

INFINITE SYSTEMS OF LINEAR EQUATIONS FOR REAL ANALYTIC FUNCTIONS

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ABSTRACT. We study the problem when an infinite system of linear functional equations

$$\mu_n(f) = b_n \quad \text{for } n \in \mathbb{N}$$

has a real analytic solution f on $\omega \subseteq \mathbb{R}^d$ for every right-hand side $(b_n)_{n \in \mathbb{N}} \subseteq \mathbb{C}$ and give a complete characterization of such sequences of analytic functionals (μ_n) . We also show that every open set $\omega \subseteq \mathbb{R}^d$ has a complex neighbourhood $\Omega \subseteq \mathbb{C}^d$ such that the positive answer is equivalent to the positive answer for the analogous question with solutions holomorphic on Ω .

INTRODUCTION

One of the first classical examples of an infinite system of linear equations is related to the so-called *moment problem* solved by Hausdorff. It is the question of finding a Borel measure μ on $[0, 1]$ such that a given sequence of reals $(b_n)_{n \in \mathbb{N}}$ is the sequence of moments of μ (i.e., $\int_0^1 t^n d\mu(t) = b_n$ for $n \in \mathbb{N}$). More generally, we can look for a functional $f \in X'$ such that for the given sequence of scalars (b_n) and of vectors (x_n) belonging to a locally convex (Banach) space X the following holds:

$$(1) \quad f(x_n) = b_n \quad \text{for } n \in \mathbb{N}.$$

First solutions for spaces $X = C[0, 1]$ or $L_p[0, 1]$ are due to F. Riesz (1909) and the functional analytic approach to the problem is contained in the famous book of Banach [1, Ch. IV §7, §8]; see also [23, pp. 106-107].

Later on, Eidelheit, a colleague of Banach, characterized in all Fréchet spaces what are now called Eidelheit sequences [9] (see [17, Th. 26.27], [12, II.38.6]). Let X be a locally convex space. We call a sequence $(\mu_n)_n \in \mathbb{N}$ of continuous linear functionals on X an *Eidelheit sequence on X* if for every sequence of scalars $(b_n)_n \in \mathbb{N}$ there is $f \in X$ satisfying

$$(2) \quad \mu_n(f) = b_n \quad \text{for } n \in \mathbb{N}.$$

The notion of Eidelheit sequences has been extensively studied in Fréchet spaces later on; see [9], [18], [20], [21] and [22], compare also [17, §26].

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We will prove a completely analogous result for the highly non-metrizable space of real analytic functions $\mathcal{A}(\omega)$ on an arbitrary open set $\omega \subseteq \mathbb{R}^d$ (Theorem 2.2). We show that for every open set $\omega \subseteq \mathbb{R}^d$ there is a domain of holomorphy $\Omega \subseteq \mathbb{C}^d$, $\Omega \cap \mathbb{R}^d = \omega$, such that every Eidelheit sequence on $\mathcal{A}(\omega)$ is automatically Eidelheit on the space $H(\Omega)$ of holomorphic functions on Ω (Theorem 2.2). The main tool in the proof is given by the so-called *distinguished sets*, which might be of independent interest (see Lemma 1.1 and Lemma 1.3).

Let $(f_{1,n})_n \in \mathbb{N}$, $(f_{2,n})_n \in \mathbb{N}$ be two Eidelheit sequences on $\mathcal{A}(\omega_1)$ and $\mathcal{A}(\omega_2)$, respectively. We consider the question if there is a continuous linear map (= *operator*) $T : \mathcal{A}(\omega_1) \rightarrow \mathcal{A}(\omega_2)$ such that $f_{1,n} = f_{2,n} \circ T$ for every $n \in \mathbb{N}$ (compare an analogous problem on Fréchet spaces due to Mityagin [18] and its solution in [21]). It turns out that for every $(f_{1,n})_{n \in \mathbb{N}}$ there is $(f_{2,n})_n \in \mathbb{N}$ without such factorization (Proposition 3.1). On the other hand, we show a positive result for sequences related to the interpolation problem for analytic functions (Theorem 3.2).

Let us recall that the space $\mathcal{A}(\omega)$, $\omega \subseteq \mathbb{R}^d$ an open subset, of real analytic functions $f : \omega \rightarrow \mathbb{C}$ is equipped with the topology of the projective limit $\text{proj}_N H(K_N)$, where (K_N) is an exhaustion of ω by a sequence of compact sets

$$K_1 \subset\subset K_2 \subset\subset \dots \subset\subset K_N \subset\subset \dots \subset \omega, \quad \bigcup_{N \in \mathbb{N}} K_N = \omega,$$

and $H(K)$ denotes the space of germs of analytic functions over K with its natural LB-space topology. It is known (see [16, Prop. 1.7, 1.2]) that this topology is equal to the inductive limit topology $\text{ind}H(U)$, where U runs over all open neighborhoods of ω in \mathbb{C}^d and $H(U)$ denotes the Fréchet space of holomorphic functions on U with the compact open topology. Thus $\mathcal{A}(\omega)$ is a complete, separable, ultra-bornological, reflexive, webbed nuclear space with the approximation property (but non-metrizable and without a basis [7]) with the dual being a complete LF-space $\mathcal{A}(\omega)'_\beta = \text{ind}H(K_N)'_\beta$. The space $\mathcal{A}(\omega)$ is the projective limit of a sequence of LB-spaces. Spaces of this type are called *PLB-spaces*. For more information on the space of real analytic functions, see [2], [4], [5], [7], [8] and [16].

For smooth functions on $\omega \subseteq \mathbb{R}^d$ we use the standard multi-index notation. So if $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, then

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}} f, \quad \delta_z(f) := f(z), \quad \delta_z^\alpha(f) := (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha} f(z),$$

where $|\alpha| = \alpha_1 + \dots + \alpha_d$.

The elements $\mu \in H(\mathbb{C}^d)'$ are called analytic functionals. A compact subset $K \subset \mathbb{C}^d$ is called a *carrier* for μ if for every neighborhood $U \subset\subset \mathbb{C}^d$ of K there is a continuity estimate

$$|\mu(f)| \leq C \sup_{z \in U} |f(z)|.$$

We may identify $\mathcal{A}(\mathbb{R}^d)'$ with the analytic functionals that are carried by some $K \subset \mathbb{R}^d$. In this case there is a smallest real carrier which is called the support of μ and denoted by $\text{supp } \mu$ (see [19, p. 44 ff.]). For open $\omega \subset \mathbb{R}^d$ we may identify $\mathcal{A}(\omega)'$ with the analytic functionals μ with $\text{supp } \mu \subset \omega$.

The other notation is standard. We refer for functional analysis to [17] and for complex analysis to [13]. For analytic functionals see [19] or [15].

1. DISTINGUISHED OPEN SETS

We will need some auxiliary notions. We call a nonnegative C^∞ -function $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ *distinguished* whenever it vanishes at infinity and

$$\sup_{x \in \mathbb{R}^d} \|H_\psi(x)\| < 1,$$

where H_ψ denotes the Hessian of ψ and $\|\cdot\|$ is the norm in the space of quadratic forms on the d -dimensional euclidean space. For every distinguished function ψ we define

$$\Omega^\psi := \{z = x + iy \in \mathbb{C}^d : |y|^2 < \psi(x)\}$$

and call it a *distinguished open set* with *basis* $\omega := \Omega^\psi \cap \mathbb{R}^d$. These sets and the construction below are variations of the proof of the Cartan-Grauert Theorem as given in [3, Prop. 1] and [10, Prop. 6 and Prop. 7].

Lemma 1.1. (a) *Every distinguished open set Ω^ψ is a domain of holomorphy and for every $\alpha > 0$, $\Omega_\alpha^\psi := \{z : |y|^2 < \psi(x) - \alpha\} \subset\subset \Omega^\psi$. Moreover, polynomials are dense in $H(\Omega^\psi)$ equipped with the compact-open topology.*

(b) *For every pair of open sets $U \subset W \subset \mathbb{C}^d$, U distinguished, there is a distinguished set Ω^ψ with the basis $W \cap \mathbb{R}^d$ such that $U \subseteq \Omega^\psi \subseteq W$. If $U \subset\subset W$, then we can choose ψ such that $U \subset\subset \Omega^\psi$.*

(c) *Every open set $\omega \subset \mathbb{R}^d$ has a basis of open neighbourhoods in \mathbb{C}^d consisting of distinguished sets.*

(d) *Every compact set in \mathbb{C}^d is contained in some bounded distinguished set.*

Parts (a) and (c) imply a corollary going back to Cartan (see [11, Cor. II. 3.15]):

Corollary 1.2. *Every open set ω in \mathbb{R}^d has a basis of neighbourhoods in \mathbb{C}^d consisting of domains of holomorphy.*

Proof of Lemma 1.1. (a): Let $f(z) := |y|^2 - \psi(x)$ for $z = x + iy \in \mathbb{C}^d$. Since $\|H_\psi(x)\| < 1$, the Levi form of f is positive definite. Thus, f is plurisubharmonic on \mathbb{C}^d . Clearly $\Omega^\psi = \{z : f(z) < 0\}$. Since ψ vanishes at infinity, $\Omega_\alpha^\psi \subset\subset \Omega^\psi$. By [13, Cor. 5.4.3], every function holomorphic on a neighbourhood of $\overline{\Omega_\alpha^\psi}$ can be approximated uniformly on this set by entire functions.

(b): Let $U = \Omega^{\psi_0}$ and

$$s(x) := \inf\{|w| : x + i(y + w) \notin W \text{ for some } y, |y|^2 \leq \psi_0(x)\}.$$

Observe that

- (i) since ψ_0 is continuous and W is open, s is lower semi-continuous;
- (ii) $V := \{x \in W \cap \mathbb{R}^d : s(x) > 0\}$ is open and contains $(W \setminus U) \cap \mathbb{R}^d$;
- (iii) if $U \subset\subset W$, then $V = W \cap \mathbb{R}^d$.

We choose a covering of V by sets

$$U_{r_j}(x_j) = \{x \in \mathbb{R}^d : |x - x_j| < r_j\} \subset\subset W \cap \mathbb{R}^d.$$

We set

$$\varphi(x) = \begin{cases} e^{\frac{1}{|x|^2-1}} & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1, \end{cases} \quad \psi_1(x) = \sum_{j=1}^\infty \varepsilon_j \varphi\left(\frac{1}{r_j}(x - x_j)\right).$$

By (i) and a suitable choice of (ε_j) , we get that the series is convergent and

- (iv) $\psi_1(x) < s^2(x)$;
- (v) $\sup_{x \in \mathbb{R}^d} \|H_{\psi_1}(x)\| < 1 - \sup_{x \in \mathbb{R}^d} \|H_{\psi_0}(x)\|$.

By (v), $\psi := \psi_0 + \psi_1$ is a distinguished function. It is immediate that $U \subseteq \Omega^\psi$ and, by (iv), we get $\Omega^\psi \subseteq W$. By (ii), the basis of ψ is equal to $W \cap \mathbb{R}^d$. Finally, if $U \subset\subset W$, then, by (iii), $\psi_1|_{\overline{U \cap \mathbb{R}^d}} > \delta > 0$. Thus $U \subset \Omega_\delta^\psi \subset\subset \Omega$.

(c): Take in (b) $U = \emptyset$ and $\omega = W \cap \mathbb{R}^d$.

(d): It suffices to observe that if φ is defined as in (b) and $\varphi_a := a\varphi(\frac{x}{a})$, then $H_{\varphi_a} = a^{-1}H_\varphi$. Therefore, for a big enough, φ_a is a distinguished function but $\varphi_a \rightarrow +\infty$ uniformly on compact sets as $a \rightarrow \infty$. □

The proof of the following lemma follows the line of, e.g., the proof of [19, Théorème 111].

Lemma 1.3. *Let ω be open and bounded in \mathbb{R}^d , Ω^ψ a distinguished set with basis ω . Then for every $\mu \in H(\Omega^\psi)' \cap \mathcal{A}(\mathbb{R}^d)'$ we have $\text{supp}\mu \subset \overline{\omega}$.*

Proof. Let $\Omega \supset \overline{\omega}$ be open and bounded in \mathbb{C}^d . Let Ω_1 be an open neighbourhood of \mathbb{R}^d in \mathbb{C}^d such that $\Omega^\psi \cap \Omega_1 \subseteq \Omega$. By Lemma 1.1, there is a distinguished set Ω^{ψ_1} with basis \mathbb{R}^d such that $\Omega^\psi \subseteq \Omega^{\psi_1} \subseteq \Omega^\psi \cup \Omega_1$. Since Ω^{ψ_1} is a domain of holomorphy with covering $\Omega_1 \cap \Omega^{\psi_1}$ and $\Omega^\psi \cap \Omega^{\psi_1} = \Omega^\psi$, the solution of the first Cousin problem yields the following exact sequence:

$$0 \longrightarrow H(\Omega^{\psi_1}) \xrightarrow{j} H(\Omega_1 \cap \Omega^{\psi_1}) \oplus H(\Omega^\psi) \xrightarrow{\delta} H(\Omega_1 \cap \Omega^\psi) \longrightarrow 0$$

where $j(f) = (f, f)$ and $\delta(f, g) = f - g$.

On $H(\Omega_1 \cap \Omega^{\psi_1}) \oplus H(\Omega^\psi)$ we define $u(f, g) = \mu f - \mu g$. Since $u \circ j = 0$ on $H(\mathbb{C}^d)$ and, by density of $H(\mathbb{C}^d)$ in $H(\Omega^{\psi_1})$ (see Lemma 1.1 (a)), we have $u \circ j = 0$ on $H(\Omega^{\psi_1})$ and u gives rise to an element $\mu \in H(\Omega_1 \cap \Omega^\psi)'$ that clearly extends the given $\mu \in \mathcal{A}(\mathbb{R}^d)'$.

Since $\Omega^\psi \cap \Omega_1 \subseteq \Omega$ this proves the result. □

2. EIDELHEIT SEQUENCES

It turns out that a condition very similar to Eidelheit’s characterization of Eidelheit sequences on Fréchet spaces is also necessary for any PLB-space.

Lemma 2.1. *If $(f_n)_{n \in \mathbb{N}}$ is an Eidelheit sequence on a PLB-space $X = \text{proj}_{N \in \mathbb{N}} X_N$, then for every $N \in \mathbb{N}$ we have*

$$(3) \quad \dim(\text{span}\{f_n : n \in \mathbb{N}\} \cap X'_N) < \infty.$$

Proof. Let us assume that for some N the condition (3) does not hold. Since X_N is an LB-space, there is a bounded set $B \subset (X_N)'_\beta$ such that $\text{span}\{f_n : n \in \mathbb{N}\} \cap B$ has infinite dimension. Clearly, for a continuous surjection $T: X \rightarrow \mathbb{C}^\mathbb{N}$, $T(g) := (f_n(g))_{n \in \mathbb{N}}$,

$$\begin{aligned} (T')^{-1}(B) &\subseteq (T')^{-1}(B^{\circ\circ}) = \{g \in (\mathbb{C}^\mathbb{N})' : T'g \in B^{\circ\circ}\} \\ &= \{g \in (\mathbb{C}^\mathbb{N})' : |g(Tx)| \leq 1 \text{ for } x \in B^\circ\} = (T(B^\circ))^\circ. \end{aligned}$$

Since T is open (as a surjective continuous operator from the webbed space onto a Fréchet space), $T(B^\circ)$ is a 0-neighbourhood and $(T(B^\circ))^\circ$ is bounded in $\varphi = (\mathbb{C}^\mathbb{N})'$.

Therefore $(T(B^\circ))^\circ$ and $(T')^{-1}(B)$ are finite dimensional. Since T' is injective we get

$$\dim(\text{Im}T' \cap B) = \dim(\text{span}\{f_n : n \in \mathbb{N}\} \cap B) < \infty, \quad \text{a contradiction.}$$

□

Remark. The condition (2) does not characterize Eidelheit sequences on an arbitrary PLB-space. In [6, Ex. 2.8] the authors constructed an example of a PLB-space X with nuclear steps such that its ultrabornological associated topology makes it a nuclear LB-space. Clearly, no sequence on X is Eidelheit (since it must be an Eidelheit sequence on its ultrabornological associated space, and therefore, on an LB-space). On the other hand, by the construction, X cannot be represented as a projective limit of LB-spaces with surjective linking maps, and therefore, there exists a sequence $(f_n)_{n \in \mathbb{N}} \in \mathbb{N} \subseteq X'$ satisfying (2).

Surprisingly, for the space $\mathcal{A}(\omega)$ the above condition is also sufficient.

Theorem 2.2. *Let $\omega \subseteq \mathbb{R}^d$ be an arbitrary domain. Then there exists a domain of holomorphy Ω in \mathbb{C}^d ($\Omega = \mathbb{C}^d$ if $\omega = \mathbb{R}^d$), $\Omega \cap \mathbb{R}^d = \omega$, such that for any sequence $(\mu_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}(\omega)'$ of linearly independent analytic functionals the following assertions are equivalent:*

- (a) $(\mu_n)_{n \in \mathbb{N}}$ is an Eidelheit sequence on $\mathcal{A}(\omega)$;
- (b) $(\mu_n)_{n \in \mathbb{N}}$ is an Eidelheit sequence on $H(\Omega)$;
- (c) for every compact set $K \subset\subset \omega$ we have $\dim(\text{span}\{\mu_n : n \in \mathbb{N}\} \cap \{\mu \in \mathcal{A}(\mathbb{R}^d) : \text{supp}\mu \subset K\}) < \infty$.

Proof. Clearly (b) \implies (a) \implies (c) follows from Lemma 2.1. Observe that by [19, Lemma 112], μ extends to $\mu \in H(K)'$ if and only if $\text{supp}\mu \subseteq K$.

(c) \implies (b): Let (ω_n) be an exhaustion of ω , $\omega_n \subset\subset \omega_{n+1}$ for $n \in \mathbb{N}$. By Lemma 1.1 (b), there exists an increasing family of distinguished sets Ω^{ψ_n} with bases ω_n , $\Omega^{\psi_n} \subset\subset \Omega^{\psi_{n+1}}$ for $n \in \mathbb{N}$. In the case of $\omega = \mathbb{R}^d$ we can take Ω^{ψ_n} containing the ball of radius n in \mathbb{C}^d (use Lemma 1.1 (d)). We define $\Omega := \bigcup_{n \in \mathbb{N}} \Omega^{\psi_n}$, which is a domain of holomorphy by Lemma 1.1 (a) and [13, 3.3.7]. By Lemma 1.3, the assumption implies that for every compact set $L \subset\subset \Omega$ we have

$$\dim(\text{span}\{\mu_n : n \in \mathbb{N}\} \cap \{\mu \in H(\Omega)' : \text{supp}\mu \subset L\}) < \infty.$$

Therefore the classical Eidelheit theorem (see [9], [17, Th. 26.27]) yields the result.

□

3. FACTORIZATION OF EIDELHEIT SEQUENCES

Proposition 3.1. *For every Eidelheit sequence $(f_n)_{n \in \mathbb{N}} \in \mathbb{N}$ on $\mathcal{A}(\omega)$ there is an Eidelheit sequence $(g_n)_{n \in \mathbb{N}} \in \mathbb{N}$ on $\mathcal{A}(\omega)$ such that there is no operator $T : \mathcal{A}(\omega) \rightarrow \mathcal{A}(\omega)$ so that $g_n \circ T = f_n$ for all $n \in \mathbb{N}$.*

Proof. By Theorem 2.2, there is a domain of holomorphy $\Omega \subseteq \mathbb{C}^d$ such that $\Omega \cap \mathbb{R}^d = \omega$ and $(f_n)_{n \in \mathbb{N}} \in \mathbb{N}$ is an Eidelheit sequence on $H(\Omega)$ as well. By [8] (compare [4]), there is a Fréchet quotient F of $\mathcal{A}(\omega)$ that is not a quojection. Since $H(\Omega)$ is quasinormable, by [21, Th. 2.4], there is an Eidelheit sequence $(h_n)_{n \in \mathbb{N}}$ on F such that for no operator $S : H(\Omega) \rightarrow F$ the factorization $h_n \circ S = f_n$ holds.

Let $q : \mathcal{A}(\omega) \rightarrow F$ be the quotient map. Observe that $g_n := h_n \circ q$ is the Eidelheit sequence we are looking for. Indeed, if $g_n \circ T = f_n$, then $h_n \circ q \circ T = f_n$, which contradicts the choice of $(h_n)_{n \in \mathbb{N}}$.

□

We will consider now Eidelheit sequences of the form $(\delta_{z_n}^\alpha)_{n \in \mathbb{N}, |\alpha| \leq k_n}$ on $\mathcal{A}(\omega)$, $\omega \subseteq \mathbb{R}^d$, where $(k_n)_{n \in \mathbb{N}}$ is an arbitrary sequence of natural numbers and (z_n) is a discrete sequence. We call such sequences *interpolation sequences*.

Theorem 3.2. *Let $(f_n^{(1)}) \subseteq \mathcal{A}(\omega_1)'$ and $(f_n^{(2)}) \subseteq \mathcal{A}(\omega_2)'$ be two arbitrary interpolation sequences. There is always an operator $T: \mathcal{A}(\omega_1) \rightarrow \mathcal{A}(\omega_2)$ such that T' maps $(f_n^{(2)})_{n \in \mathbb{N}}$ one-to-one onto $(f_n^{(1)})_n \in \mathbb{N}$.*

The proof of Theorem 3.2 follows immediately from Corollary 3.4, and Lemmas 3.5 and 3.6 below.

Lemma 3.3. *Let $\omega_1 \subseteq \mathbb{R}^d$ and $\omega_2 \subseteq \mathbb{R}^d$ be open domains, and let $(z_n) \subseteq \omega_1$ and $(y_n) \subseteq \omega_2$ be discrete sequences. Let $(k_n)_n \in \mathbb{N}$ be a sequence of natural numbers. Then there is a real analytic map $\varphi: \omega_1 \rightarrow \omega_2$ such that for every $n \in \mathbb{N}$, $\varphi(z_n) = y_n$, the Jacobian matrix $\mathcal{J}\varphi$ of φ at the points z_n is just the unit matrix and all the partial derivatives of φ at z_n of rank strictly larger than one and less than or equal to k_n are vanishing.*

Corollary 3.4. *Under the notation of the previous lemma, the composition operator $C_\varphi: \mathcal{A}(\omega_2) \rightarrow \mathcal{A}(\omega_1)$, $C_\varphi(f) := f \circ \varphi$, satisfies*

$$C'_\varphi(\delta_{z_n}^\alpha) = \delta_{y_n}^\alpha \quad \text{for } |\alpha| \leq k_n, n \in \mathbb{N}.$$

Proof of Lemma 3.3. By [5, Lemma 4.2], there is a surjective real analytic map $\psi: \mathbb{R}^d \rightarrow \omega_2$ of constant rank d . It is easily seen that for every $z \in \mathbb{R}^d$ the map C'_ψ maps $\text{span}\{\delta_z^\alpha : |\alpha| \leq k\} \subseteq \mathcal{A}(\mathbb{R}^d)'$ onto $\text{span}\{\delta_{\psi(z)}^\alpha : |\alpha| \leq k\} \subseteq \mathcal{A}(\omega_2)'$. By the classical interpolation problem on domains of holomorphy and Cor. 1.2, we find $\eta: \omega_1 \rightarrow \mathbb{R}^d$ such that $\varphi := \psi \circ \eta$ is the map we are looking for. \square

Lemma 3.5. *Let $(y_n), (x_n)$ be sequences in \mathbb{R}^d tending to infinity. Let $(\alpha(n))$ and $(\beta(n))$ be sequences of multi-indices such that if $y_n = y_k$, then $\alpha(n) \neq \alpha(k)$. Then there is an operator $T: \mathcal{A}(\mathbb{R}^d) \rightarrow \mathcal{A}(\mathbb{R}^d)$ such that*

$$\left(\frac{\partial^{\alpha(n)}}{\partial x^{\alpha(n)}} T f\right)(y_n) = \left(\frac{\partial^{\beta(n)}}{\partial x^{\beta(n)}} f\right)(x_n) \quad \text{for every } n \in \mathbb{N}.$$

Proof. Let (k_n) be an increasing sequence of natural numbers such that $k_n \geq \max\{|\alpha(l)| : y_l = y_n\}$. Let us choose $\varphi_n: \mathbb{C}^d \rightarrow \mathbb{C}^d$, $\varphi_n(\mathbb{R}^d) \subseteq \mathbb{R}^d$, holomorphic such that

$$\varphi_n(y_k) = \begin{cases} x_n & \text{for } y_k = y_n, \\ 0 & \text{for } y_k \neq y_n, \end{cases} \quad \text{and} \quad \left(\frac{\partial^\beta}{\partial x^\beta} \varphi_n\right)(y_k) = 0 \quad \text{for } 1 \leq |\beta| \leq k_n.$$

By the solution of the classical interpolation problem for entire functions, such φ_n exist and we may assume that $\varphi_n \rightarrow 0$ uniformly on compact sets. Analogously, we define functions $g_n: \mathbb{C}^d \rightarrow \mathbb{C}$ such that $\left(\frac{\partial^{\alpha(n)}}{\partial x^{\alpha(n)}} g_n\right)(y_n) = 1$ and

$$\left(\frac{\partial^\beta}{\partial x^\beta} g_n\right)(y_k) = 0 \quad \text{for } |\beta| \leq k_n \text{ and either } y_k \neq y_n \text{ or } y_k = y_n, \beta \neq \alpha(n).$$

Since the map

$$R: H(\mathbb{C}^d) \rightarrow \mathbb{C}^{\mathbb{N}}, \quad R(f) := \left(\left(\frac{\partial^\beta}{\partial x^\beta} f\right)(y_n) \right)_{n \in \mathbb{N}, |\beta| \leq k_n},$$

is surjective and open, we may choose g_n such that for every compact set $K \subseteq \mathbb{C}^d$,

$$\sup_{z \in K} \sum_{n \in \mathbb{N}} |g_n(z)| n^{k_n+1} < +\infty.$$

We can define the map

$$T : \mathcal{A}(\mathbb{R}^d) \rightarrow \mathcal{A}(\mathbb{R}^d), \quad (Tf)(z) := \sum_{n \in \mathbb{N}} g_n(z) \cdot \left(\frac{\partial^{\beta(n)}}{\partial x^{\beta(n)}} f \right) (\varphi_n(z)).$$

Clearly, the series converges for every $f \in \mathcal{A}(\mathbb{R}^d)$ and, by the Closed Graph Theorem, the map is continuous. Moreover, for each n ,

$$\begin{aligned} \left(\frac{\partial^{\alpha(n)}}{\partial x^{\alpha(n)}} Tf \right) (y_n) &= \sum_{y_k=y_n} \frac{\partial^{\alpha(n)}}{\partial x^{\alpha(n)}} \left[g_k \cdot \left(\frac{\partial^{\beta(k)}}{\partial x^{\beta(k)}} f \right) \circ \varphi_k \right] (y_k) \\ &= \sum_{y_k=y_n} \left(\frac{\partial^{\beta(k)}}{\partial x^{\beta(k)}} f \right) (x_n) \cdot \left(\frac{\partial^{\alpha(n)}}{\partial x^{\alpha(n)}} g_k \right) (y_k) \\ &= \frac{\partial^{\beta(n)}}{\partial x^{\beta(n)}} f(x_n). \end{aligned}$$

This completes the proof. \square

Lemma 3.6. *Let \mathbb{R} be embedded in a standard way into \mathbb{R}^d , and let (z_n) be a discrete sequence in \mathbb{R} . Then there are operators $T : \mathcal{A}(\mathbb{R}) \rightarrow \mathcal{A}(\mathbb{R}^d)$ and $S : \mathcal{A}(\mathbb{R}^d) \rightarrow \mathcal{A}(\mathbb{R})$ such that $T'(\delta_{z_n}) = (\delta_{z_n})$ and $S'(\delta_{z_n}) = (\delta_{z_n})$ for $n \in \mathbb{N}$.*

Proof. One takes $T(f)(x_1, \dots, x_d) := f(x_1)$, $S(f)(x) := f(x, 0, \dots, 0)$. \square

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