

CONFORMAL METRICS AND RICCI TENSORS ON THE SPHERE

ROMILDO PINA AND KETI TENENBLAT

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ABSTRACT. We consider tensors $T = fg$ on the unit sphere S^n , where $n \geq 3$, g is the standard metric and f is a differentiable function on S^n . For such tensors, we consider the problems of existence of a Riemannian metric \bar{g} , conformal to g , such that $\text{Ric } \bar{g} = T$, and the existence of such a metric that satisfies $\text{Ric } \bar{g} - \bar{K}\bar{g}/2 = T$, where \bar{K} is the scalar curvature of \bar{g} . We find the restrictions on the Ricci candidate for solvability, and we construct the solutions \bar{g} when they exist. We show that these metrics are unique up to homothety, and we characterize those defined on the whole sphere. As a consequence of these results, we determine the tensors T that are rotationally symmetric. Moreover, we obtain the well-known result that a tensor $T = \alpha g$, $\alpha > 0$, has no solution \bar{g} on S^n if $\alpha \neq n - 1$ and only metrics homothetic to g admit $(n - 1)g$ as a Ricci tensor. We also show that if $\alpha \neq -(n - 1)(n - 2)/2$, then equation $\text{Ric } \bar{g} - \bar{K}\bar{g}/2 = \alpha g$ has no solution \bar{g} , conformal to g on S^n , and only metrics homothetic to g are solutions to this equation when $\alpha = -(n - 1)(n - 2)/2$. Infinitely many C^∞ solutions, globally defined on S^n , are obtained for the equation

$$-\varphi \Delta_g \varphi + \frac{n}{2} |\nabla_g \varphi|^2 - \frac{n}{2} (\lambda + \varphi^2) = 0,$$

where $\lambda \in R$. The geometric interpretation of these solutions is given in terms of existence of complete metrics, globally defined on R^n and conformal to the Euclidean metric, for certain bounded scalar curvature functions that vanish at infinity.

1. INTRODUCTION

Consider the following problems:

- (P_1) Given a symmetric tensor T , of order two, defined on a manifold M^n , does there exist a Riemannian metric g such that $\text{Ric } g = T$?
- (P_2) Find necessary and sufficient conditions on a symmetric tensor T on a compact manifold so that one can find a metric g to satisfy $\text{Ric } g - \frac{K}{2}g = T$ where K is the scalar curvature of g .

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These are interesting problems, which are very difficult to treat in this generality. Solving problem (P_1) or (P_2) is equivalent to studying a system of nonlinear second-order partial differential equations, and (P_2) can be found in the problem section of the Seminar on Differential Geometry edited by S.T. Yau [19].

Our purpose in this paper is to study both problems on the sphere S^n , with $n \geq 3$, for tensors $T = fg$, where f is a differentiable function globally defined on S^n and g is the canonical metric on the sphere. We want to find necessary and sufficient conditions on T for the existence of metrics \bar{g} , conformal to the metric g , which solve problems (P_1) or (P_2) , i.e. we want to study the following systems:

$$(1) \quad \begin{cases} \bar{g} = \frac{1}{\varphi^2}g, \\ \text{Ric } \bar{g} = fg; \end{cases}$$

$$(2) \quad \begin{cases} \bar{g} = \frac{1}{\varphi^2}g, \\ \text{Ric } \bar{g} - \frac{\bar{K}}{2}\bar{g} = fg. \end{cases}$$

In our main results, we determine all tensors T and the corresponding metrics \bar{g} that solve these systems.

We observe that problem (1) is equivalent to obtaining all Einstein metrics on S^n , $n \geq 3$, that are also conformal to the canonical metric on the sphere. This problem is related to a question of Brinkmann of 1925 [2]: When can an Einstein space be mapped conformally on some Einstein space and in how many ways can it be mapped? Kühnel [12] showed that the existence of a nonhomothetic conformal diffeomorphism between two complete Einstein spaces implies that each space is isometric to a standard sphere with a certain radius. Our results provide a partial answer to the second part of Brinkmann's question.

Different aspects of problem (P_1) have been considered in previous articles. DeTurck showed in [6] that, whenever $n \geq 3$, problem (P_1) admits a local solution when the given tensor T is nonsingular. Results on the existence and uniqueness of solutions to problem (P_1) , whenever M^n is a two-dimensional manifold, can be found in [7] and [3]. For compact manifolds some nonexistence results can be found in [9], [10].

J. Cao and DeTurck [4] studied the existence of global solutions on balls (including R^n) and spheres for rotationally symmetric nonsingular tensors. For such tensors on balls, they showed that problem (P_1) has a unique local solution rotationally symmetric, up to homothety, and they studied the question of existence of complete metrics of this type. On the sphere S^n , they proved some nonexistence results, and they obtained necessary conditions on a rotationally symmetric tensor T for the existence of a metric g , on S^n , satisfying $\text{Ric } g = T$.

In [15], we considered special symmetric tensors T on R^n , $n \geq 3$, and we studied the problem of finding metrics \bar{g} conformal to the pseudo-Euclidean metric such that $\text{Ric } \bar{g} = T$.

With respect to problem (P_2) , we observe that if g_{ij} is the Lorentzian metric on a four-dimensional manifold, then it reduces to the Einstein field equation. DeTurck [8] considered the Cauchy problem for this equation and provided a substantial discussion for nonsingular tensors T , in several physical situations. Whenever the tensor T represents vector fields such as electromagnetic field, perfect fluid, pure radiation or vacuum ($T = 0$), then the equation has been studied by several authors

(see [11]). When the metric g is conformal to Minkowski space-time, then the solutions in the vacuum case are necessarily flat, and they were not all known explicitly, until recently (see [11] and [16]). In the remaining cases, all solutions conformal to the Minkowski metric are known. We refer to [14] for the pure radiation or the null electromagnetic field, to [5] and [17] for the non-null electromagnetic field and finally to [17] and [1] if T is a perfect fluid. As far as we know, very little is known on problem (P_2) , besides the cases mentioned above. In [16], we considered problem (P_2) for special symmetric tensors on R^n , with $n \geq 3$, and we characterized all such tensors for the existence of metrics \bar{g} , conformal to the pseudo-Euclidean metric g , such that $\text{Ric } \bar{g} - \frac{\bar{K}}{2}\bar{g} = T$. All solutions \bar{g} were given explicitly (including the vacuum case $T = 0$), and it was also shown that there are no complete metrics \bar{g} conformal and nonhomothetic to g .

In what follows, we state our main results in this paper. The proofs will be given in the next section. In Theorem 1.1, we determine all metrics \bar{g} satisfying problem (1). Moreover, we prove that such metrics are globally defined on S^n , whenever the tensor T is positive definite. In Theorem 1.2, we determine all metrics \bar{g} satisfying problem (2), and we prove that these metrics are globally defined on S^n , whenever the tensor T is negative definite. We then obtain some consequences of these results.

Theorem 1.1. *Let (S^n, g) , $n \geq 3$, be the unit sphere with the usual metric. Then there exists $\bar{g} = \frac{1}{\varphi^2}g$ such that $\text{Ric } \bar{g} = fg$, if and only if, for all $y \in S^n \subset R^{n+1}$,*

$$(3) \quad \varphi(y) = a + c + (a - c)y_{n+1} + \sum_{i=1}^n b_i y_i, \quad f(y) = \frac{-(n-1)}{\varphi^2} \lambda,$$

where a, c, b_i are real numbers and $\lambda = \sum_i b_i^2 - 4ac$. Any such metric \bar{g} is unique up to homothety.

If $\lambda < 0$, then \bar{g} is a metric on S^n that is isometric to g .

If $\lambda = 0$, then $a + c \neq 0$, \bar{g} has one point of singularity on S^n and $\text{Ric } \bar{g} = 0$.

If $\lambda > 0$, then \bar{g} has an $(n - 1)$ -dimensional sphere of singularity points on S^n .

Theorem 1.2. *Let (S^n, g) , $n \geq 3$, be the unit sphere with the usual metric. Then there exists $\bar{g} = \frac{1}{\varphi^2}g$ such that $\text{Ric } \bar{g} - \frac{\bar{K}}{2}\bar{g} = fg$, if and only if, for all $y \in S^n \subset R^{n+1}$,*

$$(4) \quad \varphi(y) = a + c + (a - c)y_{n+1} + \sum_{i=1}^n b_i y_i, \quad f(y) = \frac{(n-1)(n-2)}{2\varphi^2} \lambda,$$

where a, c, b_i are real numbers and $\lambda = \sum_i b_i^2 - 4ac$. Any such metric \bar{g} is unique up to homothety.

If $\lambda < 0$, then \bar{g} is a metric on S^n that is isometric to g .

If $\lambda = 0$, then $a + c \neq 0$, \bar{g} has one point of singularity on S^n and $\text{Ric } \bar{g} - \frac{\bar{K}}{2}\bar{g} = 0$.

If $\lambda > 0$, then \bar{g} has an $(n - 1)$ -dimensional sphere of singularity points on S^n .

Cao and DeTurck [4] considered global solutions of the equation of prescribed Ricci curvature with rotational symmetry on spheres. A symmetric covariant tensor of rank two on S^n is called *rotationally symmetric* if it remains invariant under the action of $SO(n)$ on S^n that leaves the north and south poles of S^n fixed. They provided a necessary condition for such a positive definite tensor to be the Ricci tensor of a metric. The first part of our next result determines all such tensors and

the corresponding metrics in the class of metrics conformal to the standard metric of S^n .

Corollary 1.3. *Let (S^n, g) , $n \geq 3$, be the unit sphere and $T \neq 0$ a rotationally symmetric tensor on S^n . Then there exists $\bar{g} = \frac{1}{\varphi^2}g$ such that*

- a) *Ric $\bar{g} = T$ if and only if $T = f(y)g$, $y \in S^n \subset R^{n+1}$, $\varphi(y) = a + c + (a - c)y_{n+1}$ and $f(y) = (n - 1)4ac/\varphi^2$. Any such metric is unique up to homothety.*
- b) *Ric $\bar{g} - \frac{K}{2}\bar{g} = T$ if and only if $T = f(y)g$, $y \in S^n \subset R^{n+1}$, $\varphi(y) = a + c + (a - c)y_{n+1}$ and $f(y) = -(n - 1)(n - 2)2ac/\varphi^2$. Any such metric is unique up to homothety.*

If $ac > 0$, then \bar{g} is a metric on S^n that is isometric to g .

If $ac < 0$, then \bar{g} has an $(n - 1)$ -dimensional sphere of singularity points in S^n .

If T is a rotationally symmetric nonsingular tensor on a sphere or a ball, then Cao and DeTurck [4] proved that the system Ric $\bar{g} = T$ has rotationally symmetric solutions in some neighborhood of the fixed point of the rotation group. They proved that any such tensor T has at most one rotationally symmetric metric (up to homothety) such that Ric $\bar{g} = T$. Moreover, they showed that despite the fact that local existence is guaranteed for such tensors, existence may fail to hold in the rotationally symmetric category on a domain (even a small ball) that is described in advance. Therefore, it is interesting to point out that Corollary 1.3 a) provides a two-parameter family of nonsingular rotationally symmetric tensors that admit a unique (up to homothety) rotationally symmetric metric \bar{g} such that Ric $\bar{g} = T$, given by

$$T = \frac{4ac(n - 1)}{[a + c + (a - c)y_{n+1}]^2}g,$$

where $ac \neq 0$ and $a \neq c$. The metric is defined on S^n if $ac > 0$ and on the balls $S^n \setminus \{y_{n+1} = -(a + c)/(a - c)\}$ if $ac < 0$. This family can also be rewritten in terms of rotationally symmetric tensors and metrics defined on R^n or on a region of R^n , by using the stereographic projection.

Consider the usual sphere (S^n, g) and the tensor $T_\gamma = \gamma(n - 1)g$ with $0 < \gamma \in R$. Then T_1 is the Ricci tensor of the metric g . In [9], it was shown that only the round metrics on S^n admit T_1 as a Ricci tensor, and if $\gamma > 1$, there are no metrics on S^n whose Ricci tensor is T_γ . In [4] Cao and DeTurck showed that, if $0 < \gamma < 1$ there are no rotationally symmetric metrics on S^n , whose Ricci tensor is T_γ . Moreover, using a result of Xu [18], they concluded that if $\gamma > 0$ and $\gamma \neq 1$, then the equation Ric $g = T_\gamma$ has no solution on S^n conformal to the Euclidean metric. We prove this fact as a consequence of Theorem 1.1 in Corollary 1.4 a) below. We also obtain a similar result for the system (2).

Corollary 1.4. *Let (S^n, g) , $n \geq 3$, be the unit sphere and $T = \alpha g$, $\alpha \in R \setminus \{0\}$.*

- a) *If $\alpha \neq n - 1$, then equation Ric $\bar{g} = \alpha g$ has no solution \bar{g} , conformal to g , on S^n . Among metrics pointwise conformal to g , only metrics homothetic to g admit $(n - 1)g$ as a Ricci tensor.*
- b) *If $\alpha \neq -(n - 1)(n - 2)/2$, then equation Ric $\bar{g} - \bar{K}\bar{g}/2 = \alpha g$ has no solution \bar{g} , conformal to g , on S^n . Among metrics pointwise conformal to g , only metrics homothetic to g are solutions to this equation when $\alpha = -(n - 1)(n - 2)/2$.*

As a consequence of the proof of Theorem 1.1, we get explicit solutions, globally defined on S^n and R^n , for some nonlinear differential equations given in the following result.

Corollary 1.5. *For each $\lambda \in R$ and $n \geq 3$, the equation*

$$(5) \quad -\varphi \Delta_g \varphi + \frac{n}{2} |\nabla_g \varphi|^2 - \frac{n}{2} (\lambda + \varphi^2) = 0$$

has infinitely many C^∞ solutions, globally defined on S^n , given by (3), where $\lambda = \sum b_i^2 - 4ac$. The equation on R^n ,

$$(6) \quad -\varphi \Delta \varphi + \frac{n}{2} |\nabla \varphi|^2 + \frac{2(n-2)}{1+|x|^2} \varphi \langle x, \nabla \varphi \rangle - \frac{2n}{(1+|x|^2)^2} (\lambda + \varphi^2) = 0,$$

where Δ and ∇ are the Laplacian and the gradient in the Euclidean metric, has infinitely many C^∞ solutions, globally defined on R^n , given by

$$\varphi(x) = 2(a|x|^2 + \sum b_i x_i + c)/(1 + |x|^2).$$

A geometric interpretation for equation (6) is given in terms of existence of metrics globally defined on R^n and conformal to the Euclidean metric, for certain scalar curvature functions.

2. PROOFS OF THE MAIN RESULTS

In this paper, we consider a parametrization of the unit sphere S^n by stereographic projection. Then the metric is of the form

$$(7) \quad g_{ij} = \beta(x) \delta_{ij} \quad \text{where} \quad \beta(x) = \frac{4}{(1 + |x|^2)^2},$$

and $|x|^2 = \sum_{i=1}^n x_i^2$ with $x = (x_1, x_2, \dots, x_n) \in R^n$.

For the proofs of Theorems 1.1 and 1.2 we will need the following observation. Considering

$$(8) \quad \tilde{\varphi} = (1 + |x|^2)\varphi,$$

then

$$(9) \quad |\nabla_g \varphi|^2 = \frac{1}{(1 + |x|^2)^2} (|\nabla_g \tilde{\varphi}|^2 + |x|^2 \tilde{\varphi}^2) - \frac{1}{1 + |x|^2} \sum_k x_k \tilde{\varphi} \tilde{\varphi}_{x_k},$$

$$(10) \quad (\text{Hess}_g \varphi)_{ii} = \frac{1}{1 + |x|^2} \left(\tilde{\varphi}_{x_i x_i} - \frac{2}{1 + |x|^2} \sum_k x_k \tilde{\varphi}_{x_k} + \frac{2(|x|^2 - 1)}{(1 + |x|^2)^2} \tilde{\varphi} \right),$$

$$(11) \quad (\text{Hess}_g \varphi)_{ij} = \frac{1}{1 + |x|^2} \tilde{\varphi}_{x_i x_j}, \quad 1 \leq i \neq j \leq n.$$

Proof of Theorem 1.1. Claim: Obtaining $\bar{g} = \frac{1}{\varphi^2} g$, such that $\text{Ric } \bar{g} = f g$, is equivalent to solving the following system of differential equations:

$$(12) \quad \begin{cases} \tilde{\varphi}_{x_i x_i} = \frac{2}{(1 + |x|^2)^2} \left(\frac{f \tilde{\varphi}}{n-1} + \frac{|\nabla_g \tilde{\varphi}|^2}{\tilde{\varphi}} \right), & 1 \leq i \neq j \leq n, \\ \tilde{\varphi}_{x_i x_j} = 0, \end{cases}$$

where $\tilde{\varphi}$ is given by (8).

In fact, suppose there exists such a metric \bar{g} . Since $\bar{g} = g/\varphi^2$, it is well known that

$$\text{Ric } \bar{g} - \text{Ric } g = \frac{1}{\varphi^2} \{ (n-2)\varphi \text{Hess}_g(\varphi) + (\varphi \Delta_g \varphi - (n-1)|\nabla_g \varphi|^2)g \}.$$

Since $\text{Ric } g = (n-1)g$ and we are assuming that $\text{Ric } \bar{g} = fg$, it follows that

$$(13) \quad \frac{1}{\varphi^2} ((n-2)\varphi \text{Hess}_g(\varphi)_{ij} + L(x)g_{ij}) = (f - (n-1))g_{ij},$$

where

$$(14) \quad L(x) = \varphi \Delta_g \varphi - (n-1)|\nabla_g \varphi|^2$$

and g_{ij} is given by (7). By considering $i = j$ in (13), we get

$$(15) \quad \frac{1}{\varphi^2} [(n-2)\varphi (\text{Hess}_g \varphi)_{ii} + L(x)g_{ii}] = (f - n + 1)g_{ii}.$$

For a fixed i , multiplying (15) by $-(2n-3)$ and adding the remaining equations we get

$$\frac{1}{\varphi} \left[-(2n-3)(\text{Hess}_g \varphi)_{ii} + \sum_{k \neq i} (\text{Hess}_g \varphi)_{kk} \right] - \frac{1}{\varphi^2} L(x)\beta(x) = -(f - n + 1)\beta(x),$$

where we have used the identity $-(2n-3)g_{ii} + \sum_{k \neq i} g_{kk} = (2-n)\beta(x)$, with $\beta(x)$ given by (7). Using (14), this equation is equivalent to

$$2(n-1)(\text{Hess}_g \varphi)_{ii} - \sum_k (\text{Hess}_g \varphi)_{kk} + \beta(x) \left[\Delta_g \varphi - \frac{n-1}{\varphi} |\nabla_g \varphi|^2 + (n-1-f)\varphi \right] = 0.$$

Since $\sum_k (\text{Hess}_g \varphi)_{kk} = \beta(x)\Delta_g \varphi$, this equation reduces to

$$(16) \quad (\text{Hess}_g \varphi)_{ii} + \frac{2}{(1+|x|^2)^2} \left(-\frac{|\nabla_g \varphi|^2}{\varphi} + \left(1 - \frac{f}{n-1}\right)\varphi \right) = 0.$$

Substituting (9) and (10) into (16), we get the first equation of system (12).

Now, by considering $i \neq j$ in equation (13), we have $\text{Hess}_g \varphi_{ij} = 0$. It follows from (11) that this is equivalent to $\tilde{\varphi}_{x_i x_j} = 0$. Hence system (12) is satisfied. Conversely, a straightforward computation shows that if $\tilde{\varphi}$ satisfies system (12), then $\bar{g} = g/\varphi^2$, where $\varphi = \tilde{\varphi}/(1+|x|^2)$, satisfies $\text{Ric } \bar{g} = fg$. This concludes the proof of our claim.

We now solve system (12). Since $\tilde{\varphi}_{x_i x_j} = 0, \forall 1 \leq i \neq j \leq n$, it follows that $\tilde{\varphi}(x) = \sum_k f_k(x_k)$. Observe that $\tilde{\varphi}_{x_i x_i} = \tilde{\varphi}_{x_k x_k} \forall i \neq k$. Hence, $f'_i(x_i)$ is the same constant $\forall i, 1 \leq i \leq n$. Therefore,

$$(17) \quad \tilde{\varphi}(x) = \sum_{i=1}^n 2(ax_i^2 + b_i x_i + c_i)$$

where a, c_i, b_i are real numbers (the factor 2 is unnecessary but convenient). Substituting (17) in (12), we get

$$(18) \quad 8a\tilde{\varphi} - \frac{4}{(n-1)(1+|x|^2)^2} \tilde{\varphi}^2 f - \sum_k (\tilde{\varphi}_{x_k})^2 = 0.$$

It follows, from (8) and (18), that

$$(19) \quad \varphi(x) = \frac{2(a|x|^2 + \sum_i b_i x_i + c)}{1 + |x|^2}, \quad f(x) = \frac{-(n-1)}{\varphi^2} \lambda,$$

where $c = \sum_i c_i$. By using the inverse of the stereographic projection, we have $\varphi(y)$ globally defined on S^n , given by (3). Since the constants a, b_i, c are arbitrary, we may consider any multiple of φ , without changing f . Hence the metric \bar{g} is unique up to homothety.

Moreover, if $\lambda < 0$, then $\varphi(y)$ does not vanish on S^n . Moreover, since $\bar{g} = g/\varphi^2$, it follows that

$$\text{Ric } \bar{g} = -(n-1)\lambda\bar{g}.$$

Hence, \bar{g} is an Einstein metric. Therefore, (S^n, \bar{g}) has constant sectional curvature (see [13]) and it is isometric to (S^n, g) .

$\lambda = 0$, if and only if, $f = 0$, i.e. $\text{Ric } \bar{g} = 0$. In this case, if $c + a = 0$, then $\varphi \equiv 0$. Therefore, $c + a \neq 0$ and the function $\varphi(y)$ vanishes only at the point $(-b_1, \dots, -b_n, c - a)/(c + a)$ on the sphere, where \bar{g} is singular.

If $\lambda > 0$, then $\varphi(y)$ vanishes on the $(n - 1)$ -dimensional sphere obtained by intersecting the plane $a + c + (a - c)y_{n+1} + \sum_i b_i y_i = 0$ with S^n . This corresponds to an $(n - 1)$ -dimensional sphere of singularity points of \bar{g} on S^n . □

Proof of Theorem 1.2. Claim: Obtaining $\bar{g} = g/\varphi^2$ such that $\text{Ric } \bar{g} - \frac{\bar{K}}{2}\bar{g} = fg$ is equivalent to solving the following system of equations:

$$(20) \quad \begin{cases} \tilde{\varphi}_{x_i x_i} = \frac{4}{(1 + |x|^2)^2} \left(\frac{-f(x)\tilde{\varphi}}{(n-1)(n-2)} + \frac{|\nabla_g \tilde{\varphi}|^2}{2\tilde{\varphi}} \right), & 1 \leq i \neq j \leq n, \\ \tilde{\varphi}_{x_i x_j} = 0, \end{cases}$$

where $\tilde{\varphi}$ is given by (8).

In fact, since $\text{Ric } g = (n - 1)g$ and

$$\bar{K} = (n - 1) (n\varphi^2 + 2\varphi\Delta_g\varphi - n|\nabla_g\varphi|^2),$$

obtaining such a \bar{g} is equivalent to studying the following equations, for $1 \leq i, j \leq n$:

$$(21) \quad \frac{1}{\varphi}(\text{Hess}_g\varphi)_{ij} + \left(\frac{(n-1)|\nabla_g\varphi|^2}{2\varphi^2} - \frac{\Delta_g\varphi}{\varphi} \right) g_{ij} = \left(\frac{f}{n-2} + \frac{n-1}{2} \right) g_{ij}.$$

Considering $i = j$ in (21) and multiplying, for a fixed i , the i^{th} equation by $-(n - 2)$ and adding the remaining equations we obtain

$$(22) \quad \frac{-(n-2)}{\varphi}(\text{Hess}_g\varphi)_{ii} + \frac{1}{\varphi} \sum_{k \neq i} (\text{Hess}_g\varphi)_{kk} + \left(\frac{(n-1)|\nabla_g\varphi|^2}{2\varphi^2} - \frac{\Delta_g\varphi}{\varphi} \right) \beta(x) = \left(\frac{f}{n-2} + \frac{n-1}{2} \right) \beta(x).$$

Since $\sum_k (\text{Hess}_g\varphi)_{kk} = \beta(x)\Delta_g\varphi$, it follows from (7) that equation (22) can be written as

$$(23) \quad \text{Hess}_g(\varphi)_{ii} = \frac{4}{(1 + |x|^2)^2} \left[\frac{|\nabla_g\varphi|^2}{2\varphi} - \frac{f\varphi}{(n-1)(n-2)} - \frac{\varphi}{2} \right].$$

By substituting (9) and (10) into equation (23) and $\varphi = \tilde{\varphi}/(1 + |x|^2)$, we get the first equation of (20).

Considering $i \neq j$ in equation (21), we get $\text{Hess}_g(\varphi)_{ij} = 0$. It follows from (11) that this is equivalent to $\tilde{\varphi}_{x_i x_j} = 0$. Therefore, we have proved that if $\bar{g} = g/\varphi^2$ satisfies $\text{Ric } \bar{g} - \frac{\bar{K}}{2}\bar{g} = fg$, then system (20) is satisfied. Straightforward computation shows that the converse holds.

Similarly to the proof of Theorem 1.1, we show that any solution of system (20) is of the form

$$(24) \quad \tilde{\varphi}(x) = \sum_i 2(ax_i^2 + b_i x_i + c_i),$$

where a, c_i, b_i are real numbers. Substituting (24) in (20) and using (8), we conclude that φ and f are given by

$$(25) \quad \varphi(x) = \frac{2(a|x|^2 + \sum_i b_i x_i + c)}{1 + |x|^2}, \quad f(x) = \frac{(n-1)(n-2)}{2\varphi^2}\lambda,$$

where $c = \sum c_i$. By considering the inverse of the stereographic projection, we obtain (4). The metric \bar{g} is unique up to homothety, since any multiple of φ does not change f .

If $\lambda < 0$, then $\varphi(y)$ does not vanish on S^n , and in this case, \bar{g} is complete. Moreover, straightforward computations show that $\bar{K}(y) = -(n-1)n\lambda$. Since $\text{Ric } \bar{g} - \frac{\bar{K}}{2}\bar{g} = fg$, it follows that $\text{Ric } \bar{g} = -(n-1)\lambda\bar{g}$, i.e., \bar{g} is an Einstein metric. In this case (S^n, \bar{g}) has constant sectional curvature, and hence it is isometric to (S^n, g) .

$\lambda = 0$, if and only if, $f = 0$, i.e. $\text{Ric } \bar{g} - \frac{\bar{K}}{2}\bar{g} = 0$. The same argument of the proof of Theorem 1.1 implies that $c + a \neq 0$ and that $\varphi(y)$ vanishes only at the point $(-b_1, \dots, -b_n, c - a)/(c + a)$, which is the point of singularity of \bar{g} .

If $\lambda > 0$, then φ vanishes on an $(n-1)$ -dimensional sphere contained on S^n , and hence \bar{g} is not defined on this sphere. \square

Proof of Corollary 1.3. If T is a rotationally symmetric tensor defined on S^n , then it can be considered to be given by $T = f(y)g$, where f is a rotationally symmetric function on S^n (see [4]). It follows from Theorems 1.1 and 1.2 that f is rotationally symmetric, if and only if, $b_i = 0$ for all i , $1 \leq i \leq n$. Hence Corollary 1.3 is an immediate consequence of these theorems. \square

Proof of Corollary 1.4. It follows from Theorems 1.1 and 1.2, that $f = \alpha$ is constant, if and only if, $b_i = 0$, for all i and $c = a \neq 0$. Therefore, $\varphi = 2a$.

a) It follows from (3) that $\alpha = n-1$ and $\bar{g} = g/(4a^2)$. Hence $\text{Ric } \bar{g} = (n-1)g$ and \bar{g} is homothetic to g .

b) It follows from (4) that $\alpha = -(n-1)(n-2)/2$ and $\bar{g} = g/(4a^2)$. Observe that $\bar{K} = n(n-1)4a^2$. Hence $\text{Ric } \bar{g} - \frac{\bar{K}}{2}\bar{g} = -(n-1)(n-2)g/2$ and \bar{g} is homothetic to g . \square

Proof of Corollary 1.5. It follows from the proof of Theorem 1.1 that the functions $\varphi(x)$ and $f(x)$, $x \in R^n$, given by (19), satisfy the system (13). In particular, φ also satisfies the equations

$$(26) \quad (n-2)\varphi g^{ii}\text{Hess}_g(\varphi)_{ii} + L(x) + (n-1)(\lambda + \varphi^2) = 0,$$

for all i , $1 \leq i \leq n$, where $\lambda = \sum_k b_k^2 - 4ac$ and L is given by (14). Adding the equations given by (26) we get that $\varphi(x)$, and hence $\varphi(y)$ defined by (3), satisfies equation (5), since

$$\Delta_g \varphi = \frac{1}{\beta(x)} \left(\Delta \varphi - \frac{2(n-2)}{1+|x|^2} \sum_i x_i \varphi_{x_i} \right), \quad |\nabla_g \varphi|^2 = \frac{1}{\beta(x)} |\nabla \varphi|^2,$$

where Δ and ∇ are the Laplacian and the gradient in the Euclidean metric on R^n . It follows from (5) that $\varphi(x)$ satisfies (6). \square

Remark 2.1. We now present the geometric interpretation of equation (6). We know that (see for example [15]) if R^n is the Euclidean space and $\tilde{K} : R^n \rightarrow R$ is a differentiable function on R^n , then finding a metric \tilde{g} conformal to the Euclidean metric, with conformal factor $1/\varphi^2$ and scalar curvature \tilde{K} is equivalent to studying the differential equation

$$-\varphi \Delta \varphi + \frac{n}{2} \|\nabla \varphi\|^2 + \frac{\tilde{K}}{2(n-1)} = 0.$$

Since the functions given by (19) are solutions of equation (6), then given

$$\tilde{K} = \frac{16(n-1)(n-2)h(x)}{(1+|x|^2)^4} \left[\ell(x) - h(x) \left(|x|^2 + \frac{2}{n-2} \right) \right] - \frac{4n(n-1)\lambda}{(1+|x|^2)^2}$$

where

$$h(x) = a|x|^2 + \sum_i b_i x_i + c, \quad \ell(x) = (1+|x|^2)(a|x|^2 - c), \quad \lambda = \sum_i b_i^2 - 4ac < 0,$$

then \tilde{g} provides a metric conformal to the Euclidean metric, with φ given by (19), whose scalar curvature is \tilde{K} .

We observe that \tilde{K} is a bounded function and $\lim_{|x| \rightarrow \infty} \tilde{K} = 0$. Moreover, \tilde{g} is a complete metric, since there exists a constant $m > 0$, such that for any vector $v \in R^n$, $|v|_{\tilde{g}} \geq m|v|$.

REFERENCES

- [1] Barnes, A., *Space-times of embedding class one in general relativity*, General Relativity and Gravitation, 5, (1974), 147-161. MR 53:15259
- [2] Brinkmann, H.W., *Einstein spaces which are mapped conformally on each other*, Math. Ann. 94, (1925), 119-145.
- [3] Cao, J.; DeTurck, D., *Prescribing Ricci curvature on open surfaces*, Hokkaido Math. J. 20 (1991), 265-278. MR 92k:53068
- [4] ———, *The Ricci curvature equation with rotational symmetry*, American Journal of Mathematics 116 (1994), 219-241. MR 94m:53052
- [5] Cahen, M. and Leroy, J. *Exact solutions of the Einstein-Maxwell equations*, J. Math. Mech. 16, (1966), 501-508. MR 34:3973
- [6] DeTurck, D., *Existence of metrics with prescribed Ricci curvature: Local theory*, Invent. Math. 65 (1981), 179-207. MR 83b:53019
- [7] ———, *Metrics with prescribed Ricci curvature*, Seminar on Differential Geometry, Ann. of Math. Stud. Vol. 102, (S. T. Yau, ed.), Princeton University Press, (1982), 525-537. MR 83e:53014
- [8] ———, *The Cauchy problem for Lorentz metrics with prescribed Ricci curvature*, Compositio Math. 48 (1983), 327-349. MR 85c:53041
- [9] DeTurck, D.; Koiso, N., *Uniqueness and non-existence of metrics with prescribed Ricci curvature*, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984), 351-359. MR 86i:53022
- [10] Hamilton, R.S., *The Ricci curvature equation*, Seminar on nonlinear partial differential equations, Publ. Math. Sci. Res. Inst. 2, (1984), 47-72. MR 86b:53040

- [11] Kramer, D., Stephani, H., MacCallun, M.A.H. and Herlt, E., Exact solutions of Einstein's field equations, Cambridge University Press, 1980. MR 82h:83002
- [12] Kühnel, W., *Conformal transformations between Einstein spaces*, Conformal geometry (Bonn, 1985/1986), 105-146, Aspects Math., E12, Vieweg, Braunschweig, 1988. MR 90b:53055
- [13] Lee, J.M.; Parker, T.H., *The Yamabe problem*, Bulletin (New Series) of the Amer. Math. Soc. 17, (1987), 37-91. MR 88f:53001
- [14] McLenaghan, R.G., Tarig, N. and Tupper, B.O.J. *Conformally flat solutions of the Einstein-Maxwell equations for the null electromagnetic fields*, J. Math. Phys. 16, (1975), 829-831. MR 51:12275
- [15] Pina, R., Tenenblat, K., *Conformal metrics and Ricci tensors in the pseudo-euclidean space*, Proc. Amer. Math. Soc. 129, (2001), 1149-1160. MR 2001k:53137
- [16] Pina, R., Tenenblat, K., *On metrics satisfying equation $R_{ij} - Kg_{ij}/2 = T_{ij}$ for constant tensors T_{ij}* , Journal of Geometry and Physics 40, (2002) 379-383. MR 2002g:53125
- [17] Stephani, H., *Konform flache Gravitationsfelder*, Comm. in Math. Phys. 5, (1967) 337-342.
- [18] Xu, X., *Prescribing a Ricci tensor in a conformal class of Riemannian metrics*, Proceedings of the American Mathematical Society Vol. 115, (1992), 455-459. MR 92i:53036
- [19] Yau, S.T., Ed., Seminar on Differential Geometry, Princeton University Press, 1982. MR 83a:53002

IME, UNIVERSIDADE FEDERAL DE GOIÁS, 74001-970 GOIÂNIA, GO, BRAZIL
E-mail address: romildo@mat.ufg.br

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE DE BRASÍLIA, 70910-900, BRASÍLIA, DF,
BRAZIL
E-mail address: keti@mat.unb.br