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CLOSED SIMILARITY LORENTZIAN AFFINE MANIFOLDS

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ABSTRACT. A $Sim(n-1,1)$ affine manifold is an n-dimensional affine manifold whose linear holonomy lies in the similarity Lorentzian group but not in the Lorentzian group. In this paper, we show that a compact $Sim(n-1,1)$ affine manifold is incomplete. Let $\langle,\rangle_L$ be the Lorentz form, and q the map on $\mathbb{R}^n$ defined by $q(x) = \langle x, x\rangle_L$. We show that for a compact radiant $Sim(n-1,1)$ affine manifold M, if a connected component C of $\mathbb{R}^n - q^{-1}(0)$ intersects the image of the universal cover of M by the developing map, then either C or a connected component of $C - H$, where H is a hyperplane, is contained in this image.

INTRODUCTION

An n-dimensional affine manifold M is an n-dimensional differentiable manifold endowed with an atlas whose coordinate changes are locally affine maps. The affine structure of M pulls back to its universal cover $\hat{M}$ and defines on it an affine structure determined by a local diffeomorphism $D : \hat{M} \to \mathbb{R}^n$, called the developing map. The developing map gives rise to a representation $h : \pi_1(M) \to Aff(\mathbb{R}^n)$, called the holonomy of the affine manifold. Its linear part $L(h)$, is called the linear holonomy of the affine manifold. We will say that the affine manifold is complete if and only if the developing map is a diffeomorphism. An n-affine manifold is said to be radiant if its holonomy fixes an element of $\mathbb{R}^n$.

We denote by $O(p,q)$ the subgroup of linear automorphisms of $\mathbb{R}^n$ that preserve a bilinear symmetric form of type p, q, and by $Sim(p,q)$ the group generated by $O(p,q)$ and the homotheties. An $O(p,q)$ affine manifold M is an affine manifold M such that the image of its linear holonomy $L(h)$ is a subgroup of $O(p,q)$. A $Sim(p,q)$ affine manifold M is an affine manifold M such that the image of its linear holonomy $L(h)$ is a subgroup of $Sim(p,q)$ and contains an element that is not in $O(p,q)$.

Let us consider the flat riemannian torus T^n. Bieberbach has shown that closed $O(n,0)$ affine manifolds are finitely covered by T^n. Using the notion of discompacity, Yves Carrière has shown that closed $O(n-1,1)$ affine manifolds are complete. It is obvious that a $Sim(n,0)$ affine manifold is incomplete, since an element of its holonomy that does not lie in $O(n,0)$ fixes an element of $\mathbb{R}^n$. There exist examples of complete $Sim(n-1,1)$ affine manifolds. Let us give one: Endow $\mathbb{R}^n$ with its

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basis $(e_1, ..., e_n)$ and with the Lorentzian product defined by

$$\langle e_i, e_i\rangle_L = 1; 0 < i < n; \langle e_i, e_j\rangle_L = 0; i \neq j; \langle e_n, e_n\rangle_L = -1.$$

We restrict this product to $\mathbb{R}^2$. The affine map whose linear part is

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

in the basis $(e_1 + e_2, e_1 - e_2)$, and whose translation part is $e_1 - e_2$ generates a group that acts properly and freely on $\mathbb{R}^2$.

The goal of this paper is to study closed $Sim(n-1,1)$ affine manifolds. First we show:

Theorem 1. *A compact $Sim(n-1,1)$ affine manifold is incomplete.*

After, using the notion of discompacity, we show

Theorem 2. *Let M be a closed radiant $Sim(n-1,1)$ affine manifold. If a connected component C of $\mathbb{R}^n - q^{-1}(0)$ intersects $D(\hat{M})$, then either C is contained in $D(\hat{M})$ or a connected component of $C - H$, where H is a hyperplane.*

Interesting structures of $Sim(n-1,1)$ affine manifolds can be constructed using the work of Goldman on projective structures on surfaces; see [Gol]. For instance, a $Sim(2,1)$ structure whose linear holonomy is Zariski dense in $Gl(3,\mathbb{R})$ is given in this paper.

1. Closed $Sim(n-1,1)$ affine manifolds are incomplete

The main goal of this part is to show that a closed $Sim(n-1,1)$ affine manifold cannot be complete.

Let us suppose that there exists a complete closed $Sim(n-1,1)$ affine manifold M; M is the quotient of $\mathbb{R}^n$ by a subgroup of affine transformations Γ, whose linear part is contained in $Sim(n-1,1)$.

Lemma 1.1. *Let γ be an element of Γ whose linear part has a determinant < 1. Then there exists a basis $(e_1, ..., e_n)$ of $\mathbb{R}^n$ such that the linear part of γ in this basis has the following form:*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\lambda^2} & 0 \\ 0 & 0 & \frac{1}{\lambda}B'' \end{pmatrix}$$

where λ is a real number strictly superior to 1 in absolute value, and B'' is a matrix that preserves the restriction of a euclidean product to the vector subspace generated by $e_3, ..., e_n$.

Proof. We have supposed that the determinant of the linear part $L(\gamma)$ of γ is strictly inferior to 1 in absolute value. This implies that there exists a real number $\lambda > 1$ such that $\lambda L(\gamma) = L(\gamma)'$, where $L(\gamma)'$ is an element of $O(n-1,1)$. The linear map $L(\gamma)$ has 1 as an eigenvalue, since γ acts freely. We deduce that λ is an eigenvalue of $L(\gamma)'$. We remark that $L(\gamma)'$ has another eigenvalue α whose module is different from 1 and the module of λ since the absolute value of its determinant is 1. If α is not a real number, then α and its complex conjugate $\bar{\alpha}$ are eigenvalues associated to the complex eigenvectors u_1 and u_2. In this case the restriction of $L(\gamma)'$ to the plane generated by $u_1 + u_2$ and $i(u_1 - u_2)$ is a euclidean similitude whose ratio is different from 1. This is impossible since $L(\gamma)'$ lies in $O(n-1,1)$. Let v_1 and v_2

be the eigenvectors associated to λ and α, and let $\langle , \rangle_L$ be the Lorentzian product preserved by the linear holonomy. We have:

$$\langle v_1, v_1 \rangle_L = \langle v_2, v_2 \rangle_L = 0.$$

We deduce that the restriction of $\langle , \rangle_L$ to the plane P generated by v_1 and v_2 is nondegenerate and has signature $(1,1)$. This implies that the restriction of $\langle , \rangle_L$ to the orthogonal W of P with respect to itself is a scalar product. The restriction B'' of $L(\gamma)'$ to W is an orthogonal linear map. We can suppose that its determinant is 1. We deduce that $\alpha = \frac{1}{\lambda}$. □

Up to a change of origin, we can suppose that $\gamma(0) = (a_1, 0, ..., 0)$ where a_1 is a real number. The restriction B of $L(\gamma)$ to the linear subspace generated by $(e_2, ..., e_n)$ is strictly contracting. It is easy to show that the group generated by γ is not cocompact. So Γ contains another element γ_1 different from γ.

Lemma 1.2. *Let C be the linear part of γ_1. Then $C(e_1) = e_1 + b$ where b lies in the linear subspace generated by $e_2, ..., e_n$.*

Proof. Let k be an element of $\mathbb{N}$. Consider the element $\gamma^k\gamma_1$. Its linear part has 1 as an eigenvalue. The matrix of this linear part in the basis $(e_1, ..., e_n)$ is A^kC, where A is the matrix of the linear part of γ. Let u_k be an eigenvector of A^kC associated to 1. We assume that the norm of u_k with respect to the euclidean scalar product defined by $\langle e_i, e_j \rangle = \delta_{ij}$ is 1. Let $u_k = (u_k^1, u_k^2)$. We have $C(u_k) = (v_k^1, v_k^2)$, where u_k^1 and v_k^1 are elements of $\mathbb{R}$, and u_k^2 and v_k^2 are elements of the vector space F generated by $e_2, ..., e_n$. We have $v_k^1 = u_k^1$ since $A^k(e_1) = e_1$, and A^k preserves F. Since B is strictly contracting, the norm of $A^k(0, v_k^2)$ goes to 0 with respect to the euclidean norm. So u_k goes to e_1, and $C(e_1)$, which is the limit of $C(u_k) = A^{-k}(u_k)$, is $e_1 + b$ where b is an element of F. □

Proof of the Theorem 1. Let c be the translational part of γ_1 in the basis $(e_1, ..., e_n)$. Put $c = (c_1, ..., c_n)$. We have $\gamma^k \circ \gamma_1 \circ \gamma^{-k}(0) = (c_1, B^k(-ka_1b + (c_2, ..., c_n)))$. Since B is contracting and the action of Γ is proper, we deduce that $(c_2, ..., c_n) = b = 0$. This will imply that the action of Γ preserves proper affine subspaces, which is impossible. See [FGH], Theorem 2.2. □

Remark. In contrast to the $Sim(n, 0)$ affine manifolds (see [Fr], Theorem 1), there exist compact $Sim(n-1, 1)$ affine manifolds that are not radiant. Here is an example.

Endow $\mathbb{R}^2$ with the Lorentzian product $(,)$ such that $(e_1, e_1) = (e_2, e_2) = 0$ and $(e_1, e_2) = 1$.

Consider the subgroup Γ of $Aff(\mathbb{R}^2)$ generated by the following transformations:

$$\gamma_1(x, y) = (x + 1, y),$$

$$\gamma_2(x + y) = (x, 2y).$$

The quotient of $\mathbb{R} \times (\mathbb{R} - \{0\})$ by Γ is a compact $Sim(n-1, 1)$ affine manifold.

2. On the universal cover of compact $Sim(n-1, 1)$ affine manifolds

In this part we are going to find properties of the universal cover of a closed radiant $Sim(n-1, 1)$ affine manifold. We use the notion of discompacity defined by Carrière ([Car], 2.2.1). Let us recall it.

We consider in $\mathbb{R}^n$ the unit ball B_n. The euclidean metric induces on closed subsets of $\mathbb{R}^n$ the Hausdorff distance. Let G be a subgroup of $Gl(n,\mathbb{R})$, and let $(g_p)_{p\in\mathbb{N}}$ be a sequence of elements of G. The limit of the family $(g_p(B_n)\cap B_n)_{p\in\mathbb{N}}$ converges in B_n. It is a degenerated ellipsoid (see [Car]). The codimension of this ellipsoid is the discompacity d, of the family $(g_p)_{p\in\mathbb{N}}$. The discompacity of the group with respect to the euclidean metric is the smallest d.

Obviously we cannot use the notion of discompacity in this form since the linear holonomy of our manifold may contain homotheties. Denote $q:\mathbb{R}^n\rightarrow\mathbb{R}$, $x\rightarrow\langle x,x\rangle_L$. We can define in $\mathbb{R}^n-q^{-1}(0)$ the metric

$$(u,v)\longrightarrow\langle ,u,v\rangle'=\frac{\langle u,v\rangle_{euc}}{q(x)}$$

where u and v are vectors of the tangent space at x, and $\langle,\rangle_{euc}$ is the euclidean scalar product.

Theorem 2.1. *Let $\hat{x}$ be an element of $\hat{M}$, u and v elements of $T_{\hat{x}}\hat{M}$, such that the geodesics $c_1:[0,1]\rightarrow\hat{M}$, $t\rightarrow exp_{\hat{x}}(tu)$, and the one $c_2:[0,1]\rightarrow\hat{M}$, $t\rightarrow exp_{\hat{x}}(tv)$ are defined. Suppose that the elements $exp_{\hat{x}}(u)$ and $exp_{\hat{x}}(v)$ cannot be joined by a geodesic, but for every $t,t'<1$, there is a geodesic between $exp_{\hat{x}}(tu)$ and $exp_{\hat{x}}(t'v)$. Let $c:[0,1]\rightarrow\mathbb{R}^n$, $t\rightarrow exp_{D(exp(u))}(tw)$ be the geodesic between $D(exp_{\hat{x}}(u))$ and $D(exp_{\hat{x}}(v))$, and let $U_{\hat{x}}$ be the domain of definition of $exp_{\hat{x}}$. Consider the element $t_0\in[0,1]$ such that for every $t<t_0$, $exp_{D(exp_{\hat{x}}(u))}(tw)$ is an element of $D(exp_{\hat{x}}(U_{\hat{x}}))$, but $exp_{D(exp_{\hat{x}}(u))}(t_0w)=y$ is not an element of $D(exp_{\hat{x}}(u_{\hat{x}}))$. Then y is an element of $q^{-1}(0)$.*

Proof. There is a geodesic $\hat{c}_3:[0,1[\rightarrow\hat{M}$, $t\rightarrow exp_{\hat{x}}(tb)$ such that y is an element of the adherence of $D(\hat{c}_3([0,1[))$ and such that $D(\hat{c}_3)$ is contained in the convex hull of $D(\hat{c}_1)$ and $D(\hat{c}_2)$, where $\hat{c}_1$, and $\hat{c}_2$ are geodesics of $\hat{M}$ respectively above c_1 and c_2. Set $p(\hat{x})=x$. The image c_3 of $p(\hat{c}_3)$ is a maximal incomplete geodesic of M. Since M is compact, there exists an element z of M such that the geodesic c_3 is recurrent in an affine chart U that contains z. We deduce as in Carrière the existence of a family of ellipsoids s_p of $\mathbb{R}^n$ whose centers are elements of $D(\hat{c}_3)$, such that for each p,p', there is an element $\gamma_{p,p'}$ of the holonomy such that $\gamma_{p,p'}(s_p)=s_{p'}$ and the centers x_p of s_p go to y.

Suppose that y is not an element of $q^{-1}(0)$.

Let z_p be an element of an ellipsoid s_p, and let u_p, v_p be two vectors in its tangent space. Put $\gamma_{p,p'}=\lambda_{p,p'}g_{p,p'}$ where $g_{p,p'}$ is an element of $O(n-1,1)$. We have

$$\frac{\langle\gamma_{p,p'}(u_p),\gamma_{p,p'}(v_p)\rangle_{euc}}{q(\gamma_{p,p'}(x))}=\frac{\langle g_{p,p'}(u_p),g_{p,p'}(v_p)\rangle_{euc}}{q(x)},$$

since the holonomy of M is supposed to be radiant.

The metrics $\langle,\rangle_{euc}$ and $\langle,\rangle'$ are equivalent in a neighborhood of y since $q(y)$ is different from 0. We know that the discompacity of the family of g_p with respect to the riemannian metric $\langle,\rangle_{euc}$ is 1. The family of ellipsoids s_p goes to an ellipsoid, or a codimension 1 degenerated ellipsoid centered in y. We conclude as in Carrière that y must be an element of $D(exp_{\hat{x}}(U_{\hat{x}}))$. This is not possible; so $q(y)=0$. □

A similar result is given in [Gol].

Corollary 2.2. *Let M be a compact radiant $Sim(n-1,1)$ affine manifold, and let $\hat{x}$, u, and v be respectively elements of $\hat{M}$ and $T_{\hat{x}}\hat{M}$, such that $exp_{\hat{x}}(u)$ and $exp_{\hat{x}}(v)$*

are defined. If the convex hull E of $(D(\hat{x}), D(exp_{\hat{x}}(u)), D(exp_{\hat{x}}(v))$ is contained in a connected component of $\mathbb{R}^n - q^{-1}(0)$, then it is contained in $D(exp_{\hat{x}}(U_{\hat{x}}))$.

Proof. Suppose that E is not contained in $D(exp_{\hat{x}}(U_{\hat{x}}))$. Let y and z be two elements of $E \cap D(exp_{\hat{x}}(U_{\hat{x}}))$ such that $y = D(exp_{\hat{x}}(u_1))$, $z = D(exp_{\hat{x}}(u_2))$, and for every $t_1, t_2 < 1$, $exp_{\hat{x}}$ is defined on the convex hull of $0, tu_1, tu_2$, but the elements $exp_{\hat{x}}(u_1)$ and $exp_{\hat{x}}(u_2)$ cannot be joined by a geodesic. Consider the geodesic $c : [0,1] \to \mathbb{R}^n$, $t \to exp_y(tw)$ between y and z. There exists a real number $0 < t_0 < 1$, such that for $0 < t < t_0$, $exp_y(tw)$ lies in $D(exp_{\hat{x}}(U_{\hat{x}}))$, but $exp_y(t_0 w)$ does not lie in $D(exp_{\hat{x}}(u_{\hat{x}}))$. We deduce from Theorem 2.1 that $exp_y(t_0 w)$ must lie in $q^{-1}(0)$. This is contrary to the hypothesis. □

More generally, we can determine the boundary of the image of the developing map of a compact radiant $Sim(n-1,1)$ affine manifold. More precisely, we have the following proposition, which implies Theorem 2.

Proposition 2.3. *Let M be a compact radiant $Sim(n-1,1)$ affine manifold whose developing map is injective. Then the boundary of $D(\hat{M})$ is contained in the union of $q^{-1}(0)$ and a hyperplane.*

Proof. As in [Car], p. 625, one can remark that elements of the boundary of $D(\hat{M})$ that are not elements of $q^{-1}(0)$ are limits of $(\gamma_n e)_{n\in\mathbb{N}}$, where γ_n is an element of the holonomy and e is an ellipsoid. We conclude that those elements are contained in at most two hyperplanes H_1, H_2. The case of two hyperplanes is impossible, since those hyperplanes are stable by the holonomy. The affine function α such that $\alpha(H_1) = 0$ and $\alpha(H_2) = 1$ will be invariant by the holonomy and so will define a differentiable function on M without a maximum. (It is the same argument used in [Car]). □

Proposition 2.4. *Let M be a compact radiant affine manifold. If the image of the developing map is a convex set contained in an open set of $\mathbb{R}^n - q^{-1}(0)$, then the developing map is injective.*

Proof. Let $\hat{x}$ be an element of $\hat{M}$. For all elements u and v of $U_{\hat{x}}$, the convex hull of $D(\hat{x})$ is a subset of $D(\hat{M}) \cap (\mathbb{R}^n - q^{-1}(0))$, where $y = D(exp_{\hat{x}}(u))$ and $z = D(exp_{\hat{x}}(v))$. We deduce from Corollary 2.2 that y and z are elements of $D(U_{\hat{x}})$. This implies that $U_{\hat{x}}$ is a convex set. We can conclude the proof by using [Kos]. □

A particular case of the situation of Corollary 2.4 is the following: endow a compact oriented surface S of genus > 2 with a hyperbolic structure, and consider q, the Lorentzian form defined on $\mathbb{R}^3$, by $q(x_1, x_2, x_3) = {x_1}^2 + {x_2}^2 - {x_3}^2$. The hyperbolic structure can be defined by a representation of the fundamental group of S, $\pi_1(S) \to O(2,1)$ such that the quotient of $H = q^{-1}(-1)$ by $\pi_1(S)$ is S. The quotient of $W = \{x : q(x)\langle 0, x_3 \rangle 0\}$ by the group generated by $\pi_1(S)$, and a homothety of ratio $0 < \lambda < 1$, is a compact $Sim(n-1,1)$ affine manifold whose universal cover is W.

More generally we have

Corollary 2.5. *Let M be a radiant compact affine manifold such that the image of its developing map is contained in $W = \{x : q(x)\langle 0, z \rangle 0\}$. M is the quotient of a connected component of $W - H$ by a discrete group of $Sim(n-1,1)$, where H is a hyperplane of $\mathbb{R}^n$.*

Proof. We remark that the interior of a connected component of $W - H$ is convex. This implies that the image of the developing map is a convex set. The result follows by using Propositions 2.3 and 2.4. □

Let M be a compact radiant $Sim(n-1,1)$ affine manifold. The foliation $D(\hat{\mathcal{F}}_q)$ of $\mathbb{R}^n - \{0\}$ whose leaves are the submanifolds defined by $q =$ constant is invariant by the holonomy of M. Its pullback on $\hat{M}$ defines a foliation $\hat{\mathcal{F}}_q$ of $\hat{M}$, which gives rise to a foliation $\mathcal{F}_q$ of M. If $D(\hat{N}) = D(\hat{M}) \cap q^{-1}(0)$ is not empty, then $N = p(D^{-1}(D(\hat{N})))$ is a compact submanifold of M. Note that the 1-parameter group ϕ_t, generated by the radiant vector field, preserves the foliation $\mathcal{F}_q$, and is transverse to all the leaves but not to the connected components of N.

Proposition 2.6. *If N is empty, then M is the total space of a bundle over S^1.*

Proof. If N is empty, then ϕ_t is transverse to the foliation $\mathcal{F}_q$. This implies that this foliation is a Lie foliation. We conclude the proof by using [God], Corollary 2.6, p. 154. □

Remark. If the image of the developing map is contained in $\{x/q(x) < 0\}$, one may expect that the fibers of the previous fibration are hyperbolic manifolds.

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