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ALGEBRAIC LINKS AND SKEIN RELATIONS OF THE LINKS-GOULD INVARIANT

ATSUSHI ISHII

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Abstract. We give certain skein relations of the Links-Gould invariant. We also show that the relations lead us to recursive calculation of the invariant for algebraic links. As an application we give a formula for the Links-Gould invariant of 2-bridge links.

1. Introduction

The Links-Gould invariant is a two-variable polynomial invariant of oriented links. It was derived from a one-parameter family of four-dimensional representations of the quantum superalgebra $U_q[gl(2|1)]$. In [6], Links and Gould gave the cubic skein relation of the invariant. D. De Wit, L.H. Kauffman and J.R. Links [4] showed that the Links-Gould invariant is powerful through evaluation of some links. In particular, D. De Wit [2] showed that the invariant is complete for all prime knots of up to 10 crossings. In a previous paper [5], we studied the invariant of closed 3-braids, and tried to give an improvement on the evaluation of such a powerful invariant.

In this paper, we would like to give another step toward an improvement on its evaluation. We give certain skein relations of the invariant (Theorem 3.1). By using the relations, we describe a method for recursive calculation of the invariant for algebraic links (Theorem 4.1), and give a formula for the invariant of 2-bridge links (Proposition 5.1).

2. Preliminaries

Let us recall the definition of the Links-Gould invariant. Let t_0, t_1 be complex parameters. Set $K := \mathbf{C}(t_0, t_1)$. Let V be a four-dimensional K-vector space with a basis $\{e_i\}_{i=1}^4$. A linear automorphism R of $V \otimes V$ is called an *R-matrix* if it is a solution of the Yang-Baxter equation:

$$(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R).$$

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A linear automorphism A of $V \otimes V$ is represented by the 16×16 matrix $[A_{i,j}]$ with the following rule:

$$\text{If } A(e_k \otimes e_l) = \sum_{i,j} A_{kl}^{ij} e_i \otimes e_j, \text{ then } A_{4(i-1)+j,4(k-1)+l} = A_{kl}^{ij}.$$

The linear automorphism R of $V \otimes V$ represented by the following matrix $[R_{i,j}]$ is an R-matrix [4, p. 186]:

$$\left[\begin{array}{cccc|cccc|cccc|cccc}
t_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & t_0^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_0^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\hline
\cdot & t_0^{1/2} & \cdot & \cdot & t_0-1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_0t_1-1 & \cdot & \cdot & -t_0^{1/2}t_1^{1/2} & \cdot & \cdot & -t_0^{1/2}t_1^{1/2}Y & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_1^{1/2} & \cdot & \cdot \\
\hline
\cdot & \cdot & t_0^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & t_0-1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -t_0^{1/2}t_1^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & Y & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_1^{1/2} & \cdot \\
\hline
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & -t_0^{1/2}t_1^{1/2}Y & \cdot & \cdot & Y & \cdot & \cdot & Y^2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_1^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & t_1-1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_1^{1/2} & \cdot & \cdot & t_1-1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_1
\end{array}\right],$$

where $Y = ((t_0-1)(1-t_1))^{1/2}$.

Let B_n be the n-string braid group and let $\sigma_1, \ldots, \sigma_{n-1}$ be the standard generators of the group. Then there is a unique homomorphism $B_n \to \mathrm{Aut}(V^{\otimes n})$ which transforms σ_i into $\mathrm{id}_V^{\otimes(i-1)} \otimes R \otimes \mathrm{id}_V^{\otimes(n-i-1)}$ for $i = 1, \ldots, n-1$. Denote this homomorphism by b_R^n. Let μ be the linear automorphism of V which transforms e_i into $\mu_i e_i$ for $i = 1, \ldots, 4$, where $(\mu_1, \mu_2, \mu_3, \mu_4) = (t_0^{-1}, -t_1, -t_0^{-1}, t_1)$.

To define the Links-Gould invariant, we recall the operator trace (partial trace). Let $f : V^{\otimes n} \to V^{\otimes n}$ be a homomorphism such that

$$f(e_{i_1} \otimes \cdots \otimes e_{i_n}) = \sum_{1 \le j_1, \ldots, j_n \le 4} f_{i_1,\ldots,i_n}^{j_1,\ldots,j_n} e_{j_1} \otimes \cdots \otimes e_{j_n}.$$

The operator trace $\mathrm{tr}_n^m(f)$ of f is the homomorphism $V^{\otimes(n-m)} \to V^{\otimes(n-m)}$ defined by

$$\begin{aligned}
&\mathrm{tr}_n^m(f)(e_{i_1} \otimes \cdots \otimes e_{i_{n-m}}) \\
&\quad = \sum_{1 \le j_1, \ldots, j_n \le 4} f_{i_1,\ldots,i_{n-m},j_{n-m+1},\ldots,j_n}^{j_1,\ldots,j_{n-m},j_{n-m+1},\ldots,j_n} e_{j_1} \otimes \cdots \otimes e_{j_{n-m}}.
\end{aligned}$$

For the details we refer the reader to [8].

Let $\theta^\wedge$ denote the closure of $\theta \in B_n$. For $\theta \in B_n$, the Links-Gould invariant $LG(\theta^\wedge; t_0, t_1)$ of the link $\theta^\wedge$ is defined by the following identity:

$$\mathrm{tr}_n^{n-1}\left((\mathrm{id}_V \otimes \mu^{\otimes(n-1)})(b_R^n(\theta))\right) = LG(\theta^\wedge; t_0, t_1)\mathrm{id}_V.$$

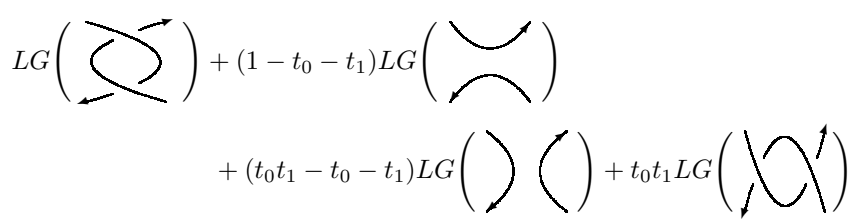
Note that $LG(\theta^\wedge; p^{-2}, p^2q^2)$ $(p = q^\alpha)$ coincides with the Links-Gould invariant in [4], where α is the complex parameter which represents a family of representations of $U_q[gl(2|1)]$. For the details we refer the reader to [4].

3. Skein relations of the Links-Gould invariant

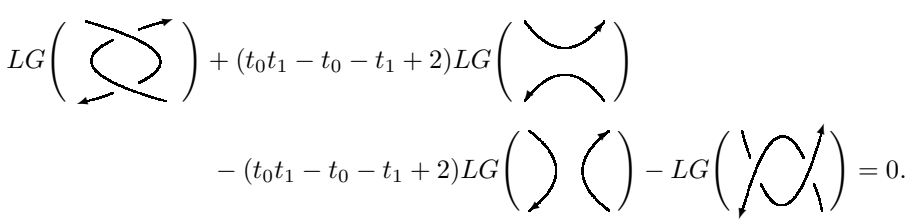
In this section, we present some useful skein relations of the Links-Gould invariant. We first present two basic skein relations of the invariant in the following theorem. One is obtained by transforming the well-known skein relation found by [6, p. 192] so that symmetrical pairs of tangles $(T(-1/2), T(-2)), (T(0), T(\infty))$ appear. (See [1] and Section 4 for the definition of the rational tangles $T(n)$.) The remaining relation was found through computer experiments by taking a hint from the symmetrized relation.

Theorem 3.1. *The Links-Gould invariant satisfies the following skein relations.*

Skein relation (i):



Skein relation (ii):

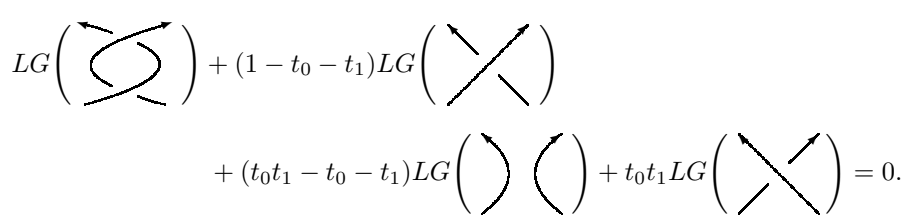


Proof. In [4, p. 170], the following skein relation is derived from the identity

$$(R - t_0\mathrm{id_V})(R - t_1\mathrm{id_V})(R + \mathrm{id_V}) = 0$$

(see also [6, p. 192]).

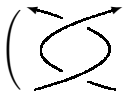
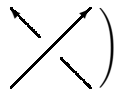
Skein relation (o):

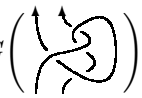


By "adding" the tangle  to skein relation (o) from the left, we obtain skein

relation (i). (Therefore skein relation (o) is equivalent to skein relation (i).)

By "adding" the tangle  to the following skein relation from the left, we obtain skein relation (ii):

$LG\Big($  $\Big) + (t_0t_1 - t_0 - t_1 + 2)LG\Big($  $\Big)$

$- (t_0t_1 - t_0 - t_1 + 2)LG\Big($  $\Big) - LG\Big($  $\Big) = 0.$

This equality is verified by checking the following equality with the aid of a computer:

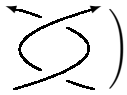
$$b_R^2(\sigma_1^2) + (t_0t_1 - t_0 - t_1 + 2)b_R^2(\sigma_1) - (t_0t_1 - t_0 - t_1 + 2)b_R^2(e) \\ - \mathrm{tr}_3^1\left((\mathrm{id}_V \otimes \mathrm{id}_V \otimes \mu)b_R^3(\sigma_2^2\sigma_1\sigma_2^{-1})\right) = 0.$$

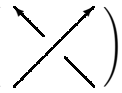
□

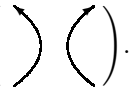
The following formulas follow from Theorem 3.1.

Corollary 3.2. *The Links-Gould invariant satisfies the following skein relations:*

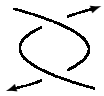
- $LG\Big($  $\Big\}$ *n half twists* $\Big)$

$$= \left(\frac{(-1)^n}{(t_0+1)(t_1+1)} + \frac{t_0^n}{(t_0+1)(t_0-t_1)} + \frac{t_1^n}{(t_1+1)(t_1-t_0)}\right) LG\Big($$  $\Big)$

$$- \left(\frac{(-1)^n(t_0+t_1)}{(t_0+1)(t_1+1)} + \frac{t_0^n(t_1-1)}{(t_0+1)(t_0-t_1)} + \frac{t_1^n(t_0-1)}{(t_1+1)(t_1-t_0)}\right) LG\Big($$  $\Big)$

$$+ \left(\frac{t_0t_1(-1)^n}{(t_0+1)(t_1+1)} - \frac{t_1t_0^n}{(t_0+1)(t_0-t_1)} - \frac{t_0t_1^n}{(t_1+1)(t_1-t_0)}\right) LG\Big($$  $\Big).$

- $LG\Big($  $\Big\}$ *n full twists* $\Big)$

$$= \frac{t_0^nt_1^n - 1}{t_0t_1 - 1} LG\Big($$  $\Big) + \left(1 - \frac{t_0^nt_1^n - 1}{t_0t_1 - 1}\right) LG\Big($  $\Big)$

$$+ \frac{2(t_0-1)(t_1-1)}{t_0t_1 - 1}\left(n - \frac{t_0^nt_1^n - 1}{t_0t_1 - 1}\right) LG\Big($$  $\Big).$

Proof. From skein relation (o), we obtain the first skein relation by induction. We obtain the following skein relation by subtracting skein relation (ii) from skein relation (i) after a half-rotation around a diagonal axis of tangles:

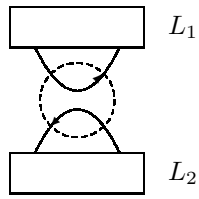
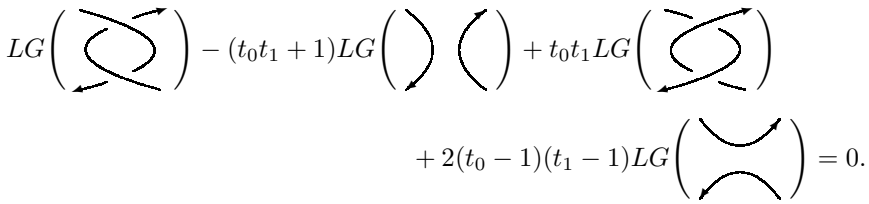


FIGURE 1.



From this skein relation, we obtain the second skein relation by induction. □

We remark that the first skein relation in Corollary 3.2 yields Proposition 1 of [6, p. 193].

To conclude this section, we summarize some of the fundamental properties of the Links-Gould invariant.

Proposition 3.3. *The Links-Gould invariant has the following properties.*

(1) $LG(\bigcirc) = 1$ *for the trivial knot* $\bigcirc$.
(2) *The Links-Gould invariant of a split link vanishes.*
(3) $LG(L\#L') = LG(L)LG(L')$, *where* $\#$ *denotes the connected sum of* L *and* L'.
(4) $LG(L^*; t_0, t_1) = LG(L; t_0^{-1}, t_1^{-1})$, *where* L^* *denotes the reflection of* L.
(5) *The Links-Gould invariant does not detect inversion, which implies the symmetry* $LG(L; t_0, t_1) = LG(L; t_1, t_0)$.

Note that the connected sum of oriented links depends upon which components are used to produce the connected sum. The property (3) is valid whatever components are selected, which easily follows through the evaluation of the (1,1)-tangle form [4]. For the properties (2),(4),(5), we refer the reader to [4, 3].

We remark that the property (2) can be deduced from skein relations (i),(ii) as follows. Let $L = L_1 \sqcup L_2$ be the split union of links L_1 and L_2. Let us apply the second skein relation in Corollary 3.2 to the tangle determined by the dotted circle in Figure 1. Then we have

$$\frac{2(t_0-1)(t_1-1)}{t_0t_1-1}\left(n - \frac{t_0^n t_1^n - 1}{t_0t_1-1}\right) LG(L) = 0.$$

Thus the Links-Gould invariant of a split link vanishes.

4. Recursive calculations of the Links-Gould invariant for algebraic links

Let us recall the definition of algebraic links. For tangles T_1 and T_2, we define new tangles T_1+T_2 and $T_1 \cdot T_2$, as in Figure 2. Here, we remark that the "multiplication" operation involves reflection of the first operand through a vertical plane containing its principal diagonal (NW-SE axis). We define tangles $T(1)$, $T(-1)$, $T(0)$

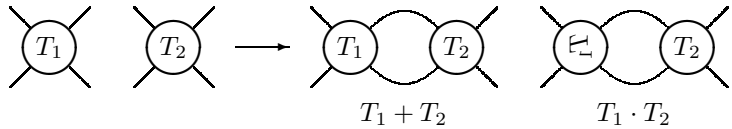


FIGURE 2.

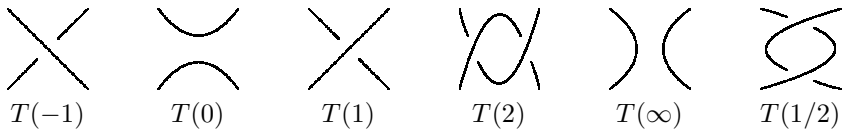


FIGURE 3.

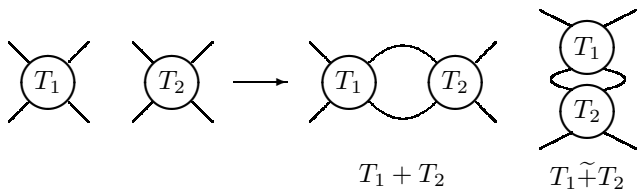


FIGURE 4.

$(= T(-1) + T(1))$, $T(2)$ $(= T(1) + T(1))$, $T(\infty)$ $(= T(0) \cdot T(0))$ and $T(1/2)$ $(= T(2) \cdot T(0))$ as in Figure 3. By an *algebraic tangle*, we mean a tangle obtained from the tangles $T(1)$ and $T(-1)$ by applying the above operations. An *algebraic link* is a link obtained from an algebraic tangle by closing off the top ends and the bottom ends respectively. For the details we refer the reader to [1].

We define certain skein modules by using skein relations (i),(ii). (For basic facts concerning skein modules, see [7].) Let $\mathcal{L}$ be the set of all oriented links in S^3 considered up to ambient isotopy. Let $\mathcal{L}_{\mathcal{A}}$ be the subset of $\mathcal{L}$ which consists of all oriented algebraic links. We fix four points on ∂B^3. Let $\mathcal{T}_{\mathcal{A}}$ be the set of all oriented algebraic tangles in B^3 whose ends are the four points. Let $K\mathcal{L}$, $K\mathcal{L}_{\mathcal{A}}$ and $K\mathcal{T}_{\mathcal{A}}$ be the free K-module generated by $\mathcal{L}$, $\mathcal{L}_{\mathcal{A}}$ and $\mathcal{T}_{\mathcal{A}}$ respectively. We define the skein modules $\mathcal{S}_{\mathcal{L}}$, $\mathcal{S}_{\mathcal{A}}^{\mathcal{L}}$ and $\mathcal{S}_{\mathcal{A}}^{\mathcal{T}}$ as the quotient of $K\mathcal{L}$, $K\mathcal{L}_{\mathcal{A}}$ and $K\mathcal{T}_{\mathcal{A}}$ by skein relations (i),(ii) respectively.

Put $\mathcal{G} = \{T(-1), T(0), T(1), T(\infty), T(1/2)\}$. Let $\vec{\mathcal{G}}$ be the set of the oriented tangles obtained from tangles in $\mathcal{G}$ by giving all possible orientations. Then we have the following theorem.

Theorem 4.1. *The skein module $\mathcal{S}_{\mathcal{A}}^{\mathcal{T}}$ is generated by the elements of $\vec{\mathcal{G}}$. The skein module $\mathcal{S}_{\mathcal{A}}^{\mathcal{L}}$ is generated by the trivial knot. In particular, we can evaluate the Links-Gould invariant of algebraic links recursively.*

Proof. Any link obtained from a tangle in $\mathcal{G}$ by closing off the ends is the trivial knot or a trivial link. Skein relations (i),(ii) imply that the Links-Gould invariant of a trivial link vanishes, as remarked at the end of previous section. Thus the first statement implies the rest.

Let us show the statement. It is easily checked that any algebraic tangle is obtained from the tangles $T(1)$ and $T(-1)$ by applying the two operations illustrated in Figure 4. Therefore it is enough to check that $LG(T_1 + T_2)$ and $LG(T_1 \widetilde{+} T_2)$,

where T_1 and T_2 run over all compatible pairs of oriented tangles in $\vec{\mathcal{G}}$, are K-linear combinations of $LG(T)$ ($T \in \vec{\mathcal{G}}$) using skein relations (i),(ii).

If $T_1 \widetilde{+} T_2$ is not trivial, then it is equivalent to $T(1) \widetilde{+} T(1) \widetilde{+} \cdots \widetilde{+} T(1)$ or $T(-1) \widetilde{+} T(-1) \widetilde{+} \cdots \widetilde{+} T(-1)$. Therefore, by using Corollary 3.2, we can reduce $LG(T_1 \widetilde{+} T_2)$ to a K-linear combination of $LG(T)$ ($T \in \vec{\mathcal{G}}$).

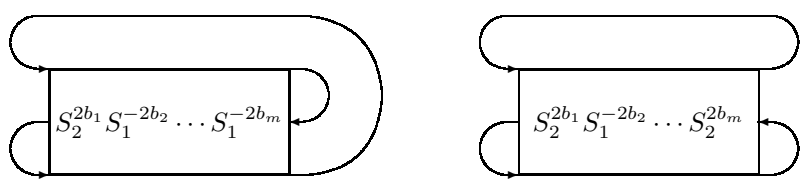
By replacing the tangle $T(1/2)$ with the tangle $T(2)$, we obtain $\mathcal{G}'$ and $\vec{\mathcal{G}}'$ from $\mathcal{G}$ and $\vec{\mathcal{G}}$ respectively. By skein relations (o),(i), it is enough to show that $LG(T_1' + T_2')$ ($T_1', T_2' \in \vec{\mathcal{G}}'$) is a K-linear combination of $LG(T')$ ($T' \in \vec{\mathcal{G}}'$), which follows from the previous case through $\pi/2$-rotation. □

Problem. Is the skein module $\mathcal{S}_{\mathcal{L}}$ generated by the trivial knot?

5. A formula for the Links-Gould invariant of 2-bridge links

A 2-bridge link is an algebraic link which is obtained from the tangles $T(1)$ and $T(-1)$ using only the "multiplication" operation. In this section we represent an oriented 2-bridge link as follows. Let S_1 and S_2 be the generators of 3-string braid group as shown in Figure 5. Let $D(b_1, b_2, \ldots, b_m)$ be the oriented 2-bridge link as shown in Figure 6. We remark that any 2-bridge link can be put in this form.

Figure 5.



m: even $\qquad$ m: odd

Figure 6.

Let $LG(b_1, b_2, \ldots, b_m)$ be the Links-Gould invariant of $D(b_1, b_2, \ldots, b_m)$.

Proposition 5.1.

$$LG(b_1, b_2, \ldots, b_m) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} A((-1)^m b_m) \cdots A(b_2) A(-b_1) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

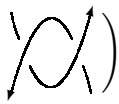
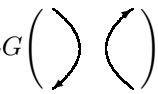
where

$$\begin{aligned} A(n) &= \begin{bmatrix} a_1(n+1) & a_2(n+1) & a_3(n+1) \\ a_1(n) & a_2(n) & a_3(n) \\ 0 & 1 & 0 \end{bmatrix}, \\ a_1(n) &= \frac{t_0^n t_1^n - 1}{t_0 t_1 - 1}, \end{aligned}$$

$$\begin{aligned} a_2(n) &= \frac{2n(t_0-1)(t_1-1)}{t_0t_1-1} - a_1(n)\left(\frac{(t_0t_1+1)(t_0-1)(t_1-1)}{t_0t_1-1}+1\right), \\ a_3(n) &= (t_0-1)(t_1-1)a_1(n)+1. \end{aligned}$$

Proof. Set $\tilde{D}(b_1, b_2, \ldots, b_m) = D(-b_1, b_2, \ldots, (-1)^m b_m)$. By skein relation (ii) and the second skein relation in Corollary 3.2, we have

$LG\Bigg($  $\Big\}\ n$ full twists $\Bigg)$

$= a_1(n)LG\Bigg($  $\Bigg) + a_2(n)LG\Bigg($  $\Bigg) + a_3(n)LG\Bigg($  $\Bigg)$.

Let us apply the skein relation to the crossings which correspond to $S_1^{-2b_m}$ or $S_2^{2b_m}$ according as m is even or odd. Then we have

$$\begin{aligned} LG(\tilde{D}(b_1, b_2, \ldots, b_m)) &= a_1(b_m)LG(\tilde{D}(b_1, b_2, \ldots, b_{m-2}, b_{m-1}+1)) \\ &\quad + a_2(b_m)LG(\tilde{D}(b_1, b_2, \ldots, b_{m-2}, b_{m-1})) \\ &\quad + a_3(b_m)LG(\tilde{D}(b_1, b_2, \ldots, b_{m-3}, b_{m-2})), \end{aligned}$$

$$\begin{aligned} LG(\tilde{D}(b_1, b_2)) &= a_1(b_2)LG(\tilde{D}(b_1+1)) + a_2(b_2)LG(\tilde{D}(b_1)) + a_3(b_2)LG(\bigcirc), \\ LG(\tilde{D}(b_1)) &= a_1(b_1)LG(\bigcirc) + a_2(b_1)LG(\bigcirc) + a_3(b_1)LG(\bigcirc\bigcirc), \end{aligned}$$

where $\bigcirc\bigcirc$ denotes a 2-component trivial link. Fix $b_1, b_2, \ldots, b_m$. Put

$$\begin{aligned} X_n &= LG(\tilde{D}(b_1, b_2, \ldots, b_n+1)), \\ Y_n &= LG(\tilde{D}(b_1, b_2, \ldots, b_n)). \end{aligned}$$

Then

$$\begin{aligned} X_n &= a_1(b_n+1)X_{n-1} + a_2(b_n+1)Y_{n-1} + a_3(b_n+1)Y_{n-2}, \\ Y_n &= a_1(b_n)X_{n-1} + a_2(b_n)Y_{n-1} + a_3(b_n)Y_{n-2}, \\ X_2 &= a_1(b_2+1)X_1 + a_2(b_2+1)Y_1 + a_3(b_2+1)1, \\ Y_2 &= a_1(b_2)X_1 + a_2(b_2)Y_1 + a_3(b_2)1, \\ X_1 &= a_1(b_1+1)1 + a_2(b_1+1)1 + a_3(b_1+1)0, \\ Y_1 &= a_1(b_1)1 + a_2(b_1)1 + a_3(b_1)0. \end{aligned}$$

Thus

$$\begin{bmatrix} X_n \\ Y_n \\ Y_{n-1} \end{bmatrix} = A(b_n)\begin{bmatrix} X_{n-1} \\ Y_{n-1} \\ Y_{n-2} \end{bmatrix}, \qquad \begin{bmatrix} X_2 \\ Y_2 \\ Y_1 \end{bmatrix} = A(b_2)A(b_1)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

We remark that

$$LG(\tilde{D}(b_1, b_2, \ldots, b_m)) = Y_m = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} X_m \\ Y_m \\ Y_{m-1} \end{bmatrix}.$$

Therefore

$$LG(b_1, b_2, \ldots, b_m) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} A((-1)^m b_m) \cdots A(b_2) A(-b_1) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

□

To conclude this paper, we give some examples.

Example 5.2.

$$\begin{aligned}
LG(3_1) &= LG(-1,1) \\
&= -t_0^2 t_1 - t_0 t_1^2 + t_0^2 + 2t_0 t_1 + t_1^2 - t_0 - t_1 + 1, \\
LG(4_1) &= LG(1,1) \\
&= 2t_0 t_1 - 3t_0 - 3t_1 + \frac{t_0}{t_1} + \frac{t_1}{t_0} - \frac{3}{t_0} - \frac{3}{t_1} + \frac{2}{t_0 t_1} + 7, \\
LG(5_1) &= LG(-1,1,-1,1) \\
&= -t_0^4 t_1 - t_0^3 t_1^2 - t_0^2 t_1^3 - t_0 t_1^4 + t_0^4 + 2t_0^3 t_1 + 2t_0^2 t_1^2 + 2t_0 t_1^3 + t_1^4 \\
&\quad - t_0^3 - 2t_0^2 t_1 - 2t_0 t_1^2 - t_1^3 + t_0^2 + 2t_0 t_1 + t_1^2 - t_0 - t_1 + 1, \\
LG(5_2) &= LG(-2,1) \\
&= -t_0^3 t_1^2 - t_0^2 t_1^3 + t_0^3 t_1 + 4t_0^2 t_1^2 + t_0 t_1^3 - 6t_0^2 t_1 - 6t_0 t_1^2 \\
&\quad + 3t_0^2 + 10t_0 t_1 + 3t_1^2 - 5t_0 - 5t_1 + 3.
\end{aligned}$$

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Department of Mathematics, Graduate School of Science, Osaka University, Machikaneyama 1-16, Toyonaka, Osaka 560-0043, Japan

E-mail address: aishii@cr.math.sci.osaka-u.ac.jp