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A COMPLETE SYSTEM OF ORTHOGONAL STEP FUNCTIONS

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ABSTRACT. We educe an orthonormal system of step functions for the interval $[0, 1]$. This system contains the Rademacher functions, and it is distinct from the Paley-Walsh system: its step functions use the Möbius function in their definition. Functions have almost-everywhere convergent Fourier-series expansions if and only if they have almost-everywhere convergent step-function-series expansions (in terms of the members of the new orthonormal system). Thus, for instance, the new system and the Fourier system are both complete for $L^p(0, 1)$; $1 < p \in \mathbb{R}$.

1. INTRODUCTION

Analytical desiderata and applications, ever and anon, motivate the elaboration of systems of orthogonal step functions—as exemplified by the Haar system, the Paley-Walsh system and the Rademacher system. Our motivation is the example-based classification of digitized images, represented by rational points of the real interval [0,1], the domain of interest in the sequel.

It is imperative to establish the "completeness" of any orthogonal system. Much is known, in this regard, for the classical systems, as this has been the subject of numerous investigations, and we use the latter results to establish analogous properties for the orthogonal system developed herein.

Definition 1. Let the *inner product* $\langle f(x), g(x)\rangle$ denote the Lebesgue integral $\int_0^1 f(x)g(x)dx$.

Definition 2. A system of orthogonal functions is *complete*, on a given domain and relative to a class of functions, whenever the vanishing of a member of the class' inner products with all the orthogonal functions implies the member is equal to zero almost everywhere (a.e.) in the domain. (Completeness is also termed *maximality* [1].)

We will employ the following standard notation.

Definition 3. (i) $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{N}_1$ denotes the set of all odd, positive integers; (ii) $L^q(0, 1)$ denotes the class of real functions whose absolute value raised to the power $q \in \mathbb{R}$ is Lebesgue integrable over the interval

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$[0,1]$; (iii) $\lfloor t \rfloor$ denotes the integer part of $t \in \mathbb{R}$; and (iv) $\mu(\ell)$ denotes the classical Möbius function; $\ell \in \mathbb{N}$ [4, §16.3].

Note that, in (ii), the conventional exponent p has been replaced by q because, in the sequel, the former exclusively denotes primes.

The Rademacher system may be taken to consist of the orthogonal functions $r_k(x) = (-1)^{\lfloor 2^k x \rfloor}$; $k \in \mathbb{N}$. This system is incomplete on [0,1] because, for instance, the $r_k(x)$'s have odd symmetry about $x = 1/2$ (i.e. $r_k(1/2 + x) = -r_k(1/2 - x)$); its classical completion is the Paley-Walsh system [3, 11].

Herein, we describe an alternative step-function completion (*cf.* Definition 5): a number-theoretic analogue of the Fourier system (*cf.* Definition 7), with the analogy reinforced by a common function-domain of completeness (*viz.* Theorem 3); Section 4 contains pertinent definitions and details. For instance, as a corollary of the Carleson-Hunt theorem [5, 6], the expansion of every function in $L^q(0,1)$; $1 < q \in \mathbb{R}$, in terms of our step functions, converges pointwise, almost everywhere (a.e.) to the respective function (*cf.* Corollary 3). Furthermore, there are Orlicz classes whose members have a.e. convergent Fourier-series expansions [5, Thm. 2], and therefore, a.e. convergent expansions in terms of our step functions. Any distinctions between the completeness properties of the new system and the aforementioned classical systems remain unresolved by means of the elementary concepts of analysis employed herein.

Members of the Paley-Walsh system have constant step heights and may, in general, be construed as having multiple step lengths. Members of the new system have constant step length and, in general, have multiple step heights. Although the new system is, by design, a discrete analogue of the Fourier system, this analogy fails in one particular aspect: Complex exponentials $\exp\{i2\pi jx\}$; $i^2 = -1$, $j \in 0 + \mathbb{N}$, are quintessential group characters. Furthermore, the members of the Paley-Walsh system, with multiplicative composition, constitute characters of the dyadic group, acting upon binary representations of respective indices [3]. Our system wants, however, a non-trivial, closed composition of its step functions: requisite for a group character.

2. Orthogonal system

Our system is constructed via linear combinations of the elementary step functions

$$c_j(x) \stackrel{\text{def}}{=} \operatorname{sgn}(\cos 2\pi jx) = (-1)^{\lfloor 2jx + 1/2 \rfloor};\ j \in 0 + \mathbb{N}, \tag{1}$$

and

$$s_j(x) \stackrel{\text{def}}{=} \operatorname{sgn}(\sin 2\pi jx) = (-1)^{\lfloor 2jx \rfloor};\ j \in \mathbb{N}, \tag{2}$$

where $\operatorname{sgn}(z)$ denotes the signum function: taking values -1, 0 and 1, for negative, vanishing and positive arguments, respectively. The Rademacher system is comprised by the $s_{2^{\ell-1}}(x) = r_\ell(x)$; $\ell \in \mathbb{N}$.

Orthogonality connotes the vanishing of the respective inner product (*cf.* Definition 1). Although every $c_j(x)$ is plainly orthogonal to every $s_j(x)$ (because of their opposite symmetries about $x = 1/2$) neither are the $c_j(x)$'s nor are the $s_j(x)$'s mutually orthogonal, in general. Furthermore, provisionally extending the compass of Definition 2 to non-orthogonal functions, the system consisting of the c's and s's would be incomplete. Consider, for instance, expanding $\sin 2\pi x$ in terms of the s's: only s_1 has a non-vanishing inner product with $\sin 2\pi x$ (*cf.* Lemma 3), and this affords a poor approximation. Nevertheless, our orthogonal system—whose members are derivable as linear combinations of the c's and the s's by canonical Gram-Schmidt orthogonalization thereof—is complete on $L^q(0,1)$; $1 < q \in \mathbb{R}$ (*viz.* Corollary 3). The members of our system are denoted d's (and t's).

Definition 4. For $i \in \mathbb{N}$, let $\underline{i}$ denote its *odd part*: the quotient of i by its largest power-of-two factor.

Definition 5. $\quad d_0(x) = 1, \quad$ *and*

$$d_j(x) = \sum_{\ell \,|\, \underline{j}} (-1)^{(\ell-1)/2} \ell^{-1} \mu(\ell) c_{j/\ell}(x);\ j \in \mathbb{N}; \tag{3}$$

$$t_j(x) = \sum_{\ell \,|\, \underline{j}} \ell^{-1} \mu(\ell) s_{j/\ell}(x);\ j \in \mathbb{N}. \tag{4}$$

Here, μ denotes the classical Möbius function; $x \in \mathbb{R}$; $c_k(x)$ and $s_k(x)$ are defined in (1) and (2); and the summations range over the divisors of $\underline{j}$.

Definitions 4 and 5 prefigure the elementary number-theoretic character of the present investigations, in which Möbius inversion is used extensively [7, Thm. 10.3], [9]. Note from (4) that $t_{2^{\ell-1}}(x) = s_{2^{\ell-1}}(x) = r_\ell(x)$; $\ell \in \mathbb{N}$, so the t's also comprise the Rademacher system.

By the symmetries of the c's and s's, every d is plainly orthogonal to every t. Orthogonality of the remaining members of the new system is established as follows.

Definition 6. $J_2(\ell)$ denotes *Jordan's totient function* of index two [10]:

$$J_2(\ell) \overset{\text{def}}{=} \sum_{d|\ell} \mu(\ell/d) d^2 = \ell^2 \prod_{p|\ell} (1 - p^{-2}), \tag{5}$$

where the product ranges over the distinct prime factors of $\ell \in \mathbb{N}$ (and with $J_2(1) = 1$).

Theorem 1.

$$\langle d_j(x), d_k(x) \rangle = \left\{ \begin{array}{cc} \delta_{jk} & \textit{if } j \textit{ or } k = 0; \\ \delta_{jk} \dfrac{J_2(\underline{j})}{\underline{j}^2} & \textit{otherwise} \end{array} \right\}; \quad j, k \in 0 + \mathbb{N},$$

and

$$\langle t_j(x), t_k(x) \rangle = \delta_{jk} \frac{J_2(\underline{j})}{\underline{j}^2};\ j, k \in \mathbb{N},$$

where δ_{jk} denotes the Kronecker delta.

The two following lemmas facilitate proof of the theorem. Recall that for j and $k \in \mathbb{Z}$, (j,k) denotes their *greatest common divisor* and $[j,k]$ denotes their *least common multiple*.

Lemma 1. *For j and k in $\mathbb{N}$,*

$$\langle c_j(x), c_k(x)\rangle = \begin{cases} (-1)^{(\underline{j}+\underline{k})/2+1}\,(j,k)/[j,k] & \text{if } j/\underline{j} = k/\underline{k}; \\ 0 & \text{otherwise}, \end{cases} \tag{6}$$

and

$$\langle s_j(x), s_k(x)\rangle = \begin{cases} (j,k)/[j,k] & \text{if } j/\underline{j} = k/\underline{k}; \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

These integrals have previously escaped notice. To facilitate exposition, a proof of Lemma 1 is postponed to the final section.

Lemma 2. *For j and $k \in \mathbb{N}$,*

$$\textstyle\sum_{m|k} \mu(m)(j,k/m)^2 = \begin{cases} J_2(k) & k|j; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Using Möbius inversion and (5), the lemma is true if and only if

$$g^2 = \sum_{m|g} J_2(m) = \sum_{m|g} m^2 \sum_{\ell|m} \ell^{-2}\mu(\ell) = \sum_{\ell|g} \mu(\ell) \sum_{r|(g/\ell)} r^2;\ g \in \mathbb{N}.$$

Here, g denotes $(j,k) \in \mathbb{N}$. Möbius inversion of $g^2 = \sum_{\ell|g} \mu(\ell) \sum_{r|(g/\ell)} r^2$ yields an identity (for all $g \in \mathbb{N}$), establishing the foregoing identity and, hence, the lemma. $\square$

Proof of Theorem 1. For $j, k \in \mathbb{N}_1$, (7) establishes that $\langle t_{2^r j}(x), t_{2^s k}(x)\rangle = 0$ whenever $r \neq s$; $r, s \in 0 + \mathbb{N}$. To evaluate the remaining inner products of pairs of t's, from Lemma 2 it follows that

$$j \sum_{m|k} \frac{\mu(m)}{m} \frac{(j,k/m)}{[j,k/m]} = \sum_{\ell|j} \delta_{k\ell} \frac{J_2(\ell)}{\ell};\ k \in \mathbb{N}.$$

Möbius inversion yields

$$\sum_{\ell|j} \mu(\ell) \left\{ \frac{j}{\ell} \sum_{m|k} \frac{\mu(m)}{m} \frac{(j/\ell, k/m)}{[j/\ell, k/m]} \right\} = \delta_{jk} \frac{J_2(j)}{j};\ j \in \mathbb{N}.$$

Note that, from (4) and (7), for $j, k \in \mathbb{N}_1$, the left-hand side plainly equals $j\langle t_{2^s j}(x), t_{2^s k}(x)\rangle$; $s \in 0 + \mathbb{N}$, establishing the claims of the theorem pertaining to the orthogonality of t's and to the inner products of their squares.

The orthogonality of the d's follows from an analogous proof, which is omitted because this result also follows from the orthogonality of the t's and Claim 2, of the next section. $\square$

3. Step-function particulars

Because $j/\underline{j}$ equals the smallest index k for a $c_k(x)$ comprised by $d_j(x)$ and for a $s_k(x)$ comprised by $t_j(x)$; $j \in \mathbb{N}$, we have the following consequence of Definition 5.

Claim 1. $d_j(x)$ and $t_j(x)$ have (minimum) period $\underline{j}/j$, i.e.,

$$d_j(x) = d_j(x \pm \underline{j}/j) \text{ and } t_j(x) = t_j(x \pm \underline{j}/j);\ j \in \mathbb{N}.$$

Translations which interconvert s's and c's are readily found from

$$s_j(x) = (-1)^k c_j(x + (2k-1)/(4j));\ j \in \mathbb{N},\ k \in \mathbb{Z}. \tag{8}$$

We now establish an analogous relationship between $t_j(x)$ and $d_j(x)$; $j \in \mathbb{N}$.

Claim 2.

$$t_j(x) = (-1)^{(k+(\underline{j}-1)/2)} d_j(x + (2k-1)\underline{j}/(4j));\ j \in \mathbb{N}, k \in \mathbb{Z}.$$

Proof. For a function of period r to contain points of even and odd symmetries, these must plainly occur alternately, at intervals of length $r/4$. Here, $r = \underline{j}/j$. From (8),

$$c_{j/\ell}(x \pm \underline{j}/(4j)) = (-1)^{((\underline{j}/\ell)\pm 1)/2} s_{j/\ell}(x),$$

for $\ell | \underline{j}$. Substituting the foregoing identity into the c's appearing on the right-hand side of (3) yields (4) – using the following modulo-four congruence: each composite element of $\mathbb{N}_1$ is plainly congruent, modulo 4, to the sum of (a) any factor thereof, (b) the quotient of the element by the factor and (c) –1, establishing the claim for $k = 1$. The extension to all other values of k results from the symmetries of the d's and t's about $x = 1/2$. $\square$

Claim 3. All the steps of $d_j(x)$ and $t_j(x)$ have length properly equal $\frac{1}{2j}$; $j \in \mathbb{N}$, using periodicity to unite the leftmost and rightmost (half) steps of $d_j(x)$.

Proof. Here, "proper" implies that consecutive step heights differ, which, from Claim 2, is equivalent to $t_j(k/2j - \epsilon) \neq t_j(k/2j)$; $k = 1, 2, \ldots, 2j$ and $0 < \epsilon$, an infinitesimal.

Assume $t_j(k/2j - \epsilon) = t_j(k/2j)$, which, from (4), implies

$$\sum_{\ell\,|\underline{j}} \ell^{-1}\mu(\ell)(-1)^{\lfloor (k-\epsilon)/\ell \rfloor} = \sum_{\ell\,|\underline{j}} \ell^{-1}\mu(\ell)(-1)^{\lfloor k/\ell \rfloor};\ k \in \{1, 2, \ldots, 2j\}.$$

For sufficiently small ϵ, the only summands which differ are those whose indices satisfy $\ell\,|k$, and these differ by a factor of –1; thus, the foregoing equality yields

$$\sum_{\ell\,|n} \ell^{-1}\mu(\ell)(-1)^{k/\ell} = 0,$$

where n denotes $(\underline{j}, k)$. As all the ℓ's are odd, this implies $\sum_{\ell\,|n} \ell^{-1}\mu(\ell) = 0$, a contradiction because the latter sum equals $\phi(n)/n > 0$, with $\phi(n)$ denoting the Euler totient function [7, (10.9)]. $\square$

Möbius inversion of formulas readily derived from (4) yield the first instances of series expansions of functions in terms of our step functions.

Claim 4.

$$s_j(x) = \sum_{\ell | j} \frac{\ell}{j} \left\{ t_\ell(x) - \delta_{2|\ell} \frac{t_{\ell/2}(x)}{2} \right\},$$

where $\delta_{2|\ell}$ equals unity if ℓ is even and equals zero otherwise.

4. Completeness and a.e. convergence

Definition 7. Let the *formal Fourier-series expansion* of $f(x)$ equal

$$\alpha_0 + \sum_{j=1}^{\infty} \{\alpha_j \cos 2\pi jx + \beta_j \sin 2\pi jx\},$$

with

$$\alpha_0 \stackrel{\text{def}}{=} \langle f(x), 1\rangle; \alpha_j \stackrel{\text{def}}{=} 2\langle f(x), \cos 2\pi jx\rangle$$

and

$$\beta_j \stackrel{\text{def}}{=} 2\langle f(x), \sin 2\pi jx\rangle;\ j \in \mathbb{N}.$$

The α's and β's are called *Fourier coefficients*.

Definition 8. The *formal series expansion* of a function $f(x)$, in terms of our step functions, equals

$$a_0 + \sum_{j=1}^{\infty} \{a_j d_j(x) + b_j t_j(x)\}, \tag{9}$$

with

$$a_j \stackrel{\text{def}}{=} \langle f(x), d_j(x)\rangle / \langle d_j(x), d_j(x)\rangle;\ j \in 0 + \mathbb{N},$$

and

$$b_j \stackrel{\text{def}}{=} \langle f(x), t_j(x)\rangle / \langle t_j(x), t_j(x)\rangle;\ j \in \mathbb{N}.$$

Recall that Theorem 1 evaluates these denominators. Note that the inner products of (9)'s numerators are guaranteed to exist for any function in $L^q(0,1)$; $1 < q \in \mathbb{R}$; a proof may be based on the lemma that a function which is locally in L^q will be locally in L^r, provided that $1 \leq r \leq q$ [2]; here, $r = 1$. Convergence is defined as follows.

Definition 9. The *convergence* of a function's series expansion at a point, such as (9), is equivalent to the existence of a limit for the partial sums of the series (i.e. the limit of the sum of summands of index $\leq n$ as $n \to \infty$). Furthermore, the convergence of a function's series expansion over a domain connotes that it converges *pointwise* at all points thereof, etc.

Theorem 2 (Carleson-Hunt)**.** *The Fourier series of every function in* $L^q(0,1)$; $1 < q \in \mathbb{R}$, *converges pointwise to the respective member a.e. (almost everywhere) in* [0,1] [5, 6].

Corollary 1. *Every function whose expansion* (9) *converges a.e. to this function has a Fourier-series expansion which converges a.e. to this function.*

Proof. Consider the Fourier-series expansion of a partial sum of the first n terms of (9). Next consider a partial sum of the first n' terms of this Fourier-series expansion, and, for fixed n, let $n' \to \infty$. Because the respective d's and t's are evidently in, say, $L^2(0,1)$, these Fourier-series expansions converge a.e. to the partial sum of (9) (by Theorem 2).

Now let $n \to \infty$. By hypothesis, these partial sums converge a.e. to the expanded function, and the Fourier-series thereof will, therefore, converge a.e. to the expanded function; the union of the complements of the two domains of a.e. convergence plainly being of measure zero. Furthermore, because, the Fourier coefficients are the same for two functions which are equal to one another almost everywhere, the Fourier coefficients yielded by this double-limit procedure equal the Fourier coefficients of the function. □

Our main result on completeness of the new system requires the converse of Corollary 1, Corollary 2, which is established by means of the following lemmas and proposition, establishing the well-approximability of $\cos 2\pi kx$ and $\sin 2\pi kx$ by their respective expansions (9); $k \in \mathbb{N}$ (*viz.* Corollary 5).

Lemma 3. $\langle c_j(x), 1\rangle = \delta_{j0}$, *and, for* j *and* $k \in \mathbb{N}$,

$$\langle c_j(x), \cos 2\pi kx\rangle = \begin{cases} \frac{(-1)^{n-1}2}{(2n-1)\pi} & if\ k = (2n-1)j,\ with\ n \in \mathbb{N}; \\ 0 & otherwise; \end{cases}$$

$$\langle s_j(x), \sin 2\pi kx\rangle = \begin{cases} \frac{2}{(2n-1)\pi} & if\ k = (2n-1)j,\ with\ n \in \mathbb{N}; \\ 0 & otherwise. \end{cases}$$

The evaluation of these integrals is straightforward; proof is omitted.

Definition 10.

$$\sigma_k = \frac{1}{\underline{k}^2} \sum_{j \in \mathbb{N}_1} \frac{1}{J_2(j)} \left\{ \sum_{\ell | (j, \underline{k})} \ell^2 \mu(j/\ell) \right\}^2; \quad k \in \mathbb{N}. \tag{10}$$

Lemma 4. *For all* $k \in \mathbb{N}$,

$$\sigma_k = \frac{\pi^2}{8}. \tag{11}$$

Outline of Proof. As the left-hand side of (11) is a function only of $\underline{k}$ it will suffice to establish it for $k \in \mathbb{N}_1$. We begin by establishing

$$\sigma_1 = \sum_{j \in \mathbb{N}_1} \frac{\mu^2(j)}{J_2(j)} = \sum_{j \in \mathbb{N}_1} \frac{\mu^2(j)}{j^2 \prod_{p|j}(1-p^{-2})} = \sum_{j \in \mathbb{N}_1} \frac{1}{j^2} = \frac{\pi^2}{8}.$$

The rightmost equality was known to Euler. The penultimate equality, from the left, follows from the right-hand equality of (5). The penultimate equality, from the right, is established by noting $\mu^2(j)$ merely annuls summands for non-square-free j's. Consider, therefore, the following mapping of natural numbers onto the square-free natural numbers: $j \mapsto j'$, the latter denoting the product of the distinct prime factors of j, and consider the middle summation to range over the square-free, positive, odd integers. Expanding the factors $(1-p^2)^{-1}$ in geometric series $1 + p^2 + p^4 \cdots$, we recover the reciprocal of the square of every j mapping to a given j'. Summing over j', the reciprocal of the square of every $j \in \mathbb{N}_1$ is evidently recovered, establishing the identity.

The following identity, used subsequently in this proof, is, in fact, a specialization of (5).

$$J_2(p_1^{i_1} p_2^{i_2} \cdots p_n^{i_n} j) = p_1^{2i_1} p_2^{2i_2} \cdots p_n^{2i_n} \left(\prod_{i \in \{1,2,\dots,n\}:\ p_i \nmid j} (1 - p_i^{-2}) \right) J_2(j), \tag{12}$$

where $p_1, p_2, \ldots, p_n$ denote distinct primes and $j, i_1, \ldots, i_n \in \mathbb{N}$. As a corollary of (12), with p prime, $m \in \mathbb{N}_1$ and $(m,p) = 1$, we obtain the following identity, also used subsequently in this proof:

$$\sum_{j\in\mathbb{N}_1} \frac{1}{J_2(p^h j)} \left\{ \sum_{\ell|(j,m)} \ell^2 \mu(pj/\ell) \right\}^2 = \frac{m^2}{p^{2h}} \sigma_m. \tag{13}$$

We proceed to establish (11) for all k of the form p^i, with p an odd prime. It is easily established, for $i \in \mathbb{N}$, using (10), that

$$\sigma_{p^i} = \frac{1}{p^{2i}} \left\{ \sigma_1 + \sum_{j\in\mathbb{N}_1} \left(2p^2 \mu(pj)\mu(j) + p^4 \mu^2(j)\right) \sum_{h=1}^{i} \frac{p^{4h-4}}{J_2(p^h j)} \right\}.$$

Transforming this formula, using (12), (13)—for $m = 1$—and elementary factorizations of the Möbius function, yields the desired identity: $\sigma_{p^i} = \sigma_1 = \pi^2/8$.

Now, consider σ_{mp^i}, where p is an odd prime, where $m \in \mathbb{N}_1$, where $i \in \mathbb{N}$ and where (m,p)=1. By induction, we may assume that $\sigma_m = \pi^2/8$. Then, from (10),

$$\begin{aligned}\sigma_{mp^i} = \frac{1}{m^2 p^{2i}} \Bigg\{ m^2 \sigma_m + \sum_{j\in\mathbb{N}_1} \Bigg(&\left\{ \sum_{\ell|(pj,pm)} \ell^2 \mu(pj/l) \right\}^2 \\ &- \left\{ \sum_{\ell|(j,m)} \ell^2 \mu(pj/l) \right\}^2 \Bigg) \sum_{h=1}^{i} \frac{p^{4h-4}}{J_2(p^h j)} \Bigg\}.\end{aligned}$$

This equation yields $\sigma_{mp^i} = \sigma_m$ upon using (12), (13) and elementary factorizations of the Möbius function. Therefore, by induction on the number of distinct primes factoring k, $\sigma_k = \pi^2/8$ for all $k \in \mathbb{N}_1$ and, hence, for all $k \in \mathbb{N}$. □

Proposition 1. *Parseval's identity (the Lyapunov-Steklov closure condition) holds for the expansions* (9) *of* $\cos 2\pi kx$; $k \in 0 + \mathbb{N}$ *and of* $\sin 2\pi kx$; $k \in \mathbb{N}$.

Proof. By definition, Parseval's identity holds for the expansion (9) of a function whenever the corresponding Bessel's inequality is satisfied as an equality. For $k = 0$, Bessel's inequality reads

$$\sum_{j=0}^{\infty} \frac{\langle d_j(x), 1\rangle^2}{\langle d_j(x), d_j(x)\rangle} \le \int_0^1 dx = 1.$$

From Lemma 3, $\langle d_j(x), 1\rangle = \delta_{j0}$, which, by Theorem 1, establishes the inequality to be an equality in this case. For the functions $\cos 2\pi kx$ and $\sin 2\pi kx$; $k \in \mathbb{N}$, Bessel's inequality yields the following inequalities, using the normalization coefficients of Theorem 1.

$$\sum_{j=1}^{\infty} \frac{\underline{j}^2}{J_2(\underline{j})} \langle d_j(x), \cos 2\pi kx\rangle^2 \le \frac{1}{2} = \langle \cos 2\pi kx, \cos 2\pi kx\rangle;\ k \in \mathbb{N}, \tag{14}$$

and

$$\sum_{j=1}^{\infty} \frac{\underline{j}^2}{J_2(\underline{j})} \langle t_j(x), \sin 2\pi kx\rangle^2 \le \frac{1}{2} = \langle \sin 2\pi kx, \sin 2\pi kx\rangle;\ k \in \mathbb{N}\,. \tag{15}$$

To establish that these inequalities are satisfied as equalities, Lemmas 3 and 4 and Definitions 5 and 10 yield, after simplification,

$$\sum_{j=1}^{\infty}\frac{\underline{j}^2}{J_2(\underline{j})}\langle t_j(x),\sin 2\pi kx\rangle^2=\sum_{j=1}^{\infty}\frac{\underline{j}^2}{J_2(\underline{j})}\langle d_j(x),\cos 2\pi kx\rangle^2=\frac{4}{\pi^2}\sigma_k=\frac{1}{2};\ k\in\mathbb{N}.$$

□

Corollary 2. *Every function whose Fourier-series expansion converges a.e. to this function has an expansion* (9) *which converges a.e. to this function.*

Sketch of Proof. [For a complete proof, *viz.* Proof of Corollary 1, *mutatis mutandis*]. Consider the partial sums of the Fourier series of such a function. As a corollary of Proposition 1, the expansions (9) of such partial sums converge a.e. to the respective partial sum. Because these partial sums converge a.e. to the function, the expansion (9) of the given function converges a.e. to the function. □

Corollaries 1 and 2 yield the following theorem.

Theorem 3. *Every function has a Fourier-series expansion which converges a.e. to this function if and only if it has an expansion* (9) *which converges a.e. to this function.*

Corollary 3. *The system consisting of the d's and t's is complete for* $L^q(0,1)$; $1<q\in\mathbb{R}$.

Proof. Thus, from Theorem 2 and Corollary 2, the expansion (9) of every member of $L^q(0,1)$; $1<q\in\mathbb{R}$ converges a.e. to the respective member. It follows that the only functions in these classes of functions whose expansions (9) converge to zero a.e. are themselves equal zero a.e.; our definition of completeness (*cf.* Definition 2). □

As the members of the Paley-Walsh and Haar systems for [0,1] are in, say, $L^2(0,1)$ and, furthermore, as our step functions have a.e. convergent expansions in either of the respective series [8], we note the following consequence of Corollary 3.

Corollary 4. *One may substitute either "Paley-Walsh-series" or "Haar-series" for "Fourier-series" in the statement of Theorem 3.*

5. Example

Definition 11. Given that $k|\ell$,

$$\tilde{J}_2(k;\ell)\stackrel{\text{def}}{=}\sum_{d|k}d^2\mu(\ell/d).$$

Corollary 5.

$$\cos 2\pi kx=\frac{-2}{\pi\underline{k}}\sum_{j\in\mathbb{N}_1}\frac{(-1)^{(j+\underline{k})/2}j\tilde{J}_2((j,\underline{k});j)d_{jk/\underline{k}}(x)}{J_2(j)};\ k\in\mathbb{N}.$$

Also,

$$\sin 2\pi kx=\frac{2}{\pi\underline{k}}\sum_{j\in\mathbb{N}_1}\frac{j\tilde{J}_2((j,\underline{k});j)t_{jk/\underline{k}}(x)}{J_2(j)};\ k\in\mathbb{N}.$$

It may be noted, in connection with Definition 11, that $\tilde{J}_2(\ell;\ell) = J_2(\ell)$ (*cf.* (5)), and

$$\tilde{J}_2(k;\ell) = \left(\frac{k}{\breve{k}}\right)^2 \mu(\ell/k) J_2(\breve{k}),$$

where $\breve{k}$ denotes the quotient of k by all its prime-power factors, for the primes factoring $(k, \ell/k)$.

6. Proof of (7)

Proof of (7) is facilitated by the introduction of new step functions. The latter may be used to obtain (6) directly, or (7) may also be used to obtain (6) as an easy corollary.

Proof of (7). Consider the functions

$$f_j(x) \stackrel{\text{def}}{=} (-1)^{\lfloor jx \rfloor}, \quad x \in [0,1], \quad j \in \mathbb{N}.$$

Note that $f_j(x) = s_{2j}(x)$ is a step function alternating between 1 and -1 on the consecutive intervals $[0,1/j)$, $[1/j, 2/j)$,..., $[(j-1)/j, 1)$. Thus, it has period $2/j$; $j \in \mathbb{N}$. For each j, let $j = \underline{j}2^{\ell_j}$. Our proof will be completed by establishing that

$$\langle f_j(x), f_k(x) \rangle = \begin{cases} 0 & \ell_j \neq \ell_k; \\ (j,k)/[j,k] & \text{otherwise.} \end{cases} \tag{16}$$

To derive (16), note that, by definition, $\langle f_j(x), f_k(x)\rangle = \sum_{s=1}^{(j,k)} \int_{\frac{s-1}{(j,k)}}^{\frac{s}{(j,k)}} f_j(x) f_k(x) dx$. From their evident periodicities, $f_j(x + \frac{s}{(j,k)}) = \pm f_j(x)$ and $f_k(x+\frac{s}{(j,k)}) = \pm f_k(x)$ for all $j, k \in \mathbb{N}$. Thus,

$$\langle f_j(x), f_k(x) \rangle = c \int_0^{\frac{1}{(j,k)}} f_j(x) f_k(x) dx,$$

where c is a constant (which we will evaluate, for non-vanishing, right-hand integrals). Now,

$$\int_0^{\frac{1}{(j,k)}} f_j(x) f_k(x) dx = \int_{-\frac{1}{2(j,k)}}^{\frac{1}{2(j,k)}} f_j\left(x + \frac{1}{2(j,k)}\right) f_k\left(x + \frac{1}{2(j,k)}\right) dx. \tag{17}$$

If $\ell_j \neq \ell_k$, then either $\frac{j}{(j,k)}$ is even and $\frac{k}{(j,k)}$ is odd or vice versa. Consequently, about $x = 0$, either $f_j(x + \frac{1}{2(j,k)})$ is odd and $f_k(x + \frac{1}{2(j,k)})$ is even or vice versa. Hence, $f_j(x+\frac{1}{2(j,k)}) f_k(x + \frac{1}{2(j,k)})$ is odd on $[-\frac{1}{2(j,k)}, \frac{1}{2(j,k)}]$ and $\langle f_j(x), f_k(x)\rangle = 0$, establishing (16) for this case.

Next, suppose that $\ell_j = \ell_k$. Then both $\frac{j}{(j,k)}$ and $\frac{k}{(j,k)}$ are odd, which implies $f_j(x + \frac{s}{(j,k)}) = (-1)^s f_j(x)$ and $f_k(x + \frac{s}{(j,k)}) = (-1)^s f_k(x)$. Thus, from (17),

$$\langle f_j(x), f_k(x) \rangle = (j,k) \int_0^{\frac{1}{(j,k)}} f_j(x) f_k(x) dx. \tag{18}$$

Thus, our proof of (16), whence (7), will be completed by establishing

$$\int_0^{\frac{1}{(j,k)}} f_j(x) f_k(x) dx = \frac{1}{[j,k]}. \tag{19}$$

Let $s = j/(j,k)$, $t = k/(j,k)$ and $\xi = \frac{1}{st(j,k)} = \frac{1}{[j,k]}$. Without loss of generality, suppose $s > t$. Partition the interval $[0, \frac{1}{(j,k)}]$ by the points in the union of $S \stackrel{\text{def}}{=} \{s\xi, 2s\xi, \ldots, (t-1)s\xi\}$ with $T \stackrel{\text{def}}{=} \{t\xi, 2t\xi, \ldots, (s-1)t\xi\}$. Note, because $(s,t) = 1$, S and T are disjoint.

The product $f_j(x)f_k(x)$ changes sign as x traverses the points in $S \cup T$. To evaluate the left-hand side of (19), one may sum the integrals of $f_j(x)f_k(x)$ over the intervals, of $[0, 1/(j,k)]$, defined by the points of T: $[(i-1)t\xi, it\xi)$, $i = 1, 2, \ldots, s$. (As the integrands are finite, whether or not the right endpoint of these intervals is included in the integration is immaterial to the value of the integral of (19).) We dichotomize these intervals into (i) those including a point of S, in their proper interior, and (ii) the complement. (Each point of S plainly falls in the proper interior of the foregoing intervals). Considering the second type of intervals—in their native order—$f_j(x)f_k(x)$ alternates between 1 and -1 thereupon because, on each intervening interval of the first kind, $f_j(x)f_k(x)$ takes both signs. Therefore, the sum of the integrals of $f_j(x)f_k(x)$ on the intervals of the second type equals $t\xi$. Now, there remain $t-1$ intervals of the first type, each of which may be represented by the respective element of S. We find

$$is = h_i t + r_i;\ 0 \le r_i < t;\ i = 1, 2, \ldots, t-1. \tag{20}$$

The integral of $f_j(x)f_k(x)$ on the ith interval of the first type, denoted by I_i, plainly equals $\pm(2r_i - t)\xi$; with the respective sign equals $(-1)^{h_i - i + 1}$. However, by (20), $h_i - i$ and r_i have the same parity. So,

$$I_i = (-1)^{r_i+1}(2r_i - t)\xi;\ i = 1, 2, \ldots, t-1.$$

Because $(s,t) = 1$, $\{r_i\} = \{1, 2, \ldots, t-1\}$. Therefore,

$$\sum_{i=1}^{t-1} I_i = \sum_{i=1}^{t-1}(-1)^{i+1}(2i - t)\xi = (1-t)\xi.$$

Adding this result to $t\xi$, the sum of the integrals over the intervals of the second type, yields (19), and, hence, with recourse to (18), we also obtain (16) for this case. □

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