TWO ESTIMATES FOR CURVES IN THE PLANE

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Abstract. We obtain a Fourier transform estimate and an $L^3(\mathbb{R}^2)$ -- $L^3(\mathbb{R}^2)$ convolution estimate for certain measures on a class of convex curves in the plane.

§1. Introduction

This is the third in a series of papers concerned with estimates for operators associated with measures on a certain class of curves in the plane. The curves are just graphs $\Gamma = \{(x, \phi(x)) : a \leq x < b\}$ where $\phi^{(j)}(a) = 0$ for $j = 0, 1, 2$ and $\phi'' > 0, \phi^{(3)} \geq 0$ on $(a,b)$. Our previous results deal with the affine arclength measure $\phi''(x)^{1/3} dx$ on $\Gamma$. They are

Theorem 1 ([7]). Writing $\lambda$ for affine arclength on $\Gamma$, there is the estimate

$$\|\lambda \ast \chi_E\|_{L^3(\mathbb{R}^2)} \leq 12^{1/3} \|\chi_E\|_{L^{3/2}}$$

for any measurable $E \subseteq \mathbb{R}^2$.

Theorem 2 ([8]). If $1 \leq p < \frac{4}{3}$ and $\frac{4}{3} + \frac{1}{p} = 1$, there is a constant $C = C(p)$ such that the estimate

$$\left( \int_a^b |\hat{f}(t, \phi(t))|^q \phi''(t)^{1/3} dt \right)^{\frac{1}{q}} \leq C(p) \|f\|_{L^p(\mathbb{R}^2)}$$

holds.

(Shortly after [8] appeared, it was pointed out to the author that Theorem 2 is a consequence of Theorem 2 in Sjölin’s paper [9].) Part of the novelty of Theorems 1 and 2 is that the estimates they provide are uniform over the class of curves under consideration. In particular, the constant $C(p)$ in Theorem 2 is independent of $\phi$. Convolution and Fourier restriction estimates like those in Theorems 1 and 2 are well known when the associated curves have nonvanishing curvature. Drury ([3]) pointed out that the damping factor $\phi''(x)^{1/3}$ could compensate for flatness in such estimates. But his and subsequent results (see, e.g., [1]) gave bounds depending on certain ancillary constants. The (quite simple) proofs of Theorems 1 and 2

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have no such dependence. The shortcoming of Theorem 1 is that it holds only for characteristic functions $\chi_E$: a more natural estimate would be

$$(1) \quad \|\lambda * f\|_{L^3(\mathbb{R}^2)} \leq C \|f\|_{L^{3/2}(\mathbb{R}^2)}$$

for nonnegative and measurable functions $f$ on $\mathbb{R}^2$ and for some absolute constant $C$. We have been unable to prove or disprove such an estimate. Our next result is a weaker substitute.

**Theorem 3.** With $\phi$ as above, write $\omega(x)$ for the function $\frac{\phi'(x)^2}{\phi(x)}$ and $\nu$ for the measure on $\Gamma$ given by $d\nu = \omega(x)^{1/3}dx$. Then there is an absolute constant $C$ such that the estimate

$$\|\nu * f\|_{L^3(\mathbb{R}^2)} \leq C \|f\|_{L^{3/2}(\mathbb{R}^2)}$$

holds for nonnegative measurable functions $f$ on $\mathbb{R}^2$.

(One can check that Theorem 3 is weaker than (1) by verifying, as we do in the proof of Theorem 4, that the inequality $\omega(x) \leq 2\phi''(x)$ follows from the hypotheses on $\phi$.)

There is also a related Fourier transform estimate.

**Theorem 4.** There is an absolute constant $C$ such that the following holds: with $\phi$ and $\omega$ as above, with $[c,d] \subseteq [a,b)$, and with $\zeta, \eta \in \mathbb{R}$, we have the inequality

$$\left| \int_c^d e^{i(\zeta x + \eta \phi(x))} \omega(x)^{1/3}dx \right| \leq \frac{C}{|\eta|^{1/2}}.$$

If $\zeta = 0$, then the change of variable $t = \phi(x)^{1/2}$ shows the conclusion of Theorem 4 to follow trivially from Van der Corput’s lemma applied to $\int e^{i\nu t^2}dt$. On the other hand, a stronger estimate

$$\left| \int_c^d e^{i(\zeta x + \eta \phi(x))} \omega(x)^{1/2+is}dx \right| \leq \frac{C(s)}{|\eta|^{1/2}},$$

with $C(s)$ growing, say, polynomially in $|s|$ (which we have been unable to obtain) would yield a proof of Theorem 3 different from the one we present. Results like Theorem 3, but for nondegenerate curves and without a uniform constant, date back at least to [5]. Fourier transform estimates such as that of Theorem 4, but for nondegenerate curves, are easy consequences of Van der Corput’s lemma. The remainder of this note is organized as follows: $\S2$ contains the proof of Theorem 3, and $\S3$ contains the proof of Theorem 4.

$\S2$. **Proof of Theorem 3**

Theorem 3 is proved by adapting the method of Drury and Guo in [4]. The proof requires an elementary lemma.

**Lemma 1.** With $\phi$ and $[a,b)$ as above, suppose $h > 0$ and $x$, $x - h \in [a,b)$. Then

$$\frac{\phi'(x)\phi'(x-h)}{\phi(x)^{1/2}\phi(x-h)^{1/2}(\phi'(x) - \phi'(x-h))} \leq \frac{5}{|h|}.$$
Proof of Lemma 1. Since \( \phi(a) = \phi'(a) = 0 \) it follows, for example, that \( \phi(x) = \int_a^x (x-t)\phi''(t)dt \). Thus the conclusion of Lemma 1 is equivalent to the inequality

\[
h \int_a^x \phi''(t)dt \int_a^{x-h} \phi''(t)dt \leq 5\left( \int_a^x (x-t)\phi''(t)dt \right)^{1/2} \left( \int_a^{x-h} (x-h-t)\phi''(t)dt \right)^{1/2} \int_{x-h}^x \phi''(t)dt.
\]

It is therefore enough to establish the two inequalities

\[
h \left( \int_a^{x-h} \phi''(t)dt \right)^2 \leq 2\left( \int_a^x (x-t)\phi''(t)dt \right)^{1/2} \left( \int_a^{x-h} (x-h-t)\phi''(t)dt \right)^{1/2} \int_{x-h}^x \phi''(t)dt
\]

and

\[
h \int_{x-h}^x \phi''(t)dt \int_a^{x-h} \phi''(t)dt \leq 3\left( \int_a^x (x-t)\phi''(t)dt \right)^{1/2} \left( \int_a^{x-h} (x-h-t)\phi''(t)dt \right)^{1/2} \int_{x-h}^x \phi''(t)dt.
\]

With no loss of generality, assume \( \phi''(x-h) = 1 \). Since \( \phi'' \) is increasing, it follows that \( h \leq \int_{x-h}^x \phi''(t)dt \). Since

\[
\int_a^{x-h} (x-h-t)\phi''(t)dt \leq \int_a^x (x-t)\phi''(t)dt,
\]

inequality (2) will follow from

\[
\left( \int_a^{x-h} \phi''(t)dt \right)^2 \leq 2 \int_a^{x-h} (x-h-t)\phi''(t)dt.
\]

To see this, let \( \epsilon = \int_a^{x-h} \phi''(t)dt \). Since \( \phi''(t) \leq \phi''(x-h) = 1 \) if \( a \leq t \leq x-h \), the RHS of (4) is minimized when \( \phi''(t) = \chi_{[x-h,x]}(t) \) on \( [a,x-h] \). This minimum is \( \epsilon^2/2 \), and so (4) holds. Now, if \( a \leq x-2h \), (3) will follow from the inequalities

\[
h \int_a^{x-2h} \phi''(t)dt \leq \left( \int_a^x (x-t)\phi''(t)dt \right)^{1/2} \left( \int_a^{x-h} (x-h-t)\phi''(t)dt \right)^{1/2}
\]

and

\[
h \int_{x-2h}^{x-h} \phi''(t)dt \leq 2\left( \int_a^x (x-t)\phi''(t)dt \right)^{1/2} \left( \int_a^{x-h} (x-h-t)\phi''(t)dt \right)^{1/2}.
\]

The first of these is clear since both of \( x-t \) and \( x-t-h \) exceed \( h \) on \( [a,x-2h] \). For the second, note that

\[
h^2 \leq 2 \int_a^{x-h} (x-t)\phi''(t)dt \leq 2 \int_a^x (x-t)\phi''(t)dt
\]

since \( \phi'' \) is nondecreasing and \( \phi''(x-h) = 1 \). Thus the desired inequality follows from (4). A slight modification handles the case \( x-2h < a \) and completes the proof of the lemma. \( \square \)
As previously mentioned, the proof of Theorem 3 is an adaptation of the method of \[4\]. For \( f \geq 0 \) we need to estimate

\[
\|\nu * f\|_{L^3(\mathbb{R}^2)}^3 = \int \cdots \int \prod_{j=1}^3 \left( f(x - t_j, y - \phi(t_j))\omega^{1/3}(t_j) \right) dt_1 dt_2 dt_3 \ dx \ dy
\]

where the \( t_j \) integrals are over \([a, b]\) and the \( x \) and \( y \) integrals are over \( \mathbb{R} \). The change of variables \( x_j = x - t_j \) leads to

\[
\int \cdots \int \prod_{j=1}^3 \left( f(x_j, y - \phi(x - x_j))\omega^{1/3}(x - x_j) \right) \ dx \ dy \ dx_1 dx_2 dx_3
\]

where the \( x_j \) integrals and the \( y \) integral are over \( \mathbb{R} \) and the \( x \) integral is over \( \bigcap_j (x_j + [a, b]) \). The idea of \[4\] is to fix temporarily the \( x_j \) and obtain an estimate of the \( xy \) integral. Accordingly, we let \( g_j(z) = f(x_j, z) \) and consider

\[
S(g_1, g_2, g_3) = \int \prod_{j=1}^3 \left( g_j(y - \phi(x - x_j))\omega^{1/3}(x - x_j) \right) \ dx \ dy.
\]

The desired estimate is

\[
S(g_1, g_2, g_3) \leq C \frac{\|g_1\|_{L^{3/2}(\mathbb{R})} \|g_2\|_{L^{3/2}(\mathbb{R})} \|g_3\|_{L^{3/2}(\mathbb{R})}}{\|x_2 - x_3\|^{1/3}}.
\]

Combining (6) with the following estimate of Christ (\[2\], Proposition 2.2),

\[
\int \int \int \frac{|h_1(x_1)h_2(x_2)h_3(x_3)|}{|(x_2 - x_3)(x_3 - x_1)(x_1 - x_2)|^{1/3}} \ dx_1 dx_2 dx_3 \leq C \frac{\|h_1\|_{L^{3/2}(\mathbb{R})} \|h_2\|_{L^{3/2}(\mathbb{R})} \|h_3\|_{L^{3/2}(\mathbb{R})}}{\|x_2 - x_3\|^{1/3}},
\]

shows that (5) is bounded by \( C \|f\|_{L^{3/2}(\mathbb{R}^2)}^3 \). Inequality (6) follows from an interpolation based on the three estimates (7.1), (7.2), and (7.3), where (7.1) is

\[
\int \int (g_1(y - \phi(x - x_1)) \prod_{j=2}^3 \left( g_j(y - \phi(x - x_j))\omega^{1/2}(x - x_j) \right) \ dx \ dy \leq C \frac{\|g_1\|_{L^\infty(\mathbb{R})} \|g_2\|_{L^1(\mathbb{R})} \|g_3\|_{L^1(\mathbb{R})}}{|x_2 - x_3|}
\]

and (7.2) and (7.3) are analogous. To see (7.1) note that

\[
\int \int \prod_{j=2}^3 \left( g_j(y - \phi(x - x_j))\omega^{1/2}(x - x_j) \right) \ dx \ dy \leq \int \int g_2(y - \phi(x - x_2))g_3(y - \phi(x - x_3)) \phi'(x - x_2) \phi'(x - x_3)
\]

\[
\phi'(x - x_2)^{1/2}\phi'(x - x_3)^{1/2}\left|\phi'(x - x_2) - \phi'(x - x_3)\right| \ dx \ dy
\]

\[
\leq C \frac{\|g_2\|_{L^1(\mathbb{R})} \|g_3\|_{L^1(\mathbb{R})}}{|x_2 - x_3|}.
\]
by Lemma 1 and the fact that the Jacobian determinant for the one-to-one mapping 
\((x, y) \mapsto (y - \phi(x - x_1), y - \phi(x - x_3))\) has absolute value \(|\phi'(x - x_2) - \phi'(x - x_3)|\).

§3. Proof of Theorem 4

We begin with a pair of lemmas.

**Lemma 2.** Suppose \(\psi\) is a real-valued continuously differentiable function on a closed interval \(I\) such that \(\psi\) and \(\psi'\) are of constant sign on \(I\). Then
\[
\left| \int_I e^{iu\psi(u)} \, du \right| \leq 5 \sup \left\{ \left| \int_J \psi \right| : J \text{ is a subinterval of } I \text{ with length } \leq \frac{1}{|\eta|} \right\}.
\]

This is Lemma 1 in [6]. The change of variable \(u = \alpha(x)\) yields the next lemma.

**Lemma 3.** With \(\psi\) as in Lemma 2, suppose that \(\alpha\) is a twice continuously differentiable function on \(I\) such that \(\alpha'\) and \(\alpha'\psi' - \psi\alpha''\) are of constant sign on \(I\). Then
\[
\left| \int_a^b e^{i\alpha(x)\psi(x)} \, dx \right| \leq 5 \sup \left\{ \left| \int_{x_0}^{x_1} \psi \right| : |\alpha(x_1) - \alpha(x_0)| \leq 1 \right\}.
\]

Theorem 4 is the estimate
\[
\left| \int_c^d e^{i(\zeta x + \eta \phi(x))} \omega(x)^{1/2} \, dx \right| \leq \frac{C}{|\eta|^{1/2}}.
\]

If \(\zeta\) and \(\eta\) have the same sign, then one can apply an easy argument based on the change of variable \(t = \phi(x)^{1/2}\). So we will assume that \(\zeta\) and \(\eta\) have opposite signs. Furthermore, it is sufficient to establish (8) under the additional hypothesis that if \(\alpha(x) = \zeta x + \eta \phi(x)\), then \(\alpha'\) is of constant sign on \([c, d]\). Of course we intend to apply Lemma 3 with \(\psi = \omega^{1/2} = \frac{\phi'}{\phi^{1/2}}\), and so we need to check that \(\alpha'\psi' - \psi\alpha''\) is of constant sign. Now
\[
\alpha'\psi' - \psi\alpha'' = (\zeta + \eta \phi') \frac{\phi'' - \frac{1}{2}(\phi')^2}{\phi^{3/2}} - \frac{\phi'}{\phi^{1/2}} (\eta \phi'') = -\frac{\eta}{2} \frac{\phi'}{\phi^{3/2}} + \zeta \frac{\phi'' - \frac{1}{2}(\phi')^2}{\phi^{3/2}}.
\]

Since
\[
\phi'(x) = \int_a^x (x - t) \phi^{(3)}(t) \, dt \\
\leq \left( \int_a^x (x - t)^2 \phi^{(3)}(t) \, dt \right)^{\frac{1}{2}} \left( \int_a^x \phi^{(3)}(t) \, dt \right)^{\frac{1}{2}} = \sqrt{2} \phi(x)^{1/2} \phi''(x)^{1/2},
\]
the fact that \(\zeta\) and \(\eta\) have opposite signs shows that \(\alpha'\psi' - \psi\alpha''\) has constant sign. To apply Lemma 2 we assume that \([x_0, x_1] \subseteq [c, d]\) and that \(|\eta(\phi(x_1) - \phi(x_0)) + \zeta(x_1 - x_0)| \leq \epsilon \leq 1\), and we will then establish the inequality
\[
\int_{x_0}^{x_1} \frac{\phi'(x)}{\phi(x)^{1/2}} \, dx \leq \frac{2}{|\eta|^{1/2}}.
\]

Assume without loss of generality that \(\eta > 0\), and assume for the moment that
\[
\phi' \geq -\frac{\zeta}{\eta}
\]
on \([x_0, x_1]\). (This inequality or its opposite must hold on \([x_0, x_1]\) because the sign of \(\alpha'\) is constant on \([c, d]\).) Then
\[
\eta(\phi(x_1) - \phi(x_0)) = -\zeta(x_1 - x_0) + \epsilon
\]
and so
\[-\zeta = \frac{\eta(\phi(x_1) - \phi(x_0)) - \epsilon}{\eta(x_1 - x_0)}.
\]
Thus (since \(\phi'\) is increasing) (10) holds on \([x_0, x_1]\) if and only if
\[
\phi'(x_0) \geq \frac{\phi(x_1) - \phi(x_0)}{x_1 - x_0} - \frac{\epsilon}{\eta(x_1 - x_0)}.
\]
which is equivalent to
\[
\frac{\epsilon}{\eta} \geq \phi(x_1) - \phi(x_0) - \phi'(x_0)(x_1 - x_0) = \int_{x_0}^{x_1} (x_1 - t)\phi''(t) \, dt.
\]
If, instead of (10), we assume
\[
\phi' \leq -\frac{\zeta}{\eta}
\]
on \([x_0, x_1]\), then it follows similarly that
\[
\frac{\epsilon}{\eta} \geq \phi(x_1) - \phi(x_0) - \phi'(x_0)(x_1 - x_0) = \int_{x_0}^{x_1} (t - x_0)\phi''(t) \, dt.
\]
Since \(\phi''\) is nondecreasing, \(\int_{x_0}^{x_1} (x_1 - t)\phi''(t) \, dt \leq \int_{x_0}^{x_1} (t - x_0)\phi''(t) \, dt\), and therefore
\[
\left(\int_{x_0}^{x_1} (x_1 - t)\phi''(t) \, dt\right)^{1/2} \leq \frac{\epsilon^{1/2}}{|\eta|^{1/2}} \leq \frac{1}{|\eta|^{1/2}}.
\]
Thus (9) will follow from
\[
(11) \quad \int_{x_0}^{x_1} \frac{\phi'(x)}{\phi(x)^{1/2}} \, dx \leq 2\left(\int_{x_0}^{x_1} (x_1 - t)\phi''(t) \, dt\right)^{1/2}.
\]
For \(x \in (a, b)\) consider
\[
\inf \left\{ \frac{\phi'(x) - \phi'(x_0)}{\left(\int_{x_0}^{x} (x - t)\phi''(t) \, dt\right)^{1/2}} : a \leq x_0 < x \right\}.
\]
A computation shows that
\[
\frac{d}{dx_0} \left(\frac{\phi'(x) - \phi'(x_0)}{\int_{x_0}^{x} (x - t)\phi''(t) \, dt}\right)^2
\]
has the same sign as \(\int_{x_0}^{x} \phi''(t)((x - x_0) - 2(x - t)) \, dt\), which is nonnegative since \(\phi''\) is positive and nondecreasing. Thus the infimum above is realized when \(x_0 = a\), and therefore
\[
(12) \quad \inf \left\{ \frac{\phi'(x) - \phi'(x_0)}{\left(\int_{x_0}^{x} (x - t)\phi''(t) \, dt\right)^{1/2}} : a \leq x_0 < x \right\} = \frac{\phi'(x)}{\phi(x)^{1/2}}.
\]
On the other hand, for fixed \(x_0\),
\[
\frac{d}{dx} \left(\int_{x_0}^{x} (x - t)\phi''(t) \, dt\right)^{1/2} = \frac{\phi'(x) - \phi'(x_0)}{2\left(\int_{x_0}^{x} (x - t)\phi''(t) \, dt\right)^{1/2}}.
\]
Thus, by (12),
\[
\int_{x_0}^{x_1} \frac{\phi'(x)}{\phi(x)^{1/2}} dx \leq \int_{x_0}^{x_1} \frac{d}{dx} \left( \int_{x_0}^{x} (x-t)\phi''(t)dt \right)^{1/2} dx = 2 \left( \int_{x_0}^{x_1} (x-t)\phi''(t)dt \right)^{1/2}.
\]
This is (11), and so the proof of Theorem 4 is complete.

References


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