ON THE GAUSS MAP OF HYPERSURFACES
WITH CONSTANT SCALAR CURVATURE IN SPHERES

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Abstract. In this work we consider connected, complete and orientable hypersurfaces of the sphere $S^{n+1}$ with constant nonnegative $r$-mean curvature. We prove that under subsidiary conditions, if the Gauss image of $M$ is contained in a closed hemisphere, then $M$ is totally umbilic.

Introduction

One of the most celebrated theorems of minimal surfaces in $\mathbb{R}^3$ is Bernstein’s theorem:

**Theorem** (Bernstein [4]). Let $M \subset \mathbb{R}^3$ be a complete minimal surface in $\mathbb{R}^3$ that is given by an entire (defined over the whole $\mathbb{R}^2$) graph of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$. Then $M$ is a plane.

The above result is also true under the weaker hypothesis that the image of the Gauss map of $M$ lies in an open hemisphere of $S^{n+1}$, as one can see in [3]. These results raise the following problem for the geometry of minimal surfaces in spheres: Does there exist a similar result for minimal hypersurfaces of the unit sphere? The answer to this question was obtained independently by E. De Giorgi ([6]) and J. Simons (see [13] - Theorem 5.2.1) as follows.

**Theorem.** If the Gauss image (see the definition below) of a compact minimal hypersurface $M^n$ in the Euclidean sphere lies in an open hemisphere of $S^{n+1}$, then $M$ must be a great hypersphere in $S^{n+1}$.

After that, K. Nomizu and Brian Smyth (see [9] - Theorem 2) were able to generalize this result to constant mean curvature hypersurfaces of $S^{n+1}$, proving the following result:

**Theorem** (Nomizu-Smyth). Let $M$ be any compact connected orientable manifold of dimension $n \geq 2$ immersed in the sphere $S^{n+1}$ with constant mean curvature. If the Gauss image of $M$ lies in a closed hemisphere of $S^{n+1}$, then $M$ is a hypersphere in $S^{n+1}$.

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The goal of this work is to extend these results to higher-order constant mean curvature hypersurfaces of the sphere. First let us fix some notation.

Let $M^n$ be a compact orientable Riemannian manifold and let $x : M^n \to S^{n+1}$ be an isometric immersion into the unit sphere $S^{n+1} \subset \mathbb{R}^{n+2}$. Since $M$ is orientable, we can choose a global unit normal field $N$. The Riemannian connections $\nabla$ and $\nabla$ of $M$ and $S^{n+1}$, respectively, are related by

$$\nabla_X Y = \nabla_X Y + (A(X), Y) N,$$

where $A$ is the shape operator of the immersion, defined by

$$\nabla_X N = A(X).$$

Let $k_1, \ldots, k_n$ be the eigenvalues of $A$. We define the $r$-mean curvature of the immersion at a point $p$ by

$$H_r = \frac{1}{\binom{n}{r}} \sum_{i_1 < \ldots < i_r} k_{i_1} \ldots k_{i_r} = \frac{1}{\binom{n}{r}} S_r,$$

where $S_r$ is the $r$-symmetric function of the $k_1, \ldots, k_n$. In order to unify the notation, we will define $H_0 = 1$ and $H_r = 0$, for all $r \geq n + 1$. For $r = 1$, $H_1 = H$ is the mean curvature of the immersion, in the case $r = 2$, $H_2$ is the scalar curvature and for $r = n$, $H_n$ is the Gauss-Kronecker curvature.

The Gauss map $\phi : M^n \to S^{n+1}$ is defined by

$$\phi(P) = N(P) \in S^{n+1}.$$

The set $\phi(M)$ is called the Gauss image of $M$. We observe that the Gauss image depends on the choice of the orientation of $M$, but the two possibilities are related by an antipodal mapping of $S^{n+1}$. Thus the statement that the Gauss image of $M$ is contained in a closed hemisphere of $S^{n+1}$ is independent of the orientation of $M$. For the case $H_r = 0$, we obtain that

**Theorem A.** Let $M^n \to S^{n+1}$ be a compact and connected hypersurface of $S^{n+1}$ with $H_r = 0$, for some $r = 1, \ldots, n - 1$. Assume that the Gauss image of $M$ is contained in a closed hemisphere and that $H_{r-1}$ does not change sign in $M$. Then $M$ is totally geodesic.

If $H_r > 0$, we were able to prove that

**Theorem B.** Let $M^n \to S^{n+1}$ be a compact and connected hypersurface of $S^{n+1}$ with constant positive $(r + 1)$-mean curvature $H_{r+1}$, for some $r = 0, \ldots, n - 2$. Assume that the Gauss image of $M$ is contained in a closed hemisphere, $H_r \geq 0$ and that the following inequality holds:

$$H_1 H_r \geq H_{r+1}.$$ 

Then $M$ is totally umbilic.

In the case of the scalar curvature, part of the hypothesis of the above theorems is trivially satisfied, and we obtain the following result.

**Theorem C.** Let $M^n$ be a compact orientable hypersurface of the sphere with constant scalar curvature $H_2 \geq 0$. In the case $H_2 = 0$, suppose also that $H_1$ does not change sign. If the Gauss image of $M$ lies in a closed hemisphere of $S^{n+1}$, then $M$ is totally umbilic.
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The authors do not know if the hypotheses of Theorems A, B and C can be weakened.

Parts of these results were obtained by R. Reilly, [11], with the strong hypothesis that the Gauss image is contained in an open hemisphere.

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1. Preliminaries

We introduce the \( r \)-th Newton tensors, \( P_r : T_pM \to T_pM \), which are defined inductively by

\[
P_0 = I, \quad P_r = S_r I - A P_{r-1}, \quad r > 1.
\]

It is easy to see that each \( P_r \) commutes with \( A \), and if \( e_i \) is an eigenvector of \( A \) associated to the principal curvature \( k_i \), then

\[
P_1(e_i) = \mu_i e_i = (S_1 - k_i)e_i.
\]

In [11], Reilly showed that the \( P_r \)'s satisfy the following.

Proposition 1.1 (see also [2] - Lemma 2.1). Let \( x : M \to N \) be an isometric immersion between two Riemannian manifolds, and let \( A \) be its second fundamental form. The \( r \)-th Newton tensor \( P_r \) associated to \( A \) satisfies:

1. \( \text{tr}(P_r) = (n - r)S_r \).
2. \( \text{tr}(AP_r) = (r + 1)S_{r+1} \).
3. \( \text{tr}(A^2 P_r) = S_1 S_{r+1} - (r + 2)S_{r+2} \).

Associated to each Newton tensor \( P_r \), we define a second-order differential operator

\[
L_r(f) = \text{tr}(P_r \text{Hess } f).
\]

We observe that for \( r = 0 \), \( L_0 \) is the Laplacian, which is always an elliptic operator. If \( N \) has constant sectional curvature, it follows from the Codazzi equation (see [12], p. 225) that \( L_r \) is

\[
L_r(f) = \text{div}_M(P_r \nabla f).
\]

Hence \( L_r \) is a self-adjoint operator. In general, for \( r \geq 1 \), \( L_r \) is not an elliptic operator. The following proposition give us a condition for \( L_r \) to be elliptic.

Proposition 1.2. Let \( M \) be a connected, compact and orientable Riemannian manifold, and let \( x : M \to S^{n+1} \) be an isometric immersion with \( H_{r+1} \) constant. If \( M \) has one point where all principal curvatures are positive, then \( L_r \) is an elliptic operator.

Proof. See the proof of Proposition 3.2 of [2].

For hypersurfaces of \( \mathbb{R}^{n+1} \) with \( H_r = 0 \), Hounie and Leite, [9], were able to give a geometric condition that is equivalent to \( L_r \) being elliptic. In fact, their proof can be generalized to hypersurfaces of the sphere, and we have the following result.

Proposition 1.3 (9 - Proposition 1.5). Let \( M \) be a hypersurface in \( \mathbb{R}^{n+1} \) or \( S^{n+1} \) with \( H_r = 0 \), \( 2 \leq r < n \). Then the operator \( L_{r-1}(f) = \text{div}(P_{r-1} \nabla f) \) is elliptic at \( p \in M \) if and only if \( H_{r+1}(p) \neq 0 \).
Since the $r$-mean curvatures of $M^n$ are symmetric means of the $n$-uple of principal curvatures of $M$, they are related by the following algebraic inequalities (see [7], p. 52, and [5], p. 285):

\begin{equation}
H_{i-1} H_{i+1} \leq H_i^2, \quad \forall i, \ 1 \leq i < n.
\end{equation}

Also, provided that the $H_r$'s are nonnegative, $r = 1, \ldots, i$,

\begin{equation}
H_1 \geq H_2^{1/2} \geq H_3^{1/3} \geq \ldots \geq H_i^{1/i}.
\end{equation}

Furthermore, the equality in (1.1) and (1.2) holds only if $k_1 = k_2 = \ldots = k_n$.

\section{Integral formula}

Consider the functions $f, g : M \to \mathbb{R}$, given by

\[ f(P) = \langle N(P), \alpha \rangle \]

and

\[ g(P) = \langle x(P), \alpha \rangle, \]

where $\alpha$ is a fixed vector of $\mathbb{R}^{n+2}$. These functions satisfy (see [2], Lemma 5.2)

\begin{align*}
L_r(g) &= -(r+1)S_{r+1}f - (n-r)S_rg, \\
L_r(f) &= -(S_1S_{r+1} - (r+2)S_{r+2}) f - (r+1)S_{r+1}g,
\end{align*}

where, in the last equation, we use the fact that $S_{r+1}$ is constant. In particular, for $r = 0$, we get

\begin{align*}
\Delta(g) &= -S_1f - ng, \\
\Delta(f) &= -(S_1^2 - 2S_2) f - S_1g = -\|A\|^2 f - S_1g.
\end{align*}

The following integral formula will be needed.

\begin{proposition}
Let $M^n \to S^{n+1}$ be a compact orientable hypersurface isometrically immersed in $S^{n+1}$, with $H_{r+1}$ constant, for some $r$ with $0 \leq r < n - 2$. Then,

\begin{equation}
\int_M [(n-r-1)S_1S_{r+1} - n(r+2)S_{r+2}] f \ dM = 0.
\end{equation}

\end{proposition}

\begin{proof}
Observe that, since $S_{r+1}$ is constant, by (2.2) and (2.4), we obtain that

\begin{align*}
L_r f - \frac{(r+1)}{n} S_{r+1} \Delta g &= -(S_1S_{r+1} - (r+2)S_{r+2}) f \\
&= -(r+1)S_{r+1}g + \frac{(r+1)}{n} S_{r+1}S_1f + \frac{(r+1)}{n} S_{r+1}ng \\
&= -S_1S_{r+1}f + (r+2)S_{r+2}f + \frac{(r+1)}{n} S_{r+1}S_1f \\
&= \frac{1}{n} [-nS_1S_{r+1}f + n(r+2)S_{r+2}f + (r+1)S_{r+1}S_1f] \\
&= \frac{1}{n} [(n-r+1)S_1S_{r+1}f + n(r+2)S_{r+2}f] \\
&= \frac{1}{n} [(n-r-1)S_1S_{r+1} - n(r+2)S_{r+2}] f.
\end{align*}

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Integrating this last expression and applying Stokes’ Theorem, one has that
\[
\int_M \left[ (n-r-1)S_1S_{r+1} - n(r+2)S_{r+2} \right] f \, dM
\]
\[
= \int_{\partial M} \langle P_r \nabla f - \frac{(r+1)}{n} S_{r+1} \nabla g, \nu \rangle \, dS = 0,
\]
where the last equality follows from the fact that \( \partial M = \emptyset \).

3. The case \( H_r = 0 \)

In this section we consider hypersurfaces of the sphere with \( H_r = 0 \). We have the following result.

**Theorem 3.1** (Theorem A of the Introduction). Let \( M^n \to S^{n+1} \) be a compact and connected hypersurface of \( S^{n+1} \) with \( H_r = 0 \), for some \( r = 1, \ldots, n-1 \). Assume that the Gauss image of \( M \) is contained in a closed hemisphere and that \( H_{r-1} \) does not change sign in \( M \). Then \( M \) is totally geodesic.

**Proof.** By (1.1) and the fact that \( H_r = 0 \), it follows that \( H_{r+1}H_{r-1} \leq 0 \).

Thus, since \( H_{r-1} \) does not change sign in \( M \), \( H_{r+1} \) also does not change sign on \( M \).

On the other hand, our hypothesis on the Gauss image implies that there exists a vector \( \alpha \in \mathbb{R}^{n+2} \) such that
\[
f(P) = \langle N(P), \alpha \rangle
\]
is nonnegative along \( M \). Hence, \( f(P)S_{r+1}(P) \) does not change sign along \( M \). The equation (2.5), in our case, reads
\[
\int_M f(P)S_{r+1}(P)dM = 0.
\]
Thus,
\[
(3.1) \quad f(P)S_{r+1}(P) = 0, \quad \forall P \in M.
\]

Let \( \mathcal{A} \subset M \) be the set of all points of \( M \) where \( S_{r+1}(P) > 0 \). In \( \mathcal{A} \), by equation (3.1), \( f = 0 \). By continuity, \( f \) is zero along \( \overline{\mathcal{A}} \), where \( \overline{\mathcal{A}} \) is the closure of \( \mathcal{A} \). On the other hand, the set \( M/\overline{\mathcal{A}} \) is an open set of \( M \) where
\[
H_r = H_{r+1} = 0.
\]
Hence equality holds in (3.1), for all \( P \in M/\overline{\mathcal{A}} \). This means that all points in \( M/\overline{\mathcal{A}} \) are umbilic. That is, for all \( P \in M/\overline{\mathcal{A}} \),
\[
k_1(P) = \ldots = k_n(P) = a(P).
\]

Thus,
\[
0 = S_r(P) = a^*(P).
\]
This implies that all points of \( M/\overline{\mathcal{A}} \) are totally geodesic, and hence \( f \) is constant along each connected component of \( M/\overline{\mathcal{A}} \). Since along the boundary of those sets, \( f = 0 \), we conclude that \( f \) is identically zero on \( M \), that is, \( M \) is totally geodesic (see Theorem 1 of [9]).

**Remark.** For the case \( r = 1 \), we observe that \( S_{r-1} = S_0 = 1 \) does not change sign. Hence, the theorem is a generalization of Theorem 2 in [9], in the minimal case.
4. The case $H_{r+1} > 0$

Let us consider the case $H_{r+1} > 0$. We have the following result:

**Theorem 4.1** (Theorem B of the Introduction). Let $M^n \rightarrow S^{n+1}$ be a compact and connected hypersurface of $S^{n+1}$ with constant positive $(r+1)$-mean curvature $H_{r+1}$, for some $r = 0, ..., n-2$. Assume that the Gauss image of $M$ is contained in a closed hemisphere, $H_r \geq 0$ and that the following inequality holds:

\[(4.1)\]

\[H_1 H_r \geq H_{r+1}.\]

Then $M$ is totally umbilic.

**Proof.** By Proposition 2.1 we have that for a fixed $\alpha \in \mathbb{R}^{n+2}$, the function $f = \langle N(P), \alpha \rangle$ satisfies

\[(4.2)\]

\[\int_M [(n-r-1)S_1 S_{r+1} - n(r+2)S_{r+2}] f \, dM = 0.\]

We are going to prove that the integrand has a fixed sign, for some $\alpha \in \mathbb{R}^{n+2}$. Since the Gauss image of $M$ lies in a closed hemisphere, there exists a vector $\alpha \in \mathbb{R}^{n+2}$ such that

\[(4.3)\]

\[f(P) = \langle N(P), \alpha \rangle \geq 0, \quad \forall P \in M.\]

On the other hand, the relation $H_1 H_r \geq H_{r+1}$ implies that $H_1 H_{r+1} \geq H_{r+2}$. In fact, by using equation (1.1), one has that

\[(4.4)\]

\[H_1 H_{r+2} \leq H_{r+1}^2 \leq H_r H_1 H_{r+1}.\]

Observe that $H_r \neq 0$; otherwise, the last inequality implies that $H_r$ and $H_{r+1}$ are equal to zero, which is a contradiction. Hence, $H_r > 0$ and we can divide (4.4) by (4.5)

\[(4.5)\]

\[H_1 H_{r+1} \geq H_{r+2}.\]

Since

\[H_t = \frac{S_t}{\binom{n}{t}},\]

by (4.5), one has

\[\frac{S_1}{n} \frac{S_{r+1}}{\binom{n}{r+1}} \geq \frac{S_{r+2}}{\binom{n}{r+2}}.\]

This implies that

\[(4.6)\]

\[(n-r-1)S_1 S_{r+1} - n(r+2)S_{r+2} \geq 0.\]

The inequalities (4.5) and (4.6) imply that

\[\frac{(n-r-1)S_1 S_{r+1} - n(r+2)S_{r+2}}{f} \geq 0.\]

Thus, by (4.2), we have that

\[\frac{(n-r-1)S_1 S_{r+1} - n(r+2)S_{r+2}}{f} = 0.\]

Observe that the function $f$ is not identically zero, since in this case, $M$ has to be totally geodesic (see Theorem 1 of [9]) and hence $H_r = 0$, which is a contradiction. Let $\mathcal{B} \subset M$ be the open and nonempty set where $f > 0$. Along $\mathcal{B}$, we have

\[\frac{(n-r-1)S_1 S_{r+1} - n(r+2)S_{r+2}}{f} = 0,\]

that is, equality holds in (4.6). This means that equality also holds in (1.1), since this inequality was used to obtain (4.6). Hence, all points of $\mathcal{B}$ are umbilic. In this
case, \( M \) has an elliptic point and \( S_r = \text{constant} > 0 \). Thus, by Proposition \ref{prop:elliptic} the operator \( L_r \) is an elliptic operator. By the principle of analytic continuation, since \( M \) is totally umbilic in an open set, it has to be totally umbilic.

Observe that in the case \( r = 2 \), part of the hypotheses of Theorems 3.1 and 4.1 is trivially satisfied, and we have the following result.

**Corollary 4.1** (Theorem C of the Introduction). Let \( M^n \) be a compact orientable hypersurface of the sphere with constant scalar curvature \( H_2 \geq 0 \). In the case \( H_2 = 0 \), suppose also that \( H_1 \) does not change sign. If the Gauss image of \( M \) lies in a closed hemisphere of \( S^{n+1} \), then \( M \) is totally umbilic.

**Proof.** The case \( H_2 = 0 \) is the statement of Theorem 3.1. For the case \( H_2 > 0 \), the hypothesis \((4.1)\) in Theorem 4.1 reads

\[
H_1^2 - H_{r+2} \leq 0,
\]

which is always true by equation \((1.1)\). The above equation also says that \( H_1 \) is different from zero on \( M \). Hence we can choose the orientation of \( M \) so that \( H_1 > 0 \). The sign of \( H_2 \) does not depend on the orientation; thus the result follows directly from Theorem 4.1.

We now give conditions that imply condition \((4.1)\). First of all, if \( H_i \) is nonnegative for \( i = 1, \ldots, r - 1 \), then \((4.1)\) holds. This fact was stated in [12], p. 232, and we are including its proof here for the sake of completeness. Let \((x_1, \ldots, x_n)\) be an \( n \)-uple of real numbers, and let \( S_r \) be the \( r \)-symmetric function of the \( x_1, \ldots, x_n \). Let \( H_r \) be defined by

\[
H_r = \frac{1}{\binom{n}{r}} S_r = \frac{1}{\binom{n}{r}} \sum_{1 \leq i_1 < \ldots < i_r} x_{i_1} \ldots x_{i_r}.
\]

**Proposition 4.1.** With the above notation, if \( H_i \geq 0 \) for all \( i = 1, \ldots, r - 1 \), then

\[
H_1 H_{i+1} \geq H_{i+2}, \quad \forall i = 1, \ldots, r - 1.
\]

Moreover,

\[
(n - i - 1)S_1 S_{i+1} - n(i + 2)S_{i+2} \geq 0, \quad \forall i = 1, \ldots, r - 1.
\]

**Proof.** By using \((4.1)\), we have that

\[
H_r H_{r-2} \geq H_{r-1}^2 \geq 0
\]

and

\[
H_{r+1} H_{r-1} \geq H_r^2 \geq 0.
\]

Since \( H_{r-2} \) and \( H_{r-1} \) are nonnegative, it follows that \( H_r \geq 0 \) and \( H_{r+1} \geq 0 \). Let us prove \((4.7)\). We will argue by induction on \( i \). By using \((1.1)\), with \( i = 1 \), and the fact that \( H_0 = 1 \), we obtain

\[
H_1^2 \geq H_0 H_2 = H_2.
\]

Hence \((4.7)\) holds for \( i = 0 \). By induction, let us suppose that

\[
H_i H_{i+1} \geq H_{i+2},
\]

This implies, using equation \((1.1)\), that

\[
H_i H_{i+2} \leq H_{i+1}^2 \leq H_{i+1} H_i.
\]
If \( H_i = 0 \), then (4.9) implies that \( H_{i+1} \leq 0 \). Since \( H_{i+1} \geq 0 \), it follows that \( H_{i+1} = 0 \). Thus we have equality in (1.2), which implies that \( x_k = 0, \forall k = 1, \ldots, n \). Hence (4.7) holds in this case.

Let us suppose \( H_i > 0 \). In this case, we can divide (4.10) by \( H_i \) and obtain
\[
H_i H_{i+1} \geq H_{i+2},
\]
and we finish the proof of (4.7). In order to obtain (4.8), just observe that
\[
H_i = \frac{S_i}{(i)}.
\]

Then, by (4.11), one has
\[
\frac{S_1 S_{i+1}}{n (\begin{array}{c} n \\ i+1 \end{array})} \geq \frac{S_{i+2}}{(\begin{array}{c} n \\ i+2 \end{array})}.
\]
This implies that
\[
(n - i - 1)S_1 S_{i+1} - n(i + 2)S_{i+2} \geq 0, \quad \forall i = 1, \ldots, r - 2.
\]

Thus, we have the following result.

**Corollary 4.2.** Let \( M^n \to S^{n+1} \) be a compact and connected hypersurface of \( S^{n+1} \) with constant positive \( r \)-mean curvature \( H_r \), for some \( r = 1, \ldots, n - 1 \). Assume that the Gauss image of \( M \) is contained in a closed hemisphere and that \( H_i \geq 0 \) for all \( i = 1, \ldots, r - 1 \). Then \( M \) is totally umbilic.

In the following proposition (see Proposition 2.3 in [2]) we have another geometric condition that gives \( H_i \geq 0 \) for all \( i = 1, \ldots, r - 1 \).

**Proposition 4.2.** Let \( M^n \) be a connected compact Riemannian manifold, and let \( x : M^n \to S^{n+1} \) be an isometric immersion. If \( H_r > 0 \) and \( x(M) \) is contained in an open hemisphere of \( S^{n+1} \), then \( H_i > 0 \) for all \( i = 1, \ldots, r - 1 \).

This and Corollary 4.2 imply

**Corollary 4.3.** Let \( x : M^n \to S^{n+1} \) be an isometric immersion of a compact and connected hypersurface of \( S^{n+1} \) with constant positive \( r \)-mean curvature \( H_r \), for some \( r = 1, \ldots, n - 1 \). Assume that the Gauss image of \( M \) is contained in a closed hemisphere and that \( x(M) \) is contained in an open hemisphere of \( S^{n+1} \). Then \( M \) is totally umbilic.

**References**


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