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CHARACTERIZATIONS OF ELEMENTS WITH COMPACT SUPPORT IN THE DUAL SPACES OF $A_p(G)$-MODULES OF $PM_p(G)$

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ABSTRACT. For a locally compact group G and $1 < p < \infty$, let $A_p(G)$ be the Figà-Talamanca-Herz algebra and let $PM_p(G)$ be its dual Banach space. For a Banach $A_p(G)$-module X of $PM_p(G)$, we denote the norm closure of the subspace of the elements in X^* with compact support by $A_{p,X}(G)$. We prove that an element u of X^* is in $A_{p,X}(G)$ if and only if for any $\epsilon > 0$, there exists a compact subset K of G such that $|\langle u, f\rangle| < \epsilon$ for all $f \in X$ with $\|f\| \leq 1$ and *supp*$(f) \subseteq G \sim K$. In particular, we have that an element b of $W_p(G)$ is in $A_p(G)$ if and only if for any $\epsilon > 0$, there exists a compact subset K of G such that $|\langle u, f\rangle| < \epsilon$ for all $f \in L^1(G \sim K)$ with $\|f\| \leq 1$. If $A_{p,X}(G)$ has an orthogonal complement $A^s_{p,X}(G)$ in X^*, we characterize $A^s_{p,X}(G)$ by the following condition: $u \in X^*$ is in $A^s_{p,X}(G)$ if and only if for any $\epsilon > 0$ and any compact subset K of G, there exists some $f \in X$ with $\|f\| \leq 1$ and *supp*$(f) \subseteq G \sim K$ such that $|\langle u, f\rangle| > \|u\| - \epsilon$. Some results of Flory (1971) and Miao (1999) can be obtained from our main theorems by taking $p = 2$ and X as some C^*-subalgebras of $PM_p(G)$.

1. Introduction and preliminaries

For any locally compact group G equipped with a fixed left Haar measure λ, let $L^p(G)$, $1 \leq p \leq \infty$, be the usual Lebesgue spaces on G with norm $\|\cdot\|_p$. If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, let $A_p(G)$ be the Figà-Talamanca-Herz algebra, i.e., the space of continuous functions u that can be represented as

$$u = \sum_{n=1}^{\infty} f_i * \check{g}_i \quad \text{with} \quad f_i \in L^q(G), g_i \in L^p(G), \quad \text{and} \quad \sum_{n=1}^{\infty} \|f_i\|_q \|g_i\|_p < \infty,$$

where $\check{g} \in L^p(G)$ is defined by $\check{g}(x) = g(x^{-1})$, $x \in G$. The norm of u is defined by

$$\|u\|_{A_p(G)} = \inf \sum_{n=1}^{\infty} \|f_i\|_q \|g_i\|_p,$$

where the infimum is taken over all the representations of u above. It is known that $A_p(G)$ is a subspace of $C_0(G)$ and, equipped with the norm $\|\cdot\|_{A_p(G)}$ above and the

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pointwise multiplication, is a regular Tauberian algebra whose Gelfand spectrum is G. Furthermore, the algebra $A_p(G)$ has a bounded approximate identity if and only if the group G is amenable (see Herz [8], Theorem 6). For $p = 2$, $A_p(G) = A(G)$, the Fourier algebra of G (see Eymard [4]).

Each element f of $L^1(G)$ defines a bounded functional on $A_p(G)$ by

$$\langle f, u\rangle = \int_G f(x)u(x)\,dx.$$

It follows that $L^1(G)$ can be considered as a subspace of $A_p(G)^*$. By definition, $PF_p(G)$ and $PM_p(G)$ are the closures of $L^1(G)$ in $A_p(G)^*$ with respect to the norm and weak* topologies of $A_p(G)^*$. As in Herz [8], $PM_p(G)$ is the smallest ultraweakly closed subspace of $CONV_p(G)$ containing the left translations, where $CONV_p(G)$ is the subspace of the bounded linear operators on $L^p(G)$ that commute with the right translations. Then the dual of $A_p(G)$ is $PM_p(G)$, and $PF_p(G)^*$ is a Banach algebra such that $A_p(G)$ is dense in the associated w^*-topology. We denote $PF_p(G)^*$ by $W_p(G)$ as in [12]. For the properties of $PM_p(G)$ and $PF_p(G)$, see Pier [12]. Let $C^*(G)$ be the group C^*-algebra. Then its dual space is the Fourier-Stieltjes algebra $B(G)$. If $p = 2$, $W_p(G)$ is a closed subalgebra of $B(G)$, $PF_p(G) = C^*_\rho(G)$ is the reduced group C^*-algebra and $PM_p(G) = VN(G)$ is the group von Neumann algebra.

It is from the general theory of group representations that the Fourier-Stieltjes algebra $B(G)$ has a Lebesgue-type decomposition $B(G) = A(G) \oplus B^s(G)$ as a direct sum of $A(G)$ and a subspace $B^s(G)$ of $B(G)$ (see Arsac [2]). A Lebesgue-type description of $A(G)$ and $B^s(G)$ was given by Flory [5] and [6] for amenable G. In Miao [10], analogous characterizations of $A(G)$ and $B^s(G)$ were established by viewing $B(G)$ as the dual of $C^*(G)$, the group C^*-algebra. Kaniuth, Lau and Schlichting [9] generalized these results to a general representation of G. Similar problems such as when an element $u \in B(G)$ is in $A(G)$ are also considered by Akemann and Walter [1] and Granirer [7].

Let X be a Banach $A_p(G)$-module of $PM_p(G)$ (see the definition below). The main purpose of this paper is to give the same descriptions for the norm closed subspace $A_{p,X}(G)$ containing all elements in X^* with compact support and its orthogonal complement $A^s_{p,X}(G)$ in X^* if there is any. This generalizes the main results in Miao [10] to an arbitrary p. If we take $p = 2$ and $X = C^*_\delta(G)$, the C^*-subalgebras of the group von Neumann algebra $VN(G)$ generated by the point measures $\delta_x,\ x \in G$, then Flory's theorem in [5] can be derived from our main theorems.

For a Banach space X, let X^* be the conjugate Banach space of X. For $x \in X$ and $f \in X^*$, the value of f at $x, f(x)$, is sometimes denoted by $\langle f, x\rangle$ or $\langle x, f\rangle$ in duality. For a subset E of G, let 1_E denote the characteristic function of E. If u is a function on G, the support of G in the usual sense, i.e., the closure of $\{x \in G : u(x) \neq 0\}$ is denoted by $supp_G\, u$.

For $a \in A_p(G)$ and $f \in PM_p(G)$, then $af \in PM_p(G)$ is defined by $\langle af, u\rangle = \langle f, au\rangle$ for $u \in A_p(G)$. We call a closed subspace X of $PM_p(G)$ a Banach $A_p(G)$-module if $af \in X$ for $a \in A_p(G)$ and $f \in X$. Similarly, we can define a Banach $A(G)$-module of $B(G)^*$.

The following definition is due to Herz [8].

Definition 1.1. Let $f \in PM_p(G)$ ($B(G)^*$, respectively). The **support** of f is the subset *supp* (f) of G defined by $x \notin supp\,(f)$ if and only if there exists a neighborhood U of x in G such that $\langle u, f\rangle = 0$ for all $u \in A_p(G)$ ($A(G)$, respectively) with $supp_G\, u \subseteq U$.

Remarks 1.2. (a) As in Herz [8] (page 101), for any f in $PM_p(G)$ or $B(G)^*$, we have that $supp\, f$ is the smallest closed subset $E \subseteq G$ such that $f \perp J_E$, where J_E is the set of $u \in A_p(G)$ whose support $supp_G\, u$ is compact and disjoint from E by the existence of suitable partitions of unity (see Eymard [4], page 222).

(b) Let $f \in PM_p(G)$ and $u \in A_p(G)$ ($f \in B(G)^*$ and $u \in A(G)$, respectively). If $supp_G\, u$, denoted by K, is compact and $supp\, f \subseteq G \sim K$, then $\langle uf, a\rangle = \langle f, ua\rangle = 0$ for all $a \in A_p(G)$ ($B(G)$, respectively) since $supp_G\, ua$ is compact and disjoint from $supp\, f$. Thus, $uf = 0$ as an element of $PM_p(G)$ ($B(G)^*$, respectively).

(c) If $f \in B(G)^*$ such that $f = 0$ on $A(G)$, then $supp\, f = \emptyset$.

The following proposition is about the relation between the "support" of a function f in $L^1(G)$ and $supp\, f$ defined above if f is considered as either in $PM_p(G)$ or $B(G)^*$.

Proposition 1.3. *Let E be a closed subset of G and $f \in L^1(G)$. Then $f(x) = 0$ a.e. on $G \sim E$ if and only if $supp\, f \subseteq E$, where f is considered either as an element of $PM_p(G)$ or $B(G)^*$.*

Proof. This follows from the regularity of $A_p(G)$. □

Definition 1.4. Let X be a Banach $A_p(G)$-module of $PM_p(G)$ ($A(G)$-module of $B(G)^*$, respectively). We say that the *support* of an element $u \in X^*$ is contained in a compact subset K of G if $\langle u, f\rangle = 0$ for all $f \in X$ with $supp\,(f) \subseteq G \sim K$. In this case, we denote $supp\, u \subseteq K$. Let $A_{p,X}(G)$ denote the closed subspace of X^* generated by all elements in X^* whose supports are contained in some compact subset.

Remarks 1.5. (a). Let $u \in A_p(G)$ and $supp_G\, u \subseteq K$ for some compact subset K of G. If u_X denotes the restriction of u onto X, then $supp\, u_X \subseteq K$ by definition. Hence, $A_p(G) \subseteq A_{p,X}(G)$; that is, the restriction of an element in $A_p(G)$ to X belongs to $A_{p,X}(G)$. Furthermore, if $w \in X^*$, then $uw \in X^*$ is defined by $\langle uw, f\rangle = \langle w, uf\rangle$ for $f \in X$ and $supp\,(uw) \subseteq K$. In fact, if $f \in X$ with $supp\,(f) \subseteq G \sim K$, then $uf = 0$ as an element of X (see Remarks 1.2 (b)). Thus, $\langle uw, f\rangle = \langle w, uf\rangle = 0$.

(b). Let $X = PF_p(G)$. Then $X^* = W_p(G)$ and $A_p(G) \subseteq A_{p,X}(G)$ as above. Conversely, let $u \in A_{p,X}(G)$ and $supp\, u \subseteq K$ for some compact subset K of G. Then $supp_G\, u \subseteq K$. In fact, for every compact subset E of $G \sim K$, we have $1_E \in L^1(G)$. By Proposition 1.3, $supp\, 1_E \subseteq E \subseteq G \sim K$. Hence $\langle u, 1_E\rangle = 0$. Since E is an arbitrary compact subset of $G \sim K$ and u is continuous, we have $u = 0$ on $G \sim K$. Hence $supp_G\, u \subseteq K$ and $u \in A_p(G)$. Therefore, $A_p(G) = A_{p,X}(G)$.

(c). Let X be the $A(G)$-module of $B(G)^*$ consisting of all $f \in B(G)^*$ such that $f = 0$ on $A(G)$. Then it is clear that $A_{2,X}(G) = \{0\}$.

2. Main results

The motivation of the following theorem is from the main results in Miao [10] where the case of $p = 2$ and $X = C^*_\rho(G)$ or $C^*(G)$ are considered. The technique of the proof is also an improvement of the proof for the main theorem in Miao [10].

Theorem 2.1. *Let X be a Banach $A_p(G)$-module of $PM_p(G)$ ($A(G)$-module of $B(G)^*$, respectively) and $u \in X^*$. Then $u \in A_{p,X}(G)$ if and only if for any $\epsilon > 0$, there exists a compact subset K of G such that $|\langle u, f\rangle| < \epsilon$ for all $f \in X$ with $\|f\| \leq 1$ and $supp\,(f) \subseteq G \sim K$ (condition $(\triangle)$).*

Proof. We prove the theorem for the case of an $A_p(G)$-module of $PM_p(G)$ only. The proof for the other case is simply a modification of this proof. Let $u \in A_{p,X}(G)$ and $\epsilon > 0$. Then there exists a $v \in X^*$ such that $\|u - v\| < \epsilon$ and $supp\, v \subseteq K$ for some compact subset K of G. For any $f \in X$ with $\|f\| \leq 1$ and $supp\,(f) \subseteq G \sim K$, we have $\langle v, f\rangle = 0$. Hence,

$$|\langle u, f\rangle| = |\langle u - v, f\rangle| \leq \|u - v\| < \epsilon.$$

Conversely, let $u \in X^*$ satisfy the condition $(\triangle)$ and $\epsilon > 0$. We assume that $\|u\| = 1$ without loss of generality. If $u \notin A_{p,X}(G)$, then it follows from the Hahn-Banach theorem that there exists $F \in X^{**}$ such that $\|F\| = 1$, $\langle F, u\rangle = \eta > 0$ and $\langle F, a\rangle = 0$ for all $a \in A_{p,X}(G)$. By applying the Goldstine theorem, we obtain a net f_α in X such that $\|f_\alpha\| \leq 1$ and $f_\alpha \to F$ in the $\sigma(X^{**}, X^*)$-topology. Hence

$$\langle f_\alpha, u\rangle \to \langle F, u\rangle = \eta.$$

We assume that $\langle f_\alpha, u\rangle$ is real and that $\langle f_\alpha, u\rangle > \eta - \epsilon$ for all α without loss of generality. Let K be a compact subset of G satisfying the condition $(\triangle)$. We can choose an open subset U of G such that $K \subseteq U$, and the closure of U is compact since G is a locally compact group. Then there is an $a_K \in A_p(G)$ such that $a_K = 1$ on U and $supp_G\,(a_K)$ is compact (see Pier [12]). Since X is a Banach $A_p(G)$-module and $a_K \in A_p(G)$, we have $a_K f_\alpha \in X$ for all α. For every $w \in X^*$, since $a_K w \in A_{p,X}(G)$ by Remarks 1.5 (a), we have

$$\langle a_K f_\alpha, w\rangle = \langle f_\alpha, a_K w\rangle \to \langle F, a_K w\rangle = 0.$$

Thus, $a_K f_\alpha \to 0$ weakly in X. Then there exist positive numbers $\beta_1, \beta_2, \ldots, \beta_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\sum_{i=1}^n \beta_i = 1$ and $\|\sum_{i=1}^n \beta_i a_K f_{\alpha_i}\| < \epsilon$. Let

$$f = \sum_{i=1}^n \beta_i f_{\alpha_i} - a_K \sum_{i=1}^n \beta_i f_{\alpha_i}.$$

Then f satisfies the following conditions:

(i) $\|f\| \leq 1 + \epsilon$;

(ii) $|\langle f, u\rangle| > \eta - 2\epsilon$ since $\langle \sum_{i=1}^n \beta_i f_{\alpha_i}, u\rangle > \eta - \epsilon$ by assumption and

$$|\langle f, u\rangle| > \eta - \epsilon - |\langle u, \sum_{i=1}^n \beta_i a_K f_{\alpha_i}\rangle| \geq \eta - \epsilon - \|u\|\,\|\sum_{i=1}^n \beta_i a_K f_{\alpha_i}\|;$$

(iii) $supp\,(f)$ is contained in $G \sim K$. In fact, let $x \notin G \sim K$. Then $x \in U$. For any $a \in A_p(G)$ with $supp_G\, a \subseteq U$, since $a a_K = a$, we have

$$\langle a, f\rangle = \langle a, \sum_{i=1}^n \beta_i f_{\alpha_i} - a_K \sum_{i=1}^n \beta_i f_{\alpha_i}\rangle = 0.$$

Hence $x \notin supp\, f$. Thus, $|\langle f, u\rangle| = |\langle \frac{f}{\|f\|}, u\rangle|\,\|f\| < \epsilon(1 + \epsilon)$ by $(\triangle)$ and (i). It follows from (ii) that $\eta < 2\epsilon + \epsilon(1 + \epsilon)$. Since ϵ is arbitrary, $\eta = 0$. This is a contradiction. □

Remark 2.2. Let Y be a norm dense subspace of X such that $af \in Y$ for $a \in A_p(G)$ and $f \in Y$. It is obvious from the proof of the theorem that the condition "for all $f \in X$" can be replaced by "for all $f \in Y$".

Theorem 2.3. *Let $u \in W_p(G)$. Then $u \in A_p(G)$ if and only if for any $\epsilon > 0$, there exists a compact subset K of G such that $|\langle u, f\rangle| < \epsilon$ for all $f \in L^1(G \sim K)$ with $\|f\| \leq 1$ (condition $(\triangle^*)$).*

Proof. Let $X = PF_p(G)$ in Theorem 2.1. Then $X^* = W_p(G)$ and $A_{p,X}(G) = A_p(G)$ (see Remarks 1.5 (b)). The necessary condition of this theorem is proved in Proposition 6.1 of Miao [10].

Conversely, let u satisfy $(\triangle^*)$. For any $\epsilon > 0$, there exists a compact subset K of G such that $|\langle u, f\rangle| < \epsilon$ for all $f \in L^1(G \sim K)$ with $\|f\| \leq 1$. Since G is locally compact, there is an open subset U of G such that $K \subseteq U$ and its closure $\bar{U}$ is compact. For any $f \in L^1(G)$ with $supp\, f \subseteq G \sim U$ and $\|f\| \leq 1$, it follows from Proposition 1.3 that $f = 0$ a.e. on U. Thus, $f \in L^1(G \sim K)$. Hence $|\langle u, f\rangle| < \epsilon$ by $(\triangle^*)$. If we apply Theorem 2.1 (see Remark 2.2), we have $u \in A_p(G)$. □

Remark 2.4. If G is amenable, then $W_p(G)$ is the multiplier algebra of $A_p(G)$ as in Miao [10]. Our Theorem 2.3 generalizes Corollary 6.2 in Miao [10]. When $p = 2$, we have that $W_p(G) = B_\rho(G)$, the dual space of the reduced C^*-algebra $C^*_\rho(G)$. We obtain the characterization of $A(G)$ within $B_\rho(G)$ as in Miao [10].

Theorem 2.5. *Let X be a Banach $A_p(G)$-module of $PM_p(G)$ ($A(G)$-module of $B(G)^*$, respectively). If $A_{p,X}(G)$ has an orthogonal complement $A^s_{p,X}(G)$ in X^* and $u \in X^*$, then u is an element of $A^s_{p,X}(G)$ if and only if for any $\epsilon > 0$ and any compact subset K of G, there exists some $f \in X$ with $\|f\| \leq 1$ and $supp\,(f) \subseteq G \sim K$ such that $|\langle u, f\rangle| > \|u\| - \epsilon$ (condition $(\triangle\triangle)$).*

Proof. Again, we prove the case of an $A_p(G)$-module of $PM_p(G)$ only. Suppose that $u \in X^*$ satisfies $(\triangle\triangle)$. Then $u = u_c + u_s$ with $\|u\| = \|u_c\| + \|u_s\|$ for some $u_c \in A_{p,X}(G)$ and $u_s \in A^s_{p,X}(G)$ by assumption. For any $\epsilon > 0$, there exists a compact subset K of G such that $|\langle u_c, f\rangle| < \epsilon$ for any $f \in X$ with $supp\,(f) \subseteq G \sim K$ by Theorem 2.1. There exists an $f \in X$ with $\|f\| \leq 1$ and $supp\,(f) \subseteq G \sim K$ such that $|\langle u, f\rangle| > \|u\| - \epsilon$ by applying $(\triangle\triangle)$. Hence,

$$\|u_s\| \geq |\langle u_s, f\rangle| = |\langle u - u_c, f\rangle| = |\langle u, f\rangle - \langle u_c, f\rangle| \geq \|u\| - 2\epsilon.$$

Since $\|u\| = \|u_c\| + \|u_s\|$ and ϵ is arbitrary, $\|u_c\| = 0$. So $u = u_s$ is in $A^s_{p,X}(G)$.

Conversely, let $u \in A^s_{p,X}(G)$ and $\|u\| = 1$. Then $u \notin A_{p,X}(G)$. For any $\epsilon > 0$ and compact subset K of G, by following the proof of Theorem 2.1 with $\eta = 1$, we have an $f \in X$ with $\|f\| \leq 1 + \epsilon$ such that $supp\,(f) \subseteq G \sim K$ and $|\langle f, u\rangle| > 1 - 2\epsilon$, which are (i), (ii) and (iii) in the proof of Theorem 2.1. Hence, $(\triangle\triangle)$ is satisfied. □

Remark 2.6. Again as in Remark 2.2, if Y is a norm dense subspace of X such that $af \in Y$ for $a \in A_p(G)$ and $f \in Y$, then the condition "there exists some $f \in X$" in the theorem can be replaced by "there exists some $f \in Y$". We do not know if $A_p(G)$ has an orthogonal complement in $W_p(G)$ or not.

3. The case of $p = 2$

We assume $p = 2$. The Fourier-Stieltjes algebra $B(G)$ is the dual Banach space of the group C^*-algebra $C^*(G)$, and $C^*(G)$ is a Banach $A(G)$-module of $B(G)^*$.

In this section we will apply our main theorems from section 2 to obtain and to generalize some of the early results in Flory [5] and Miao [10]. As in Miao [10], there is a subspace $B^s(G)$ of $B(G)$ such that $B(G) = A(G) \oplus B^s(G)$ as a direct sum of $A(G)$ and $B^s(G)$. If we take $X = C^*(G)$ in Theorem 2.1 and Theorem 2.5 together with Proposition 1.3, Remark 2.2 and Remark 2.6, we obtain the following theorem in Miao [10].

Theorem 3.1 (Miao)**.** *Let $u \in B(G)$. Then*

(i) *$u \in A(G)$ if and only if for any $\epsilon > 0$, there exists a compact subset K of G such that $|\langle u, f\rangle| < \epsilon$ for all $f \in L^1(G \sim K)$ with $\|f\| \leq 1$;*

(ii) *$u \in B^s(G)$ if and only if for any $\epsilon > 0$ and compact subset K of G, there exists $f \in L^1(G \sim K)$ with $\|f\| \leq 1$ such that $|\langle u, f\rangle| > \|u\| - \epsilon$.*

Let $X = C^*_\delta(G)$ be the C^*-algebra generated by the point measures δ_x in $VN(G)$ for $x \in G$. Then $C^*_\delta(G)$ is a Banach $A(G)$-module of $VN(G)$, and $C^*_\delta(G)^*$ is a closed subspace of $B(G_d)$, where G_d is the group G equipped with the discrete topology. We denote the closed subalgebra of $C^*_\delta(G)^*$ generated by the elements with compact support in the normal sense by $A_{\delta,c}(G)$. Then $A_{\delta,c}(G)$ is a closed subspace of $B(G_d)$. Note that $A_{\delta,c}(G)$ is also two-sided translation invariant under the action of G. Hence, by Théorème 3.18 in Arsac [2], there is a subspace $A_{\delta,s}(G)$ of $C^*_\delta(G)^*$ such that

$$C^*_\delta(G)^* = A_{\delta,c}(G) \oplus A_{\delta,s}(G).$$

In the following, we show that the support of an element either in $C^*_\delta(G)$ or in $C^*_\delta(G)^*$ defined in section 1 and the support of the element in the usual sense are the same.

Proposition 3.2. (i) *If $f = \sum_{i=1}^n \alpha_i \delta_{x_i} \in VN(G)$ for nonzero $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $x_1, x_2, \ldots, x_n$ in G, then $supp\, f = \{x_1, x_2, \ldots, x_n\}$, where $supp\, f$ is defined in Definition* 1.1.

(ii) *$A_{\delta,c}(G) = A_{2,X}(G)$, where $X = C^*_\delta(G)$ and $A_{2,X}(G)$ is as in Definition* 1.4.

Proof. (i) This follows from the regularity of $A_p(G)$.

(ii) Let $u \in A_{\delta,c}(G)$. Suppose the support of u is contained in a compact subset K, i.e., $u(x) = 0$ if $x \notin K$. Take an open set U such that $K \subseteq U$ and its closure $\bar{U}$ is compact. Extend u to an element, denoted by u^*, of $VN(G)^*$. By the Goldstine theorem, there is a net $\{a_\alpha\}$ is $A(G)$ such that $a_\alpha \to u^*$ in the $\sigma(VN(G)^*, VN(G))$ topology. Take $a \in A(G)$ with $a = 1$ on K and $a = 0$ on $G \sim U$ (see Eymard [4], Lemme 3.2). For every $f \in C^*_\delta(G)$ with $supp\, f \subseteq G \sim \bar{U}$, since $au = u$, we have

$$\langle f, aa_\alpha\rangle = \langle af, a_\alpha\rangle \to \langle af, u^*\rangle = \langle af, u\rangle = \langle f, u\rangle.$$

On the other hand, since $supp_G\,(aa_\alpha) \subseteq U$, we have $\langle f, aa_\alpha\rangle = 0$ for all α (see Remarks 1.2). So $\langle f, u\rangle = 0$. Therefore $supp\, u \subseteq \bar{U}$. Thus, $u \in A_{2,X}(G)$ and $A_{\delta,c}(G) \subseteq A_{2,X}(G)$. Conversely, let $u \in A_{2,X}(G)$ and $supp\, u \subseteq K$ for some compact subset K of G. For every $x \in G \sim K$, we have $\delta_x \in X$ and $supp\, \delta_x = \{x\} \subseteq G \sim K$ by (i). So $\langle \delta_x, u\rangle = u(x) = 0$ by definition. Hence $supp_G \subseteq K$ and $u \in A_{\delta,c}(G)$. Therefore, $A_{\delta,c}(G) = A_{2,X}(G)$. □

If we apply Theorem 2.1, Theorem 2.5 and Proposition 3.2 with $X = C^*_\delta(G)$ and the norm dense subspace of X consisting of all linear combinations of $\delta_x\,(x \in G)$, we have the following result.

Theorem 3.3. *Let $u \in C_\delta^*(G)^*$. Then*

(i) *$u \in A_{\delta,c}(G)$ if and only if for any $\epsilon > 0$, there exists a compact subset K of G such that $|\langle u, f\rangle| < \epsilon$ for all $f \in \ell^1(G \sim K)$ with finite support and $\|f\| \leq 1$, i.e., condition ($\triangle$).*

(ii) *$u \in A_{\delta,s}(G)$ if and only if for any $\epsilon > 0$ and compact subset K of G there exists an $f \in \ell^1(G \sim K)$ with finite support and $\|f\| \leq 1$ such that $|\langle u, f\rangle| > \|u\| - \epsilon$, i.e., condition ($\triangle\triangle$).*

Question. Let $C(G)$ be the space of bounded continuous functions on G. Is it true that $A_{\delta,c}(G) \cap C(G) = A(G)$?

Let $MA(G)$ be the space of pointwise multipliers of $A(G)$ equipped with the multiplier norm $\|u\|_M = \sup\{\|uv\|_{A(G)} : v \in A(G), \|v\|_{A(G)} \leq 1\}$, i.e., the space of all continuous functions u on G such that the pointwise multiplication $uv \in A(G)$ for every $v \in A(G)$ and in this way $u : A(G) \to A(G)$ is a bounded operator on $A(G)$. It is obvious that $A(G) \subseteq MA(G)$ and $\|u\|_M \leq \|u\|_{A(G)}$ if $u \in A(G)$ (see De Cannière and Haagerup [3] and Miao [11]).

Lemma 3.4. *For every $\phi \in MA(G)$, the pointwise multiplication ϕu for $u \in C_\delta^*(G)^*$ defines a multiplier from $C_\delta^*(G)^*$ to $C_\delta^*(G)^*$. Furthermore, the multiplier norm $\|\cdot\|_M$ on $A(G)$ and the multiplier norm $\|\cdot\|_{C_\delta^*(G)}$ on $C_\delta^*(G)^*$ are the same.*

Proof. For every $u \in C_\delta^*(G)^*$, let $\tilde{u}$ be an extension of u onto $VN(G)$ with $\|\tilde{u}\| = \|u\|$. By Goldstine's theorem, there exists a net a_α in $A(G)$ such that $\|a_\alpha\| \leq \|u\|$ and $a_\alpha \to \tilde{u}$ in the $\sigma(VN(G)^*, VN(G))$ topology. Hence we have $\phi a_\alpha \to \phi\tilde{u}$ in the $\sigma(VN(G)^*, VN(G))$ topology since $\phi VN(G) \subseteq VN(G)$. Thus, for every $f \in C_\delta^*(G)$, we have

$$\langle \phi a_\alpha, f\rangle \to \langle \phi u, f\rangle,$$

which implies that $\phi u \in C_\delta^*(G)^*$ and

$$\|\phi u\| \leq \|\phi a_\alpha\| \leq \|\phi\|_M \|a_\alpha\| \leq \|\phi\|_M \|u\|.$$

Hence $\|\phi\|_{C_\delta^*(G)} \leq \|\phi\|_M$. It is obvious that $\|\phi\|_M \leq \|\phi\|_{C_\delta^*(G)}$ since, for any element in $A(G)$, the two norms on $A(G)$ and on $C_\delta^*(G)$ are the same. □

We say that G is weakly amenable if there is a bounded net $\{u_\alpha\}$ in $A(G)$ with respect to the multiplier norm such that $\|u_\alpha a - a\|_{A(G)} \to 0$ for every $a \in A(G)$. If G is amenable, then G is weakly amenable. The free group F_2 on two generators is weakly amenable but not amenable (see De Cannière and Haagerup [3] and Miao [11]).

Theorem 3.5. *Let G be weakly amenable. Then $A_{\delta,c}(G) \cap C(G) = A(G)$.*

Proof. It is trivial that $A(G) \subseteq A_{\delta,c}(G) \cap C(G)$. Let $u \in A_{\delta,c}(G) \cap C(G)$. Then $u \in B(G)$ (see Eymard [4]). Let $\{a_\alpha\}$ be a net in $A(G)$ and $C > 0$ be such that $\|a_\alpha\|_M \leq C$ for all α and $\|a_\alpha a - a\|_{A(G)} \to 0$ for all $a \in A(G)$. For any $\epsilon > 0$, there exists a $u_K \in A_{\delta,c}(G)$ such that $supp_G\, u_K = K$ is compact and $\|u_K - u\| < \epsilon$. Choose $v_K \in A(G)$ such that $v_K = 1$ on K. Thus,

$$\|a_\alpha u_K - u_K\| = \|(a_\alpha v_K - v_K)u_K\| \leq \|a_\alpha v_K - v_K\|\|u_K\| \to 0$$

since $v_K \in A(G)$. Hence, we have

$$\|a_\alpha u - u\| \leq \|a_\alpha u - a_\alpha u_K\| + \|a_\alpha u_K - u_K\| + \|u_K - u\| < C\epsilon + \|a_\alpha v_K - v_K\|\,\|u_K\| + \epsilon,$$

where $\|a_\alpha u - a_\alpha u_K\| \leq \|a_\alpha\| \|u - u_K\| \leq C\epsilon$ by Lemma 3.4. Choose α such that $\|a_\alpha v_K - v_K\| \leq \epsilon$. Hence $\|a_\alpha u - u\| < C\epsilon + \epsilon(\|u\| + \epsilon) + \epsilon$, where $\|u_K\| \leq \|u_K - u\| + \|u\| < \epsilon + \|u\|$. Since $a_\alpha u \in A(G)$ and ϵ is arbitrary, $u \in A(G)$. □

It is clear that if $A_{\delta,c}(G) \cap C(G) = A(G)$, then $A_{\delta,s}(G) \cap C(G) = B^s(G)$. We obtain the following result immediately from Theorem 3.3 and Theorem 3.5.

Corollary 3.6. *Let G be weakly amenable and let $u \in B(G)$. Then*

(i) *$u \in A(G)$ if and only if for any $\epsilon > 0$, there exists a compact subset K of G such that $|\langle u, f\rangle| < \epsilon$ for all $f \in \ell^1(G \sim K)$ with finite support and $\|f\| \leq 1$, i.e., condition ($\triangle$).*

(ii) *$u \in B^s(G)$ if and only if for any $\epsilon > 0$ and compact subset K of G there exists an $f \in \ell^1(G \sim K)$ with finite support and $\|f\| \leq 1$ such that $|\langle u, f\rangle| > \|u\| - \epsilon$, i.e., condition ($\triangle\triangle$).*

Remark. This generalizes Flory's theorem in [5] and [6] (see Pier [12], page 210).

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