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TWO ESTIMATES FOR CURVES IN THE PLANE

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Abstract. We obtain a Fourier transform estimate and an $L^{3/2}(\mathbb{R}^2)-L^3(\mathbb{R}^2)$ convolution estimate for certain measures on a class of convex curves in the plane.

§1. Introduction

This is the third in a series of papers concerned with estimates for operators associated with measures on a certain class of curves in the plane. The curves are just graphs $\Gamma = \{(x, \phi(x)) : a \leq x < b\}$ where $\phi^{(j)}(a) = 0$ for $j = 0, 1, 2$ and $\phi'' > 0$, $\phi^{(3)} \geq 0$ on (a, b). Our previous results deal with the affine arclength measure $\phi''(x)^{1/3}dx$ on Γ. They are

Theorem 1 ([7]). *Writing λ for affine arclength on Γ, there is the estimate*

$$\|\lambda * \chi_E\|_{L^3(\mathbb{R}^2)} \leq 12^{1/3}\ \|\chi_E\|_{3/2}$$

for any measurable $E \subseteq \mathbb{R}^2$.

Theorem 2 ([8]). *If $1 \leq p < \frac{4}{3}$ and $\frac{1}{p} + \frac{1}{3q} = 1$, there is a constant $C = C(p)$ such that the estimate*

$$\Big(\int_a^b |\widehat{f}(t, \phi(t))|^q \phi''(t)^{\frac{1}{3}} dt\Big)^{\frac{1}{q}} \leq C(p)\ \|f\|_{L^p(\mathbb{R}^2)}$$

holds.

(Shortly after [8] appeared, it was pointed out to the author that Theorem 2 is a consequence of Theorem 2 in Sjölin's paper [9].) Part of the novelty of Theorems 1 and 2 is that the estimates they provide are uniform over the class of curves under consideration. In particular, the constant $C(p)$ in Theorem 2 is independent of ϕ. Convolution and Fourier restriction estimates like those in Theorems 1 and 2 are well known when the associated curves have nonvanishing curvature. Drury ([3]) pointed out that the damping factor $\phi''(x)^{1/3}$ could compensate for flatness in such estimates. But his and subsequent results (see, e.g., [1]) gave bounds depending on certain ancillary constants. The (quite simple) proofs of Theorems 1 and 2

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have no such dependence. The shortcoming of Theorem 1 is that it holds only for characteristic functions χ_E: a more natural estimate would be

$$\|\lambda * f\|_{L^3(\mathbb{R}^2)} \leq C\ \|f\|_{L^{3/2}(\mathbb{R}^2)} \tag{1}$$

for nonnegative and measurable functions f on $\mathbb{R}^2$ and for some absolute constant C. We have been unable to prove or disprove such an estimate. Our next result is a weaker substitute.

Theorem 3. *With ϕ as above, write $\omega(x)$ for the function $\frac{\phi'(x)^2}{\phi(x)}$ and ν for the measure on Γ given by $d\nu = \omega(x)^{1/3}dx$. Then there is an absolute constant C such that the estimate*

$$\|\nu * f\|_{L^3(\mathbb{R}^2)} \leq C\ \|f\|_{L^{3/2}(\mathbb{R}^2)}$$

holds for nonnegative measurable functions f on $\mathbb{R}^2$.

(One can check that Theorem 3 is weaker than (1) by verifying, as we do in the proof of Theorem 4, that the inequality $\omega(x) \leq 2\phi''(x)$ follows from the hypotheses on ϕ.)

There is also a related Fourier transform estimate.

Theorem 4. *There is an absolute constant C such that the following holds: with ϕ and ω as above, with $[c,d] \subseteq [a,b)$, and with $\zeta,\ \eta \in \mathbb{R}$, we have the inequality*

$$\Big| \int_c^d e^{i\big(\zeta x + \eta\phi(x)\big)} \omega(x)^{1/2} dx \Big| \leq \frac{C}{|\eta|^{1/2}}.$$

If $\zeta = 0$, then the change of variable $t = \phi(x)^{1/2}$ shows the conclusion of Theorem 4 to follow trivially from Van der Corput's lemma applied to $\int e^{i\eta t^2} dt$. On the other hand, a stronger estimate

$$\Big| \int_c^d e^{i\big(\zeta x + \eta\phi(x)\big)} \omega(x)^{1/2+is} dx \Big| \leq \frac{C(s)}{|\eta|^{1/2}},$$

with $C(s)$ growing, say, polynomially in $|s|$ (which we have been unable to obtain) would yield a proof of Theorem 3 different from the one we present. Results like Theorem 3, but for nondegenerate curves and without a uniform constant, date back at least to [5]. Fourier transform estimates such as that of Theorem 4, but for nondegenerate curves, are easy consequences of Van der Corput's lemma. The remainder of this note is organized as follows: §2 contains the proof of Theorem 3, and §3 contains the proof of Theorem 4.

§2. Proof of Theorem 3

Theorem 3 is proved by adapting the method of Drury and Guo in [4]. The proof requires an elementary lemma.

Lemma 1. *With ϕ and $[a,b)$ as above, suppose $h > 0$ and $x,\ x-h \in [a,b)$. Then*

$$\frac{\phi'(x)\phi'(x-h)}{\phi(x)^{1/2}\phi(x-h)^{1/2}\big(\phi'(x)-\phi'(x-h)\big)} \leq \frac{5}{|h|}.$$

Proof of Lemma 1. Since $\phi(a) = \phi'(a) = 0$ it follows, for example, that $\phi(x) = \int_a^x (x-t)\phi''(t)dt$. Thus the conclusion of Lemma 1 is equivalent to the inequality

$$h\int_a^x \phi''(t)dt \int_a^{x-h} \phi''(t)dt$$
$$\leq 5\Big(\int_a^x (x-t)\phi''(t)dt\Big)^{1/2}\Big(\int_a^{x-h}(x-h-t)\phi''(t)dt\Big)^{1/2}\int_{x-h}^x \phi''(t)dt.$$

It is therefore enough to establish the two inequalities

$$h\Big(\int_a^{x-h}\phi''(t)dt\Big)^2$$
$$\leq 2\Big(\int_a^x (x-t)\phi''(t)dt\Big)^{1/2}\Big(\int_a^{x-h}(x-h-t)\phi''(t)dt\Big)^{1/2}\int_{x-h}^x \phi''(t)dt \tag{2}$$

and

$$h\int_{x-h}^x \phi''(t)dt\int_a^{x-h}\phi''(t)dt$$
$$\leq 3\Big(\int_a^x (x-t)\phi''(t)dt\Big)^{1/2}\Big(\int_a^{x-h}(x-h-t)\phi''(t)dt\Big)^{1/2}\int_{x-h}^x \phi''(t)dt. \tag{3}$$

With no loss of generality, assume $\phi''(x-h) = 1$. Since ϕ'' is increasing, it follows that $h \leq \int_{x-h}^x \phi''(t)dt$. Since

$$\int_a^{x-h}(x-h-t)\phi''(t)dt \leq \int_a^x (x-t)\phi''(t)dt,$$

inequality (2) will follow from

$$\Big(\int_a^{x-h}\phi''(t)dt\Big)^2 \leq 2\int_a^{x-h}(x-h-t)\phi''(t)dt. \tag{4}$$

To see this, let $\epsilon = \int_a^{x-h}\phi''(t)dt$. Since $\phi''(t) \leq \phi''(x-h) = 1$ if $a \leq t \leq x-h$, the RHS of (4) is minimized when $\phi''(t) = \chi_{[x-h-\epsilon,x-h]}$ on $[a, x-h]$. This minimum is $\epsilon^2/2$, and so (4) holds. Now, if $a \leq x-2h$, (3) will follow from the inequalities

$$h\int_a^{x-2h}\phi''(t)dt \leq \Big(\int_a^x (x-t)\phi''(t)dt\Big)^{1/2}\Big(\int_a^{x-h}(x-h-t)\phi''(t)dt\Big)^{1/2}$$

and

$$h\int_{x-2h}^{x-h}\phi''(t)dt \leq 2\Big(\int_a^x (x-t)\phi''(t)dt\Big)^{1/2}\Big(\int_a^{x-h}(x-h-t)\phi''(t)dt\Big)^{1/2}.$$

The first of these is clear since both of $x-t$ and $x-t-h$ exceed h on $[a, x-2h]$. For the second, note that

$$h^2 \leq 2\int_{x-h}^x (x-t)\phi''(t)dt \leq 2\int_a^x (x-t)\phi''(t)dt$$

since ϕ'' is nondecreasing and $\phi''(x-h) = 1$. Thus the desired inequality follows from (4). A slight modification handles the case $x-2h < a$ and completes the proof of the lemma. □

As previously mentioned, the proof of Theorem 3 is an adaptation of the method of [4]. For $f \geq 0$ we need to estimate

$$\|\nu * f\|^3_{L^3(\mathbb{R}^2)} = \int \cdots \int \prod_{j=1}^{3} \Big(f\big(x - t_j, y - \phi(t_j)\big)\omega^{1/3}(t_j)\Big) dt_1 dt_2 dt_3 dx\ dy \tag{5}$$

where the t_j integrals are over $[a, b)$ and the x and y integrals are over $\mathbb{R}$. The change of variables $x_j = x - t_j$ leads to

$$\int \cdots \int \prod_{j=1}^{3} \Big(f\big(x_j, y - \phi(x - x_j)\big)\omega^{1/3}(x - x_j)\Big)\ dx\ dy\ dx_1 dx_2 dx_3$$

where the x_j integrals and the y integral are over $\mathbb{R}$ and the x integral is over $\bigcap_j \big(x_j + [a, b)\big)$. The idea of [4] is to fix temporarily the x_j and obtain an estimate of the xy integral. Accordingly, we let $g_j(z) = f(x_j, z)$ and consider

$$S(g_1, g_2, g_3) = \int \int \prod_{j=1}^{3} \Big(g_j\big(y - \phi(x - x_j)\big)\omega^{1/3}(x - x_j)\Big)\ dx\ dy.$$

The desired estimate is

$$S(g_1, g_2, g_3) \leq \frac{C\ \|g_1\|_{L^{3/2}(\mathbb{R})}\|g_2\|_{L^{3/2}(\mathbb{R})}\|g_3\|_{L^{3/2}(\mathbb{R})}}{|(x_2 - x_3)(x_3 - x_1)(x_1 - x_2)|^{1/3}}. \tag{6}$$

Combining (6) with the following estimate of Christ ([2], Proposition 2.2),

$$\begin{aligned}\int \int \int \frac{|h_1(x_1)h_2(x_2)h_3(x_3)|}{|(x_2 - x_3)(x_3 - x_1)(x_1 - x_2)|^{1/3}} dx_1 dx_2 dx_3 \\ \leq C\ \|h_1\|_{L^{3/2}(\mathbb{R})}\|h_2\|_{L^{3/2}(\mathbb{R})}\|h_3\|_{L^{3/2}(\mathbb{R})},\end{aligned}$$

shows that (5) is bounded by $C\ \|f\|^3_{L^{3/2}(\mathbb{R}^2)}$. Inequality (6) follows from an interpolation based on the three estimates (7.1), (7.2), and (7.3), where (7.1) is

$$\begin{aligned}\int \int \big(g_1\big(y - \phi(x - x_1)\big) \prod_{j=2}^{3} \Big(g_j\big(y - \phi(x - x_j)\big)\omega^{1/2}(x - x_j)\Big)\ dx\ dy \\ \leq \frac{C\ \|g_1\|_{L^\infty(\mathbb{R})}\|g_2\|_{L^1(\mathbb{R})}\|g_3\|_{L^1(\mathbb{R})}}{|x_2 - x_3|}\end{aligned}$$

and (7.2) and (7.3) are analogous. To see (7.1) note that

$$\begin{aligned}&\int \int \prod_{j=2}^{3} \Big(g_j\big(y - \phi(x - x_j)\big)\omega^{1/2}(x - x_j)\Big)\ dx\ dy \\ &= \int \int g_2\big(y - \phi(x - x_2)\big) g_3\big(y - \phi(x - x_3)\big) \\ &\quad \cdot \frac{\phi'(x - x_2)\phi'(x - x_3)}{\phi(x - x_2)^{1/2}\phi(x - x_3)^{1/2}|\phi'(x - x_2) - \phi'(x - x_3)|} \\ &\quad \cdot |\phi'(x - x_2) - \phi'(x - x_3)| dx\ dy \\ &\leq \frac{5\ \|g_2\|_{L^1(\mathbb{R})}\|g_3\|_{L^1(\mathbb{R})}}{|x_2 - x_3|}\end{aligned}$$

by Lemma 1 and the fact that the Jacobian determinant for the one-to-one mapping $(x,y) \mapsto \big(y-\phi(x-x_2), y-\phi(x-x_3)\big)$ has absolute value $|\phi'(x-x_2)-\phi'(x-x_3)|$.

§3. Proof of Theorem 4

We begin with a pair of lemmas.

Lemma 2. *Suppose ψ is a real-valued continuously differentiable function on a closed interval I such that ψ and ψ' are of constant sign on I. Then*

$$\Big|\int_I e^{iru}\psi(u)\ du\Big| \le 5\sup\Big\{\Big|\int_J \psi\Big| : J \textit{ is a subinterval of } I \textit{ with length } \le \frac{1}{|r|}\Big\}.$$

This is Lemma 1 in [6]. The change of variable $u=\alpha(x)$ yields the next lemma.

Lemma 3. *With ψ as in Lemma 2, suppose that α is a twice continuously differentiable function on I such that α' and $\alpha'\psi'-\psi\alpha''$ are of constant sign on I. Then*

$$\Big|\int_I e^{i\alpha(x)}\psi(x)\ dx\Big| \le 5\sup\Big\{\Big|\int_{x_0}^{x_1}\psi\Big| : |\alpha(x_1)-\alpha(x_0)|\le 1\Big\}.$$

Theorem 4 is the estimate

$$\Big|\int_c^d e^{i\left(\zeta x+\eta\phi(x)\right)}\omega(x)^{1/2}dx\Big| \le \frac{C}{|\eta|^{1/2}}. \tag{8}$$

If ζ and η have the same sign, then one can apply an easy argument based on the change of variable $t=\phi(x)^{1/2}$. So we will assume that ζ and η have opposite signs. Furthermore, it is sufficient to establish (8) under the additional hypothesis that if $\alpha(x)=\zeta x+\eta\phi(x)$, then α' is of constant sign on $[c,d]$. Of course we intend to apply Lemma 3 with $\psi=\omega^{1/2}=\frac{\phi'}{\phi^{1/2}}$, and so we need to check that $\alpha'\psi'-\psi\alpha''$ is of constant sign. Now

$$\alpha'\psi'-\psi\alpha'' = (\zeta+\eta\phi')\frac{\phi\phi''-\frac{1}{2}(\phi')^2}{\phi^{3/2}} - \frac{\phi'}{\phi^{1/2}}(\eta\phi'') = -\frac{\eta}{2}\frac{(\phi')^3}{\phi^{3/2}} + \zeta\frac{\phi\phi''-\frac{1}{2}(\phi')^2}{\phi^{3/2}}.$$

Since

$$\phi'(x) = \int_a^x (x-t)\phi^{(3)}(t)\ dt$$
$$\le \Big(\int_a^x (x-t)^2\phi^{(3)}(t)\ dt\Big)^{\frac{1}{2}}\Big(\int_a^x \phi^{(3)}(t)\ dt\Big)^{\frac{1}{2}} = \sqrt{2}\phi(x)^{1/2}\phi''(x)^{1/2},$$

the fact that ζ and η have opposite signs shows that $\alpha'\psi'-\psi\alpha''$ has constant sign. To apply Lemma 2 we assume that $[x_0,x_1]\subseteq[c,d]$ and that $|\eta\big(\phi(x_1)-\phi(x_0)\big)+\zeta(x_1-x_0)| \doteq \epsilon \le 1$, and we will then establish the inequality

$$\int_{x_0}^{x_1}\frac{\phi'(x)}{\phi(x)^{1/2}}dx \le \frac{2}{|\eta|^{1/2}}. \tag{9}$$

Assume without loss of generality that $\eta>0$, and assume for the moment that

$$\phi' \ge -\frac{\zeta}{\eta} \tag{10}$$

on $[x_0,x_1]$. (This inequality or its opposite must hold on $[x_0,x_1]$ because the sign of α' is constant on $[c,d]$.) Then

$$\eta\big(\phi(x_1)-\phi(x_0)\big) = -\zeta(x_1-x_0)+\epsilon$$

and so

$$-\frac{\zeta}{\eta} = \frac{\eta\big(\phi(x_1) - \phi(x_0)\big) - \epsilon}{\eta(x_1 - x_0)}.$$

Thus (since ϕ' is increasing) (10) holds on $[x_0, x_1]$ if and only if

$$\phi'(x_0) \geq \frac{\phi(x_1) - \phi(x_0)}{x_1 - x_0} - \frac{\epsilon}{\eta(x_1 - x_0)},$$

which is equivalent to

$$\frac{\epsilon}{\eta} \geq \phi(x_1) - \phi(x_0) - \phi'(x_0)(x_1 - x_0) = \int_{x_0}^{x_1} (x_1 - t)\phi''(t)\ dt.$$

If, instead of (10), we assume

$$\phi' \leq -\frac{\zeta}{\eta}$$

on $[x_0, x_1]$, then it follows similarly that

$$\frac{\epsilon}{\eta} \geq \phi(x_1) - \phi(x_0) - \phi'(x_0)(x_1 - x_0) = \int_{x_0}^{x_1} (t - x_0)\phi''(t)\ dt.$$

Since ϕ'' is nondecreasing, $\int_{x_0}^{x_1}(x_1 - t)\phi''(t)\ dt \leq \int_{x_0}^{x_1}(t - x_0)\phi''(t)\ dt$, and therefore

$$\Big(\int_{x_0}^{x_1} (x_1 - t)\phi''(t)\ dt\Big)^{1/2} \leq \frac{\epsilon^{1/2}}{|\eta|^{1/2}} \leq \frac{1}{|\eta|^{1/2}}.$$

Thus (9) will follow from

$$\int_{x_0}^{x_1} \frac{\phi'(x)}{\phi(x)^{1/2}} dx \leq 2\Big(\int_{x_0}^{x_1} (x_1 - t)\phi''(t)\ dt\Big)^{1/2}. \tag{11}$$

For $x \in (a, b)$ consider

$$\inf \left\{ \frac{\phi'(x) - \phi'(x_0)}{\big(\int_{x_0}^{x} (x - t)\phi''(t) dt\big)^{1/2}} : a \leq x_0 < x \right\}.$$

A computation shows that

$$\frac{d}{dx_0}\left(\frac{\big(\phi'(x) - \phi'(x_0)\big)^2}{\int_{x_0}^{x} (x - t)\phi''(t) dt}\right)$$

has the same sign as $\int_{x_0}^{x} \phi''(t)\big((x - x_0) - 2(x - t)\big)dt$, which is nonnegative since ϕ'' is positive and nondecreasing. Thus the infimum above is realized when $x_0 = a$, and therefore

$$\inf \left\{ \frac{\phi'(x) - \phi'(x_0)}{\big(\int_{x_0}^{x} (x - t)\phi''(t) dt\big)^{1/2}} : a \leq x_0 < x \right\} = \frac{\phi'(x)}{\phi(x)^{1/2}}. \tag{12}$$

On the other hand, for fixed x_0,

$$\frac{d}{dx}\Big(\int_{x_0}^{x} (x - t)\phi''(t) dt\Big)^{1/2} = \frac{\phi'(x) - \phi'(x_0)}{2\Big(\int_{x_0}^{x} (x - t)\phi''(t) dt\Big)^{1/2}}.$$

Thus, by (12),

$$\int_{x_0}^{x_1} \frac{\phi'(x)}{\phi(x)^{1/2}}dx \leq 2\int_{x_0}^{x_1} \frac{d}{dx}\Big(\int_{x_0}^{x}(x-t)\phi''(t)dt\Big)^{1/2}dx = 2\Big(\int_{x_0}^{x_1}(x-t)\phi''(t)dt\Big)^{1/2}.$$

This is (11), and so the proof of Theorem 4 is complete.

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