$. One checks that $h \in A$. Define $B = \overline{hAh}$. Clearly B is not unital. Let $g_n \in C_0((0, 1])$ such that $g_n = f_{\varepsilon_n}$ for some decreasing sequence of $\{\varepsilon_n\}$ with $0 < \varepsilon_{n+1} < \varepsilon_n/2 < 1/2n$. Define $d_n = g_n(h)$. It follows that $d_{n+1}d_n = d_{n+1}d_n = d_n$, $n = 1, 2....$ Moreover, for each $m > n$,

$$d_m - d_n \ \tilde{<} \sum_{k=n}^{l(n,m)} b_{2k}$$

for some $l(n, m) > m > n$. It follows that B has a continuous scale. □

It was proved in [Ln1] that, for a non-unital separable simple C^*-algebra $A \not\cong \mathcal{K}$, $M(A)/A$ is a simple C^*-algebra if and only if A has a continuous scale. Furthermore, $M(A)/A$ is always simple if $A \not\cong \mathcal{K}$ is a non-unital and σ-unital simple C^*-algebra with a continuous scale. The following theorem shows that the converse also holds for the non-separable case. Furthermore, the second part of the theorem shows that the definition of continuous scale can be strengthened slightly.

Theorem 2.4. *Let $A \not\cong \mathcal{K}$ be a non-unital and σ-unital simple C^*-algebra. Then $M(A)/A$ is simple if and only if A has a continuous scale.*

Moreover, if A has a continuous scale, then for any approximate identity $\{e_n\}$ with $e_{n+1}e_n = e_ne_{n+1} = e_n$ (for all n) and any $a \in A_+$ with $0 \leq a \leq 1$, there exists an integer n_0 such that

$$e_m - e_n \ \tilde{<}\ a$$

for all $m > n \geq n_0$.

Proof. By 2.8 in [Ln1], we may assume that $M(A)/A$ is simple. Let $\{e_n\}$ be an approximate identity such that $e_{n+1}e_n = e_ne_{n+1} = e_n$. Fix a nonzero element $a \in A_+$. It follows from p. 67 in [AS] that there exists an element $b \in \overline{aAa}$ such that $\mathrm{sp}(b) = [0,1]$. Thus one obtains a sequence of mutually orthogonal nonzero elements $\{b_n\} \subset \overline{aAa}$. For each n, there are n mutually orthogonal nonzero elements $c_n^{(1)}, ..., c_n^{(n)}$ in $\overline{b_nAb_n}$. Since A is simple, by Lemma 2.3 in [Ln1], there are nonzero positive elements $0 \le d_n^{(0)}, d_n^{(1)}, ..., d_n^{(n)} \le 1$ and $w_n^{(0)}, w_n^{(1)}, ..., w_n^{(n)} \in A$ such that $(w_n^{(i)})^*w_n^{(i)} = d_n^{(0)}$ and $w_n^{(i)}(w_n^{(i)})^* = d_n^{(i)}$, $i = 0, 1, 2, ..., n$, $d_n^{(i)} \in \overline{c_n^{(i)}Ac_n^{(i)}}$ for $i = 1, 2, ..., n$ and $d_n^{(0)} \in \overline{(e_{2n} - e_{2n-1})A(e_{2n} - e_{2n-1})}$. Define

$$c = \sum_{n=1}^{\infty} d_n^{(0)}.$$

It is easy to verify that $c \in M(A)\backslash A$. Since $M(A)/A$ is simple, by 3.3.6 in [Ln3] there are $x_1, ..., x_m \in M(A)/A$ such that

$$\sum_{j=1}^{m} x_j^*\pi(c)x_j = 1.$$

Thus there are $z_1, z_2, ..., z_m \in M(A)$ such that

$$1 - \sum_{j=1}^{m} z_i^*cz_i \in A.$$

Moreover,

$$1 - \sum_{j=1}^{m} z_j^*(1 - e_{m+1})c(1 - e_{m+1})z_j \in A.$$

Let $1/4 > \varepsilon > 0$. Choose $0 < \delta < \varepsilon/4$. There exists $N(> m)$ such that

$$\|(e_k - e_N) - (e_k - e_N)^{1/2}[\sum_{j=1}^{m} z_j^*(1 - e_{m+1})c(1 - e_{m+1})z_j](e_k - e_N)^{1/2}\| < \delta/2$$

and

$$\|(e_k - e_N)^{1/2}[\sum_{j=1}^{m} z_j^*(e_{m(k)} - e_{m+1})c(e_{m(k)} - e_{m+1})z_j](e_k - e_N)^{1/2}$$
$$-(e_k - e_N)^{1/2}[\sum_{j=1}^{m} z_j^*(1 - e_{m+1})c(1 - e_{m+1})z_j](e_k - e_N)^{1/2}\| < \delta/2$$

for all $k \ge N$ and some $m(k) \ge k$. Hence

$$\|(e_k - e_N)^{1/2}[\sum_{j=1}^{m} z_j^*(e_{m(k)} - e_{m+1})c(e_{m(k)} - e_{m+1})z_j](e_k - e_N)^{1/2} - (e_k - e_N)\| < \delta.$$

It follows from 2.2 in [Ro2] that

$$f_\varepsilon(e_k - e_N) \ \stackrel{\sim}{<} (e_k - e_N)^{1/2}[\sum_{j=1}^{m} z_j^*(e_{m(k)} - e_{m+1})c(e_{m(k)} - e_{m+1})z_j](e_k - e_N)^{1/2}$$

for all k. It follows from 1.4 that for any $l > N + 1$,

$$e_l - e_{N+1} \stackrel{\sim}{<} f_\varepsilon(e_{l+1} - e_N).$$

On the other hand,

$$(e_{m(k)} - e_{m+1})c(e_{m(k)} - e_{m+1}) \underset{\sim}{<} \sum_{j=m}^{m(k)+1} d_j^{(0)}$$

and by 1.3

$$\sum_{j=1}^{m} z_j^*(e_{m(k)} - e_{m+1})c(e_{m(k)} - e_{m+1})z_j \underset{\sim}{<} \sum_{i=1}^{m}(\sum_{j=m}^{m(k)+1} d_j^{(i)}) = \sum_{j=m}^{m(k)+1} (\sum_{i=1}^{m} d_j^{(i)})$$
$$\underset{\sim}{<} \sum_{j=1}^{m(k)+1} b_j \underset{\sim}{<} b \underset{\sim}{<} a.$$

It follows that

$$e_l - e_{N+1} \underset{\sim}{<} a$$

for all $l > N$. Thus A has a continuous scale, and the last part of the theorem follows. □

3. Infiniteness of $M(A)/A$

The following lemma significantly simplifies the proof of 3.2, and it will also be used in [Ln4].

Lemma 3.1. *Let A be a σ-unital C^*-algebra and $D \subset M(A)$ be a separable C^*-algebra. Then A admits an approximate identity $\{e_n\}$ satisfying the following:*

$$e_{n+1}e_n = e_n = e_ne_{n+1} \quad \textit{and} \quad \lim_{n\to\infty} \|e_nd - de_n\| = 0 \ \textit{for} \ d \in D.$$

Proof. Fix a strictly positive element $a \in A$ with $0 \le a \le 1$. Then $\{f_{1/n}(a)\}$ forms an approximate identity for A.

Let

$$C = \{\sum_{i=1}^{j} \alpha_i f_{1/n_i} : 0 \le \alpha_i \le 1, \sum_{i=1}^{j} \alpha_i = 1\}.$$

It follows from the proof of 3.12.14 in [P] (see also 3.12.15 in [P] and [AP]) that there exists a sequence of elements $\{a_n\} \subset C$ such that $\{a_n\}$ forms an approximate identity for A and

$$\lim_{n\to\infty} \|a_nx - xa_n\| = 0$$

for all $x \in M(A)$.

Let $\{d_n\} \subset D$ be a dense subset of the unit ball of D. Note for each n, $a_n = g_n(a)$ for some $g_n \in C_0((0,1])$ with $0 \le g_n \le 1$. Furthermore, $g_n(t) = 0$ if $0 < t \le t_n$ for some $t_n > 0$ and $g_n(t) > 0$ if $t > t_n$, $n = 1, 2, ...$ Since $\{a_n\}$ forms an approximate identity, $g_n(t) \to 1$ for $t \in (0,1]$, when $n \to \infty$.

Define $e_1 = a_1$. Choose a_{n_2} such that

$$\|a_{n_2}e_1 - e_1\| < 1/8 \quad \text{and} \quad \|a_{n_2}d_i - d_ia_{n_2}\| < 1/4, \ \ i = 1, 2.$$

We may assume that $t_{n_2} < t_1/2$ and

$$g_{n_2}(t) > 1 - 1/8 \ \text{ for } \ t > t_1/2.$$

Find $h_2 \in C_0((0,1])$ with $0 \le g_{n_2} \le h_2 \le 1$ such that
(1) $h_2(t) = 1$ if $t > t_1$,
(2) $\|h_2 - g_{n_2}\| < 1/8$ and

(3) $h_2(t) = 0$ if $0 < t \le t_{n_2}$.
Put $e_2 = h_2(a)$. Then $e_2e_1 = e_1e_2 = e_1$. Also

$$\begin{aligned}\|e_2d_i - d_ie_2\| &\le \|(e_2 - a_{n_2})d_i\| + \|a_{n_2}d_i - d_ia_{n_2}\| + \|d_i(e_2 - a_{n_2})\| \\ &< 1/8 + 1/4 + 1/8 = 1/2, \quad i = 1, 2.\end{aligned}$$

Choose a_{n_3} such that

$$\|a_{n_3}e_2 - e_2\| < 1/4^2 \quad \text{and} \quad \|a_{n_3}d_i - d_ia_{n_3}\| < 1/4^2, \;\; i = 1, 2, 3.$$

Find $h_3 \in C_0((0, 1])$ with $0 \le g_{n_3} \le h_3 \le 1$ such that
(1′) $h_3(t) = 1$ if $t > t_{n_2}$,
(2′) $\|h_3 - g_{n_3}\| < 1/4^2$ and
(3′) $h_3(t) = 0$ if $0 < t \le t_{n_3}$.
Put $e_3 = h_3(a)$.
Then $e_3e_2 = e_2e_3 = e_2$ and

$$\begin{aligned}\|e_3d_i - d_ie_3\| &< \|(e_3 - a_{n_3})d_i\| + \|a_{n_3}d_i - d_ia_{n_3}\| + \|d_i(a_{n_3} - e_3)\| \\ &< 1/4^3 + 1/4^2 + 1/4^3 = 1/4, \quad i = 1, 2, 3.\end{aligned}$$

By induction, we obtain a sequence of positive elements $\{e_k\}$ such that

$$1 \ge e_k \ge a_{n_k}, e_{k+1}e_k = e_ke_{k+1} = e_k \quad \text{and} \quad \|e_nd_i - d_ie_n\| < 1/2^n,$$

$i = 1, 2, ..., n$ and $n = 1, 2, ...$ Since $a_{n_k} = g_k(a)$, and $t_{n_k} \to 0$, we see that $\{e_n\}$ is an approximate identity for A. Since $\{d_k\}$ is dense in the unit ball of D, we conclude that

$$\lim_{n\to\infty} \|e_nd - de_n\| = 0 \;\text{ for }\; d \in D.$$

□

Theorem 3.2. *Let A be a non-unital and σ-unital simple C^*-algebra. Suppose that A has a continuous scale. Then $M(A)/A$ is a purely infinite simple C^*-algebra.*

Proof. Fix a nonzero positive element $x \in M(A)/A$ with $0 \le x \le 1$. Let $a \in M(A)_+$ with $0 \le a \le 1$ such that $\pi(a) = x$, where $\pi : M(A) \to M(A)/A$ is the quotient map. Let $\{e_n\}$ be an approximate identity such that

$$e_{n+1}e_n = e_ne_{n+1} = e_n \quad \text{and} \quad \lim_{n\to\infty} \|e_na - ae_n\| = 0$$

(by 3.1). By Theorem 2.4, for any nonzero element $d \in A_+$, there exists an integer $n(d) > 0$ such that

$$e_m - e_n \stackrel{\sim}{<} d \;\text{ for }\; \text{all} \;\; m > n \ge n(d).$$

Without loss of generality, by passing to a subsequence, we may assume that

$$\|(e_{n+1} - e_n)^{1/2}a - a(e_{n+1} - e_n)^{1/2}\| < 1/2^n$$

for all n. Set

$$a_n = (e_{n+1} - e_n)^{1/2}[(e_{n+1} - e_n)^{1/2}a - a(e_{n+1} - e_n)^{1/2}].$$

Then $a_n \in A$ and $\|a_n\| < 1/2^n$. Thus $\sum_{n=0}^{\infty} a_n \in A$. Therefore (with $e_0 = 0$)

$$a - \sum_{n=0}^{\infty}(e_{n+1}-e_n)^{1/2}a(e_{n+1}-e_n)^{1/2} = \sum_{n=0}^{\infty}[(e_{n+1}-e_n)a-(e_{n+1}-e_n)^{1/2}a(e_{n+1}-e_n)^{1/2}]$$
$$= \sum_{n=0}^{\infty} a_n \in A.$$

Let $b = \sum_{n=0}^{\infty}(e_{n+1}-e_n)^{1/2}a(e_{n+1}-e_n)^{1/2}$. Therefore $\pi(b) = \pi(a) = x$. Define

$$b_1 = \sum_{k=1}^{\infty}(e_{4k+1}-e_{4k})^{1/2}a(e_{4k+1}-e_{4k})^{1/2}$$

and

$$b_2 = \sum_{k=1}^{\infty}(e_{4k+3}-e_{4k+2})^{1/2}a(e_{4k+3}-e_{4k+2})^{1/2}.$$

Then $b_1, b_2 \in M(A)$ and $b_1b_2 = b_2b_1 = 0$. Put $c_k = (e_{k+1}-e_k)^{1/2}a(e_{k+1}-e_k)^{1/2}$.

It is important to note that

$$c_k c_{k'} = c_{k'} c_k = 0$$

if $k = 4n$ and $k' = 4m$ with $n \neq m$, or $k = 4n+2$ and $k' = 4m+2$ with $n \neq m$. (This follows from the fact that $e_{n+1}e_n = e_ne_{n+1} = e_n$.)

Let $y_1 = \sum_{k=1}(e_{2k+1}-e_{2k})$ and $y_2 = \sum_{k=1}(e_{2k}-e_{2k-1})$. Then y_1, $y_2 \in M(A)$. For each k, there is $n(k)$ such that

$$e_{m+1} - e_m \ \tilde{<}\ c_k \ \text{ for } \ m \geq n(k).$$

Thus, by induction, we can find a partition of $\{n(1), n(1)+1, n(1)+2, ...\}$ into finite subsets $N_1, N_2, ...$ (of consecutive integers) such that for each $k = 1, 2, ...,$

$$\sum_{n\in N_k}(e_{2k+1}-e_{2k}) \ \tilde{<}\ c_{4k}.$$

Hence there is $z_k \in A$ such that

$$z_k^* z_k = \sum_{I\in N_k} e_{2i+1} - e_{2i} \quad \text{and} \quad z_k z_k^* \in \overline{c_{4k}Ac_{4k}}, \ \ k = 1, 2, ...$$

Note that $\|z_k\| \leq 1$. Let $z = \sum_{k=n(1)}^{\infty} z_k$. We are ready to verify that the sum converges in the strict topology and $z \in M(A)$. Moreover, one verifies that

$$z^*z = \sum_{k\geq n(1)}(e_{2k+1}-e_{2k}) \quad \text{and} \quad zz^* \in \overline{b_1Ab_1}.$$

It follows that

$$\pi(y_1) \ \tilde{<}\ \pi(b_1).$$

The same argument shows that

$$\pi(y_2) \ \tilde{<}\ \pi(b_2).$$

Since $b_1b_2 = b_2b_1 = 0$, by 1.3,

$$1 = \pi(y_1) + \pi(y_2) \ \tilde{<}\ \pi(b_1) + \pi(b_2) \ \tilde{<}\ \pi(b) = x.$$

Now, for any $y \in (M(A)/A)_+$, $y \stackrel{\sim}{<} 1$. Thus

$$y \stackrel{\sim}{<} x.$$

It follows from [LZ] that $M(A)/A$ is purely infinite. □

Corollary 3.3. *Let $A \not\cong \mathcal{K}$ be a non-unital and σ-unital simple C^*-algebra. Then the following are equivalent:*
(1) *A has a continuous scale;*
(2) *$M(A)/A$ is simple;*
(3) *$M(A)/A$ is a purely infinite simple C^*-algebra.*

Lemma 3.4. *Let A be a non-unital and σ-unital C^*-algebra and $b \in M(A)\backslash A$ be a positive element. Let $B = \overline{bAb}$ and $C = \overline{bM(B)b}$. Then $C \subset M(A)$. Let $\pi : M(A) \to M(A)/A$ and $\pi' : M(B) \to M(B)/B$ be the quotient maps, respectively. Then $\pi(C) = \pi'(C)$.*

Proof. We view $M(A)$ and $M(B)$ as subalgebras of A'', the enveloping von-Neumann algebra of A. For any $x \in C$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|xf_\delta(b) - x\| < \varepsilon \quad \text{and} \quad \|f_\delta(b)x - x\| < \varepsilon.$$

For any $a \in A$ with $\|a\| \leq 1$, we have

$$\|ax - af_\delta(b)x\| < \varepsilon \quad \text{and} \quad \|xa - xf_\delta(b)a\| < \varepsilon.$$

Let $af_\delta(b) = u|f_\delta(b)a^*af_\delta(b)|^{1/2}$ be the polar decomposition in A''. Then we have that $u|f_\delta(b)a^*af_\delta(b)|^{1/4} \in A$ and $|f_\delta(b)a^*af_\delta(b)|^{1/4} \in B$. Thus $|f_\delta(b)a^*af_\delta(b)|^{1/4}x \in B \subset A$. So $af_\delta(b)x \in A$. Therefore $a^*f_\delta(b)x^* \in A$ and consequently $xf_\delta(b)a \in A$. It follows that $xa, ax \in A$. This proves that $C \subset M(A)$. Note that $C \cap A = B$ and $C \cap B = B$. Thus $C/A = C/B$. Therefore $\pi(C) = \pi'(C)$. □

Corollary 3.5. *Let A be a non-unital separable simple C^*-algebra. Then $M(A)/A$ contains an infinite projection, and any nonzero projection in $M(A)/A$ is infinite.*

Proof. Let I_0 be as in Lemma 2.1 in [Ln1] and I be its closure. As in 2.3, there is a nonzero positive element $b \in I_0 \setminus A$ such that $B = \overline{bAb}$ has a continuous scale. It follows from 3.2 that $M(B)/B$ is a purely infinite simple C^*-algebra. Note that $b \in M(B)$. Let $1/2 > \delta > 0$ and $\pi' : M(B) \to M(B)/B$ be the quotient map. Let $D = \pi'(f_\delta(b))(M(B)/B)\pi'(f_\delta(b))$. Then there are projections $q_1 \leq q_2 \in D$ such that $q_2 - q_1 \neq 0$, and there exists another element $s \in D$ such that

$$s^*s = q_2 \quad \text{and} \quad ss^* = q_1.$$

Let c_1, c_2 and c_3 in $\overline{f_\delta(b)M(B)f_\delta}$ such that $\pi'(c_1) = q_1$, $\pi'(c_2) = q_2$ and $\pi'(c_3) = s$. It follows from 3.4 that $\pi(c_1)$ and $\pi(c_2)$ are projections with $\pi(c_2) \geq \pi(c_1)$ and $\pi(c_2) - \pi(c_1) \neq 0$. Moreover,

$$\pi(c_3)^*\pi(c_3) = \pi(c_2) \quad \text{and} \quad \pi(c_3)\pi(c_3) = \pi(c_1).$$

Thus $\pi(c_2)$ is an infinite projection.

We have shown that $M(A)/A$ contains an infinite projection. Suppose that $p \in M(A)/A$ is a nonzero projection. By replacing b by p, the above shows that there is an infinite projection $q \leq p$. It follows that p itself is infinite. □

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