

ON THE HOMOTOPY TYPE OF ESCHENBURG SPACES WITH POSITIVE SECTIONAL CURVATURE

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ABSTRACT. A rigidity theorem is proved for principal Eschenburg spaces of positive sectional curvature. It is shown that for a very large class of such spaces the homotopy type determines the diffeomorphism type.

1. INTRODUCTION

The Eschenburg spaces introduced in [2], [3], [4] are quotients of free circle actions on $SU(3)$ and provide examples of non-homogeneous manifolds that admit a Riemannian metric with positive sectional curvature. In this work we shall be concerned with the family of *principal* Eschenburg spaces $E_{l,m}$ that fiber as circle bundles over the inhomogeneous Eschenburg flag manifold F' of [4] and are obtained as quotient spaces of actions of the circle on $SU(3)$ given by

$$z \cdot X = \text{diag}(z^{l+m}, z^l, z^m)X \text{diag}(z^{2l+2m}, 1, 1)^{-1},$$

where l and m are relatively prime integers and $\text{diag}(z^r, z^s, z^t)$ denotes the diagonal matrix in $U(3)$ with entries z^r, z^s and z^t . The topological and differentiable classification of this family was settled in [1] where it is shown that there are abundant examples of homeomorphic but not diffeomorphic principal Eschenburg spaces. Eschenburg also showed [4] that under the assumption that the product lm is positive, the spaces $E_{l,m}$ admit a natural metric with positive sectional curvature. We shall refer to these spaces as *principal positively curved Eschenburg spaces*. Such spaces are very rigid, since it has been recently shown by Shankar [8] that homeomorphic principal positively curved Eschenburg spaces are in fact diffeomorphic. In this note we address the question of whether the homotopy type determines a unique principal positively curved Eschenburg space.

In Section 2, after characterizing the Eschenburg spaces $E_{l,m}$ as a set of integers in the quadratic field $\mathbb{Q}(\sqrt{5})$, we determine the orders of the fourth cohomology groups that can occur in this family. In Section 3 we first determine the precise number of principal positively curved Eschenburg spaces $E_{l,m}$ with prescribed fourth cohomology group. We then show that when the order of the fourth cohomology group satisfies a very general condition, the homotopy classification is enough

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to distinguish the diffeomorphism types of principal positively curved Eschenburg spaces.

2. ESCHENBURG SPACES AS QUADRATIC INTEGERS

It will be convenient to represent the Eschenburg spaces of type $E_{l,m}$ described above as integers in the quadratic field $\mathbb{Q}(\sqrt{5})$. We begin by recalling some well-known facts about this field and refer the reader to [9] for details. The symbol $\mathbb{Q}(\sqrt{5})$ denotes the set of numbers $a + b\sqrt{5}$ where a and b are arbitrary rational numbers. If $\alpha = a + b\sqrt{5}$, then the *conjugate* of α is the number $\bar{\alpha} = a - b\sqrt{5}$ and the *norm* of α is given by $N(\alpha) = \alpha\bar{\alpha} = a^2 - 5b^2$.

We will require as well some essentials about the arithmetic of $\mathbb{Q}(\sqrt{5})$. A *quadratic integer* is a number $\alpha = \frac{a}{2} + \frac{b}{2}\sqrt{5}$ in which a and b are (rational) integers with $a \equiv b \pmod{2}$. Most of the basic properties of the divisibility of the integers generalize to the quadratic integers of $\mathbb{Q}(\sqrt{5})$, but one of the main differences lies in the fact that the group of units in $\mathbb{Q}(\sqrt{5})$ is infinite and, hence, every integer α in $\mathbb{Q}(\sqrt{5})$ has an infinite number of *associates* $\alpha\varepsilon$, with ε a unit. However, it is not difficult to establish the following elementary result.

Lemma 2.1. *Every quadratic integer α in $\mathbb{Q}(\sqrt{5})$ has a unique pair of associates $\alpha_0 = \frac{a_0}{2} + \frac{b_0}{2}\sqrt{5}$ and $\alpha_1 = \frac{a_1}{2} + \frac{b_1}{2}\sqrt{5}$ satisfying $a_0 > 0$ and $-a_0 < 5b_0 < a_0$, as well as $b_1 > |a_1|$.*

Definition 1. An integer $\alpha = \frac{a}{2} + \frac{b}{2}\sqrt{5}$ in $\mathbb{Q}(\sqrt{5})$ will be called *basic* if $a > 0$ and $-a < 5b \leq a$.

Since the integers in $\mathbb{Q}(\sqrt{5})$ form a unique factorization domain, it follows from Lemma 2.1 that any quadratic integer α can be decomposed uniquely as a product $\alpha = \varepsilon\pi_1\pi_2 \cdots \pi_r$, where ε is a unit and the π_i are basic primes. The next result, which is a direct consequence of Gauss's law of quadratic reciprocity, describes the basic primes in $\mathbb{Q}(\sqrt{5})$.

Proposition 2.2. *Every basic prime in $\mathbb{Q}(\sqrt{5})$ is of one of the following types:*

- (i) $\alpha_p = p$, where p is a rational prime with $p \equiv \pm 2 \pmod{5}$.
- (ii) α_p and $\bar{\alpha}_p$, where $N(\alpha_p) = N(\bar{\alpha}_p) = p$, for p an odd rational prime with $p \equiv \pm 1 \pmod{5}$.
- (iii) $\alpha_5 = \frac{5}{2} + \frac{1}{2}\sqrt{5}$.

The family of Eschenburg spaces $E_{l,m}$ is parametrized by pairs (l, m) of relatively prime integers. The fourth cohomology group $H^4(E_{l,m}; \mathbb{Z})$ is cyclic of order N , where N is the absolute value of $l^2 + m^2 + 3lm$. If we set $a = l - m$ and $b = l + m$, then there is a one-to-one correspondence between the Eschenburg spaces $E_{l,m}$ and the set of quadratic integers $\alpha = \frac{a}{2} + \frac{b}{2}\sqrt{5}$ in $\mathbb{Q}(\sqrt{5})$ with $\gcd(b + a, b - a) = 2$.

Our first goal will be to determine which integers N appear as orders of fourth cohomology groups of the family $E_{l,m}$. Observe first that since

$$N(\alpha) = \frac{a^2}{4} - 5\frac{b^2}{4} = -(l^2 + m^2 + 3lm),$$

the norm of α provides the order of the fourth cohomology group of the corresponding Eschenburg space.

Let $\alpha = \frac{a}{2} + \frac{b}{2}\sqrt{5}$ be a quadratic integer with $\gcd(b+a, b-a) = 2$ and let $p \neq 5$ be an odd rational prime such that $p|N(\alpha)$. If $p \equiv \pm 2 \pmod{5}$, then $p \left| \frac{a}{2} + \frac{b}{2}\sqrt{5} \right.$, implying that $p|a$ and $p|b$. This contradicts the assumption that $\gcd(b+a, b-a) = 2$. Similarly, $N(\alpha)$ is odd, for otherwise $2 \left| \frac{a}{2} + \frac{b}{2}\sqrt{5} \right.$, implying that 4 divides both $b+a$ and $b-a$. It follows that $\alpha = \frac{a}{2} + \frac{b}{2}\sqrt{5}$ is (up to a unit) a product of primes α_p and $\bar{\alpha}_p$, with $p \equiv \pm 1 \pmod{5}$, and possibly α_5 . However, for each p dividing $N(\alpha)$, only one of these two basic primes α_p and $\bar{\alpha}_p$ appears in the factorization of α , for otherwise $p = \alpha_p \bar{\alpha}_p$ would divide $\frac{a}{2} + \frac{b}{2}\sqrt{5}$. Finally, the prime 5 turns out to be special in this respect, since $\alpha_5^2 = 5(\frac{3}{2} + \frac{1}{2}\sqrt{5})$ cannot appear as a factor of α . Thus we have the following result.

Proposition 2.3. *Let N be the order of the fourth cohomology group $H^4(E_{l,m}; \mathbb{Z})$. Then $N = 5^m N'$, where m is 0 or 1 and N' is a product of powers of primes congruent to $\pm 1 \pmod{5}$.*

3. HOMOTOPY INVARIANTS OF ESCHENBURG SPACES WITH POSITIVE SECTIONAL CURVATURE

The geometric properties of the spaces $E_{l,m}$ were studied by Eschenburg [4], who established that these spaces admit a natural metric with positive sectional curvature provided that the product lm is positive. It is immediate that a quadratic integer $\alpha = \frac{a}{2} + \frac{b}{2}\sqrt{5}$ represents a principal positively curved Eschenburg space if and only if $|b| > |a|$. We can now determine the precise number of principal positively curved Eschenburg spaces $E_{l,m}$ with a given fourth cohomology group.

Theorem 3.1. *Let N be as in Proposition 2.3, and let k be the number of odd rational primes congruent with $\pm 1 \pmod{5}$ that appear in the decomposition of N . Then there are exactly 2^{k+1} principal positively curved Eschenburg spaces $E_{l,m}$ such that $H^4(E_{l,m}; \mathbb{Z})$ is cyclic of order N .*

Proof. We need to determine how many quadratic integers $\alpha = \frac{a}{2} + \frac{b}{2}\sqrt{5}$ satisfy $|N(\alpha)| = N$ with $\gcd(b+a, b-a) = 2$ and $|b| > |a|$. The condition $\gcd(b+a, b-a) = 2$ implies that at most one of α_p and $\bar{\alpha}_p$ can appear in the decomposition of α into primes. There are 2^k quadratic integers of the form $a_5^m \pi_1^{n_1} \pi_2^{n_2} \cdots \pi_k^{n_k}$, where each of the π_i is either an α_p or an $\bar{\alpha}_p$. By Lemma 2.1 each of these quadratic integers has a unique associate $\frac{a_j}{2} + \frac{b_j}{2}\sqrt{5}$ satisfying $b_j > |a_j|$. The other 2^k integers are obtained by multiplying each $\frac{a_j}{2} + \frac{b_j}{2}\sqrt{5}$ by -1 .

We now address the question of finding two Eschenburg spaces E_{l_0, m_0} and E_{l_1, m_1} that admit metrics of positive sectional curvature with the same homotopy type. The homotopy classification of Eschenburg spaces was established by Milgram in [6]. (See also [5] for a partial classification.) For principal Eschenburg spaces, Milgram’s result can be stated in terms of quadratic integers as follows.

Theorem 3.2. *Two quadratic integers $\alpha_0 = \frac{a_0}{2} + \frac{b_0}{2}\sqrt{5}$ and $\alpha_1 = \frac{a_1}{2} + \frac{b_1}{2}\sqrt{5}$ represent Eschenburg spaces with the same homotopy type if and only if*

- (i) $N(\alpha_0) = N(\alpha_1) = N$,
- (ii) $b_0 \equiv b_1 \pmod{3}$,
- (iii) $b_0^3 \equiv b_1^3 \pmod{N}$ and
- (iv) $\gcd(b_i + a_i, b_i - a_i) = 2$.

Since we are only interested in Eschenburg spaces admitting a metric with positive sectional curvature and since $E_{l,m}$ is diffeomorphic to both $E_{m,l}$ and $E_{-l,-m}$, it is enough to consider pairs (l_0, m_0) and (l_1, m_1) with $l_i > m_i > 0$. These inequalities are equivalent to

(v) $b_i > a_i > 0$.

Let $\alpha_0 = \frac{a_0}{2} + \frac{b_0}{2}\sqrt{5}$ and $\alpha_1 = \frac{a_1}{2} + \frac{b_1}{2}\sqrt{5}$ be two quadratic integers satisfying conditions (i) to (v). The first and the last conditions imply that

$$(3.3) \quad |b_1 - b_0| < \sqrt{N} - \sqrt{\frac{4N}{5}} = \frac{\sqrt{5} - 2}{\sqrt{5}}\sqrt{N}.$$

Let p be a rational prime with $p^n | N$. By Proposition 2.3 we know that $p \geq 5$. Condition (iv) implies that $\gcd(b_0, p) = \gcd(b_1, p) = 1$. Hence, $\gcd(b_0^3, p^n) = 1$. It is known (see [7], p. 104) that the congruence $x^3 \equiv b_0^3 \pmod{p^n}$ has either one or three solutions, according to whether $p \equiv 2$ or $p \equiv 1 \pmod{3}$, respectively. Write $b_1 = b_0 + d$. Then $d \equiv 0 \pmod{3}$; say $d = 3e$. Since $b_0^3 \equiv b_1^3 \pmod{p^n}$, if $p \equiv 2 \pmod{3}$, then $b_0 \equiv b_1 \pmod{p^n}$, implying that $d \equiv 0 \pmod{p^n}$. Thus,

$$(3.4) \quad e \equiv 0 \pmod{p^n}.$$

We can now prove our main result.

Theorem 3.5. *Suppose that E_{l_0, m_0} and E_{l_1, m_1} are principal positively curved Eschenburg spaces of the same homotopy type, and such that $l_i > m_i > 0$. Let $N = 5^m p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, where m is 0 or 1 and $p_i \equiv \pm 1 \pmod{5}$, and write $N = xy$, where $x = 5^m p_1^{n_1} \cdots p_r^{n_r}$ and $y = p_{r+1}^{n_{r+1}} \cdots p_k^{n_k}$ with $p_i \equiv 2 \pmod{3}$ for $1 \leq i \leq r$ and $p_i \equiv 1 \pmod{3}$ for $r + 1 \leq i \leq k$. If $\frac{x}{y} > \frac{9 - 4\sqrt{5}}{45}$, then $l_0 = l_1$ and $m_0 = m_1$.*

Remark 3.6. Observe that since $\frac{9 - 4\sqrt{5}}{45} \approx 0.00124$, the hypothesis $\frac{x}{y} > \frac{9 - 4\sqrt{5}}{45}$ is not very restrictive.

Proof of 3.5. Write $b_1 = b_0 + 3e$ as above to obtain $\frac{\sqrt{5} - 2}{3\sqrt{5}}\sqrt{N} > e$ from (3.3).

It follows from (3.4) that $x|e$. If we multiply by x the inequality $x > \frac{9 - 4\sqrt{5}}{45}y$, we get $x^2 > \frac{9 - 4\sqrt{5}}{45}N$. Taking square roots now gives $x > \frac{\sqrt{5} - 2}{3\sqrt{5}}\sqrt{N} > e$. But since $x|e$, then e must be zero.

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