

## LATTICE POLYGONS AND GREEN'S THEOREM

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ABSTRACT. Associated to an  $n$ -dimensional integral convex polytope  $P$  is a toric variety  $X$  and divisor  $D$ , such that the integral points of  $P$  represent  $H^0(\mathcal{O}_X(D))$ . We study the free resolution of the homogeneous coordinate ring  $\bigoplus_{m \in \mathbb{Z}} H^0(mD)$  as a module over  $Sym(H^0(\mathcal{O}_X(D)))$ . It turns out that a simple application of Green's theorem yields good bounds for the linear syzygies of a projective toric surface. In particular, for a planar polytope  $P = H^0(\mathcal{O}_X(D))$ ,  $D$  satisfies Green's condition  $N_p$  if  $\partial P$  contains at least  $p + 3$  lattice points.

### 1. GREEN'S THEOREM AND HYPERPLANE SECTIONS

For a curve  $C$  of genus  $g$ , a divisor  $D$  of degree  $d \geq 2g + 1$  is very ample, so gives an embedding of  $C$  into projective space. In fact, when  $d \geq 2g + 1$ , work of Castelnuovo, Mattuck and Mumford shows that the embedding is *projectively normal*, which means that  $S = Sym(H^0(\mathcal{O}_X(D)))$  surjects onto  $\bigoplus_{m \in \mathbb{Z}} H^0(mD) = R$ . When  $d \geq 2g + 2$ , results of Fujita and St. Donat show that the homogeneous ideal of  $I_C$  is generated by quadrics. Let  $F_\bullet$  be a minimal free resolution of  $R$  over  $S$ . A very ample divisor is said to satisfy property  $N_p$  if  $F_0 = S$  and  $F_q \simeq \bigoplus S(-q - 1)$  for all  $q \in \{1, \dots, p\}$ . Thus,  $N_0$  means projectively normal,  $N_1$  means that the homogeneous ideal is generated by quadrics,  $N_2$  means that the minimal syzygies on the quadrics are linear, and so on. In [7], Green used Koszul cohomology to give a beautiful generalization of the classical results above: if  $\deg(D) \geq 2g + p + 1$ , then  $D$  satisfies  $N_p$ .

In this brief note, we investigate the  $N_p$  property for toric varieties. For any divisor  $D$  and variety  $X$  such that  $R$  is arithmetically Cohen-Macaulay, it is natural to slice with hyperplanes until  $X$  has been reduced to a curve, and then apply Green's theorem. Results of Hochster [8] show that projectively normal toric varieties are always arithmetically Cohen-Macaulay. So it makes sense to apply the technique in this setting. In [4], Ewald and Wessels prove that if  $D$  is an ample divisor on a toric variety of dimension  $n$ , then  $(n - 1)D$  is very ample and satisfies  $N_0$ . Bruns, Gubeladze and Trung [2] give another proof and also show that  $nD$  satisfies property  $N_1$ . While it is often difficult to determine if a given divisor satisfies  $N_0$ , for a lattice polygon  $P$  and corresponding divisor on a toric surface, the property  $N_0$  holds "for free".

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In [6] Gallego and Purnaprajna give criteria for the  $N_p$  property for smooth rational surfaces. Toric varieties are rational, and in the case of smooth surfaces the result we obtain is a toric restatement of the result in [6]. However, the proof is simpler in the toric case, applies to singular surfaces, and extends several results in the toric literature. For example, in [10] Koelman proves that a toric surface defined by  $P$  satisfies  $N_1$  iff  $\partial P$  contains at least four lattice points, and Ewald and Schmeink [3] prove that certain polytopes associated to smooth toric varieties with  $\text{Pic}(X) = 2$  satisfy  $N_1$ .

**Theorem 1.1.** *Let  $P$  be an  $n$ -dimensional lattice polytope, and  $X, D$  the associated projective toric variety and ample divisor; so  $P = H^0(\mathcal{O}_X(D))$ . If  $D$  satisfies  $N_0$ , then  $D$  satisfies  $N_p$  if  $P$  satisfies*

$$\sum_{\text{facets } F_i} \text{vol}(F_i) \geq n(n-2)\text{vol}(P) + \frac{p+3}{(n-1)!}.$$

*Proof.* Hochster's results mentioned earlier show that  $R$  is arithmetically Cohen-Macaulay. In [9], Khovanskii shows that a toric variety  $X$  defined by a lattice polytope  $P$  is *normal* iff the Hilbert polynomial of  $X$  and the Ehrhart polynomial of  $P$  agree. Projective normality implies normality, and so  $X$  is normal. Hence, the singular locus of  $X$  is of codimension at least two. So a general member of  $|D|$  is smooth. Slicing with  $n-1$  general hyperplanes, we obtain a smooth curve  $C$  with the same minimal free resolution as  $X$ . By Khovanskii's result,

$$\chi(\mathcal{O}_X(mD)) = |mP \cap \mathbb{Z}^n| = am^n + bm^{n-1} + \dots.$$

After slicing with  $n-1$  general hyperplanes, the resulting curve  $C$  has

$$\chi(\mathcal{O}_C(m)) = n!am + (n-1)!b - (n-1)! \binom{n}{2} a.$$

The first two coefficients of the Ehrhart polynomial are

$$\begin{aligned} a &= \text{vol}(P), \\ b &= \frac{1}{2} \sum_{\text{facets } F_i} \text{vol}(F_i). \end{aligned}$$

Thus, applying Green's theorem, the divisor  $D$  associated to  $P$  satisfies  $N_p$  if

$$\sum_{\text{facets } F_i} \text{vol}(F_i) \geq n(n-2)\text{vol}(P) + \frac{p+3}{(n-1)!}.$$

□

## 2. APPLICATIONS

In [12], Wills shows that an  $n$ -dimensional lattice polytope  $P$  that contains an interior point satisfies  $n \cdot \text{vol}(P) \geq \sum_{\text{facets}} \text{vol}(F_i)$ . So at first glance the bound above seems useless. However, when  $n=2$  the term  $n(n-2)\text{vol}(P)$  vanishes, and by [4] the divisor associated to a lattice polygon  $P$  satisfies  $N_0$ . So we obtain:

**Corollary 2.1.** *The divisor  $D$  associated to a lattice polygon  $P$  satisfies  $N_p$  if*  

$$\# \text{ integral points in } \partial P \geq p+3.$$

**Example 2.2.** If  $P$  is the unit lattice two-simplex, then  $dP$  defines the  $d$ -uple Veronese embedding of  $\mathbb{P}^2$ . By Corollary 2.1,  $dP$  satisfies  $N_p$  if  $p \leq 3d-3$ , recovering a result of [1]. In fact, Ottaviani and Paoletti [11] show that this bound is tight.

**Example 2.3.** The ideal sheaf of a projective toric surface  $X$  is two-regular iff  $N_p$  holds for all  $p \leq \text{codim}(X)$ . By Corollary 2.1, this is true if  $P$  has no interior points. In this case  $R$  is level with  $a$ -invariant  $-2$ , which gives half of Theorem 1.27 of [2]. If  $P$  has no interior points, then the corresponding divisor has arithmetic genus zero ([5], p. 91). Thus  $X$  is a surface of minimal degree. So if  $X$  is smooth, then it must be a rational normal scroll or the Veronese surface in  $\mathbb{P}^5$ .

If  $P$  is three-dimensional, then  $P$  satisfies  $N_p$  if  $2 \sum \text{vol}(F_i) - 6\text{vol}(P) - 3 \geq p$  and  $N_0$  holds. In order to obtain a useful bound, we require that  $P$  have no interior points, so that the Ehrhart polynomial evaluated at  $-1$  is zero. For such a polytope, this implies that  $\sum \text{vol}(F_i) = \#$  integral points in  $P - 2$ , which yields:

**Corollary 2.4.** *A lattice three-polytope  $P$  with no interior points satisfies  $N_p$  if  $D$  is projectively normal and  $\#$  integral points in  $P \geq 3\text{vol}(P) + \frac{p+7}{2}$ .*

**Example 2.5.** Polytopes corresponding to smooth torics with  $\text{Pic}(X) = 2$  are studied in [3]; for threefolds there are only two families. Ewald and Schmei\nc show that the polytopes below satisfy  $N_1$ :

$$\begin{aligned} P_1(a) &= \text{conv}\{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + (a+1)\mathbf{e}_3, \mathbf{e}_1 + (a+1)\mathbf{e}_2\}, \\ P_2(a, b) &= \text{conv}\{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + (a+1)\mathbf{e}_3, \mathbf{e}_2 + (b+1)\mathbf{e}_3\}. \end{aligned}$$

A calculation shows that

$$\begin{aligned} \text{vol}(P_1(a)) &= \frac{a^2+3a+3}{6}, \quad \# \text{ integral points in } P = \frac{a^2+5a+12}{2}, \\ \text{vol}(P_2(a, b)) &= \frac{a+b+3}{6}, \quad \# \text{ integral points in } P = a + b + 6. \end{aligned}$$

Thus,  $P_1(a)$  satisfies  $N_p$  if  $p \leq 2a + 2$ , and  $P_2(a, b)$  satisfies  $N_p$  if  $p \leq a + b + 2$ .

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