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REMARK ON WELL-POSEDNESS FOR THE FOURTH ORDER NONLINEAR SCHRÖDINGER TYPE EQUATION

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Abstract. We consider the initial value problem for the fourth order nonlinear Schrödinger type equation (4NLS) related to the theory of vortex filament. In this paper we prove the time local well-posedness for (4NLS) in the Sobolev space, which is an improvement of our previous paper.

1. Introduction

We consider the initial value problem for the fourth order nonlinear Scrödinger type equation (4NLS) of the form:

$$\begin{cases} i\partial_t u + \partial_x^2 u + \nu\partial_x^4 u = F(u,\overline{u},\partial_x u,\partial_x\overline{u},\partial_x^2 u,\partial_x^2\overline{u}), & (t,x)\in\mathbf{R}\times\mathbf{R}, \\ u(0,x) = u_0(x), & x\in\mathbf{R}, \end{cases} \tag{1.1}$$

where $u(x,t):\mathbf{R}\times\mathbf{R}\to\mathbf{C}$ is an unknown function. The nonlinear term F is given by

$$\begin{aligned} F(u,\overline{u},\partial_x u,\partial_x\overline{u},\partial_x^2 u,\partial_x^2\overline{u}) = &-\frac{1}{2}|u|^2u + \lambda_1|u|^4u + \lambda_2(\partial_x u)^2\overline{u} + \lambda_3|\partial_x u|^2u \\ &+ \lambda_4 u^2\partial_x^2\overline{u} + \lambda_5|u|^2\partial_x^2 u, \end{aligned} \tag{1.2}$$

where ν, μ are real constants satisfying $\lambda_1 = 3\mu/4, \lambda_2 = 2\mu-\nu/2, \lambda_3 = 4\mu+\nu, \lambda_4 = \mu, \lambda_5 = 2\mu-\nu$.

The equation in (1.1) describes the three-dimensional motion of an isolated vortex filament embedded in an inviscid incompressible fluid filling an infinite region. This equation is proposed by Fukumoto and Moffatt [8] as some detailed model taking account of the effect from the higher order corrections of the Da Rios model (cubic nonlinear Schrödinger equation):

$$i\partial_t u + \partial_x^2 u = -\frac{1}{2}|u|^2u.$$

For the physical background we refer to [7] and [8].

To motivate our problem in this paper, we state briefly our previous result associated with the well-posedness of the initial value problem (1.1). The notion of well-posedness used here includes the existence, uniqueness of a solution and

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continuous dependence upon the initial data. In [14], we proved the time local well-posedness as the initial value problem (1.1) in the usual Sobolev spaces $H^s(\mathbf{R})$ with $s \geq 1/2$ by imposing the condition $\lambda_5 = 2\mu - \nu = 0$ on the coefficients. It was not clear whether this restriction has any physical interpretations. In the present paper, we eliminate this restriction and guarantee the time local well-posedness for (1.1) in Sobolev spaces not lacking the term $|u|^2\partial_x u$, i.e., $\lambda_5 = 2\mu - \nu \neq 0$. Consequently our result improved to include the non-completely integrable case, which appears in the real model.

To state our main result precisely, we introduce some notation and function spaces. For a function $u(x,t)$, we denote by $\hat{u} = \mathcal{F}_x u$ the Fourier transforms in the x variable. We denote by $\widehat{\widehat{u}}(\tau,\xi) = \mathcal{F}_t\mathcal{F}_x u(\tau,\xi)$ the space-time Fourier transform. The operator D_x and $\langle D_x\rangle$ are given by $D_x = \mathcal{F}_x^{-1}|\xi|\mathcal{F}_x$ and $\langle D_x\rangle = \mathcal{F}_x^{-1}\langle\xi\rangle\mathcal{F}_x$, respectively, where $\langle x\rangle = (1+|x|^2)^{1/2}$. We abbreviate $L_t^p(\mathbf{R}; L_x^q(\mathbf{R}))$ as $L_t^p(L_x^q)$ and $H_t^p(\mathbf{R}; H_x^q(\mathbf{R}))$ as $H_t^p(H_x^q)$, respectively. Let $\psi(t)$ be a smooth cut-off function to the interval $[-1,1]$, i.e., $\psi \in C_0^\infty(\mathbf{R})$, $\psi(t) \equiv 1$ for $|t| \leq 1$, and $\psi(t) \equiv 0$ for $|t| \geq 2$. For $\delta > 0$, we set $\psi_\delta(t) = \psi(t/\delta)$. $W_\nu(t)$ is the unitary group generated by the linear equation of (1.1). For a real number s, let $s+$ denotes a fixed constant larger than s.

The equation (1.1) is rewritten as the following integral equation:

$$u(t) = W_\nu(t)u_0 - i\int_0^t W_\nu(t-t')F(u,\overline{u},\partial_x u,\partial_x\overline{u},\partial_x^2 u,\partial_x^2\overline{u})(t')dt', \tag{1.3}$$

for $t \in [-T,T]$. Then our main result is the following:

Theorem 1.1. *Let* $\nu < 0$. *If* $s > 7/12$, $b \in (1/2, 3/4)$, *then for* $u_0 \in H^s(\mathbf{R})$, *there exist* $T = T(\|u_0\|_{H^s}) > 0$ *and a unique solution* $u(t)$ *of the initial value problem* (1.1) *satisfying*

$$\begin{aligned} u &\in C([-T,T]; H^s(\mathbf{R})),\\ \psi_T W_\nu(-t)u &\in H_t^b(\mathbf{R}; H_x^s(\mathbf{R})),\\ \psi_T W_\nu(-t)F &\in H_t^{b-1}(\mathbf{R}; H_x^s(\mathbf{R})). \end{aligned}$$

Moreover, given $T' \in (0,T)$, *two maps* $u_0 \longmapsto u$ *from* $H^s(\mathbf{R})$ *to* $C([-T',T']; H^s(\mathbf{R}))$ *and* $u_0 \longmapsto \psi_{T'}W_\nu(-t)u$ *from* $H^s(\mathbf{R})$ *to* $H_t^b(\mathbf{R}; H_x^s(\mathbf{R}))$ *are Lipschitz continuous, respectively.*

Remark. By employing an analogous method in Molinet-Saut-Tzvetkov [13], we showed in [14] that the initial value problem for (1.1) cannot be solved by the Picard iterative succession via the corresponding integral equation in the Sobolev space $H^s(\mathbf{R})$ with $s < 1/2$. Therefore there is a gap between the index $s = 7/12+$ of the Sobolev spaces in Theorem 1.1 and the $s = 1/2$ suggested by the counterexample. However, assuming that $\lambda_5 = 2\mu - \nu = 0$, we could solve the local well-posedness of the initial value problem (1.1) in $H^s(\mathbf{R})$ with $s \geq 1/2$ ([14]).

When $\mu + \nu/2 = 0$, it should be remarked that (1.1) is the completely integrable equation (see [7]) and has infinitely many conserved quantities (see [9]); for example,

$$\Phi_1(u) = \frac{1}{2}\int_{\mathbf{R}}|u|^2dx, \qquad \Phi_2(u) = -\frac{i}{2}\int_{\mathbf{R}}(\partial_x u)\overline{u}dx,$$

$$\Phi_3(u) = -\frac{1}{2}\int_{\mathbf{R}}(\partial_x^2 u)\overline{u}dx - \frac{1}{8}\int_{\mathbf{R}}|u|^4dx, \cdots.$$

Therefore, if $\mu + \nu/2 = 0$, combining the above properties and the Gagliardo-Nirenberg inequality:

$$\|u\|_{L^4}^4 \leq C\|u\|_{L^2}^3 \|\partial_x u\|_{L^2} \leq \frac{C^2}{2}\|u\|_{L^2}^6 + \frac{1}{2}\|\partial_x u\|_{L^2}^2,$$

we can show an a priori bound of the solution in $H^1(\mathbf{R})$ for all $t > 0$. Hence we have the global well-posedness of the solution in our theorem.

To prove Theorem 1.1, we use the method of Fourier restriction norm introduced by Bourgain [4] and Kenig-Ponce-Vega [11], [12]. We define the Fourier restriction space $X_{s,b}^{\nu}$ with $b, s, \nu \in \mathbf{R}$ associated with the equation (1.1) as follows: We denote $\phi_\nu(\xi) = \xi^2 - \nu\xi^4$. Then $\tau + \phi_\nu(\xi)$ represents the symbol of the linearized equation of (1.1). Let

$$\begin{aligned} X_{s,b}^{\nu} &\equiv \{u \in \mathcal{S}'(\mathbf{R}^2); \|u\|_{X_{s,b}^{\nu}} < \infty\}, \\ \|u\|_{X_{s,b}^{\nu}} &\equiv \|\langle \tau + \phi_\nu(\xi)\rangle^b \langle \xi\rangle^s \widehat{\widehat{u}}(\xi,\tau)\|_{L_\xi^2(L_\tau^2)} \\ &= \|W_\nu(-t)u(t)\|_{H_t^b(H_x^s)}. \end{aligned} \tag{1.4}$$

Then we introduce a new estimate of the maximal function related to the unitary group of the fourth order Schrödinger equation (see Proposition 2.2 in Section 2 below). This estimate enables us to handle the worst term $|u|^2\partial_x^2 u$ in the nonlinear terms, and we can show the crucial trilinear estimate relevant to this term.

In the next section, we list some linear estimates including the estimate for the maximal function. In the last section, we show the crucial nonlinear estimate and prove Theorem 1.1.

2. Linear estimates

In this section, we give the linear estimates needed for the proof of crucial nonlinear estimates (see Propositions 3.1 and 3.2 below). It is convenient to use the following notation for the proof of Proposition 3.2 below: For $b \in \mathbf{R}$ let us define F_b by

$$\widehat{\widehat{F_b}}(\tau,\xi) = \frac{f(\tau,\xi)}{\langle \tau \pm \phi_\nu(\xi)\rangle^b}, \tag{2.1}$$

for $f \in \mathcal{S}(\mathbf{R}^2)$.

Lemma 2.1. *Let $b > 1/2$ and $b' > 1/4$. For any $f \in L_\tau^2(L_\xi^2)$, we have*

$$\|D_x^{-1/4} F_b\|_{L_x^4(L_t^\infty)} \leq C\|f\|_{L_\tau^2(L_\xi^2)} \quad \text{(Kenig-Ruiz estimate)}, \tag{2.2}$$

$$\|D_x^{3/2} F_b\|_{L_x^\infty(L_t^2)} \leq C\|f\|_{L_\tau^2(L_\xi^2)} \quad \text{(Kato type smoothing effect)}, \tag{2.3}$$

$$\|D_x^{3/4} F_{b'}\|_{L_x^4(L_t^2)} \leq C\|f\|_{L_\tau^2(L_\xi^2)} \quad \text{(Kato type smoothing effect)}, \tag{2.4}$$

where F_b is defined by (2.1).

Proof of Lemma 2.1. The estimates (2.2) and (2.3) are due to Kenig-Ponce-Vega [10]. For the proof of those estimates, see Theorem 2.5 and Theorem 4.1 in [10], respectively. The inequality (2.4) follows from the interpolation between (2.3) and the Plancherel identity $\|F_0\|_{L_x^2(L_t^2)} = \|f\|_{L_\xi^2(L_\tau^2)}$. □

The next proposition plays an important role in the proof of our main theorem (see the proof of Proposition 3.2 below).

Proposition 2.2 (Estimate for the maximal function)**.** *Let* $\rho > 1,\ T > 0$. *For any* $u_0 \in L^2_x(\mathbf{R})$ *and* $F \in L^2_x(L^1_T)$, *we have*

$$\|\langle D_x\rangle^{-\rho}W(t)u_0\|_{L^2_x(L^\infty_T)} \le C\|u_0\|_{L^2_x}, \tag{2.5}$$

$$\|\langle D_x\rangle^{-2\rho}\int_0^t W_\nu(t-t')F(t')dt'\|_{L^2_x(L^\infty_T)} \le C\|F\|_{L^2_x(L^1_T)}, \tag{2.6}$$

where $C > 0$ *is a constant depending on* T *and* ρ.

Corollary 2.3. *Let* $\rho > 1/2,\ T \in (0,1),\ b > 1/2$, *and* $f \in L^2_\tau(L^2_\xi)$ *with*

$$supp\mathcal{F}_t^{-1}\mathcal{F}_x^{-1}f \subset (-T,T).$$

Then, for any F_b *defined by* (2.1), *we have*

$$\|\langle D_x\rangle^{-\rho}F_b\|_{L^2_x(L^\infty_t)} \le C\|f\|_{L^2_\tau(L^2_\xi)}. \tag{2.7}$$

A similar result to Proposition 2.2 for the Schrödinger equation is obtained by Constantin-Saut [6], Sjölin [15] and Vega [16]. The estimate (2.5) is proved by applying the duality argument to the estimate (2.6).

For the purpose of the proof for the inequality (2.6), it suffices to show that the integral kernel of $\langle D_x\rangle^{-2\rho}W_\nu(t-t')$ belongs to $L^1_x(L^\infty_T)$. More precisely , we require the following lemma.

Lemma 2.4. *Let* $\rho > 1$. *We define the integral kernel of* $\langle D_x\rangle^{-2\rho}W_\nu(t-t')$ *as* $K(t-t', x-y)$. *Then for some* $\varepsilon > 0$, *we have*

$$|K(t-t', x-y)| \le C\langle x-y\rangle^{-1-\varepsilon}, \tag{2.8}$$

where $C > 0$ *is a constant depending only on* T, ρ *and independent of* $t, t' \in [0,T]$.

A simple application of Young's inequality and Lemma 2.4 yield (2.6).

Proof of Lemma 2.4. For simplicity, we only show the case $\nu = -1$. Let $\phi(\xi) \equiv \phi_{-1}(\xi) = \xi^2 + \xi^4$. We note that the integral kernel of $\langle D_x\rangle^{-2\rho}W(t-s)$ is given by

$$K(\sigma, z) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{iz\xi - i\sigma\phi(\xi)}\langle\xi\rangle^{-2\rho}d\xi,$$

where $\sigma = t - t'$ and $z = x - y$.

By differentiating the phase, we have

$$\frac{d}{d\xi}(z\xi - \sigma\phi(\xi)) = z - \sigma(\phi'(\xi)) = -4\sigma\Big(\xi^3 + \frac{\xi}{2} - \frac{z}{4\sigma}\Big) = -4\sigma\alpha^3\Big(\eta^3 + \frac{\eta}{2\alpha^2} - 1\Big),$$

where we put $\frac{z}{4\sigma} = \alpha^3,\ \xi = \alpha\eta,\ \alpha \in \mathbf{R}$, and $\phi' = \frac{d\phi}{d\xi}$.

Let p_j $(j = 0,1,2)$ be the roots of the algebraic equation $\eta^3 + \frac{1}{2\alpha^2}\eta - 1 = 0$, and let p_0 be the unique real root. We note that the p_j's are depending on α, and it is easy to see that $p_0 \to 1,\ p_1 \to e^{\frac{\pi i}{3}},\ p_2 \to e^{\frac{2\pi i}{3}}$ as $|\alpha| \to \infty$. Indeed, we have a more precise estimate by Rouché's Theorem as follows: Let $|\alpha| > 2$. Then, we have

$$|p_j - e^{\frac{\pi j i}{3}}| < \frac{1}{|\alpha|^2}, \qquad j = 0,1,2. \tag{2.9}$$

Remark. Since $\eta^3 + \frac{1}{2\alpha^2}\eta - 1 = (\eta - p_0)(\eta - p_1)(\eta - p_2)$, (2.9) yields

$$C_1|\eta - p_0|\langle\eta\rangle^2 \le \Big|\eta^3 + \frac{1}{2\alpha^2}\eta - 1\Big| \le C_2|\eta - p_0|\langle\eta\rangle^2 \qquad \text{for } \eta \in \mathbf{R}, |\alpha| > 2 \tag{2.10}$$

where $C_1,\ C_2$ are independent of η and α. We often use the above inequality when we consider the estimate of $K(\sigma, z)$.

We separate into two cases: $|z| > 64T$ and $|z| \le 64T$.

The case $|z| \le 64T$. It directly follows from the definition of $K(\sigma, z)$ that

$$|K(z,\sigma)| \le \frac{1}{2\pi}\int_{\mathbf{R}} \langle\xi\rangle^{-2\rho} d\xi \le C. \tag{2.11}$$

The case $|z| > 64T$. We note that $|\alpha| = \left|\frac{z}{4\alpha}\right|^{1/3} > \left|\frac{64T}{8T}\right|^{1/3} = 2$. By the identity,

$$\frac{d}{d\xi}(\xi - \alpha p_0)e^{iz\xi - i\sigma\phi(\xi)} = \{1 - 4\sigma i(\xi - \alpha p_0)\prod_{j=0}^{2}(\xi - \alpha p_j)\}e^{iz\xi - i\sigma\phi(\xi)},$$

and integrating by parts, we have

(2.12)

$$\begin{aligned}
&K(\sigma,z)\\
&= -\frac{1}{2\pi}\int_{\mathbf{R}} e^{iz\xi - i\sigma\phi(\xi)}(\xi - \alpha p_0)\frac{d}{d\xi}\left\{\frac{1}{1 - 4\sigma i(\xi - \alpha p_0)\prod_{j=0}^{2}(\xi - \alpha p_j)}\langle\xi\rangle^{-2\rho}\right\}d\xi\\
&=\frac{1}{2\pi}\int_{\mathbf{R}} e^{iz\xi - i\sigma\phi(\xi)}\left\{\frac{8\sigma i\prod_{j=0}^{2}(\xi - \alpha p_j)}{\{1 - 4\sigma i(\xi - \alpha p_0)\prod_{j=0}^{2}(\xi - \alpha p_j)\}^2}\right.\\
&\qquad\qquad\left. + \frac{4\sigma i(\xi - \alpha p_0)^2(2\xi - \alpha p_1 - \alpha p_2)}{\{1 - 4\sigma i(\xi - \alpha p_0)\prod_{j=0}^{2}(\xi - \alpha p_j)\}^2}\right\}\langle\xi\rangle^{-2\rho}d\xi\\
&\quad - \frac{1}{2\pi}\int_{\mathbf{R}} e^{iz\xi - i\sigma\phi(\xi)}\frac{\xi - \alpha p_0}{1 - 4\sigma i(\xi - \alpha p_0)\prod_{j=0}^{2}(\xi - \alpha p_j)}\frac{d}{d\xi}\langle\xi\rangle^{-2\rho}d\xi\\
&\equiv M_1(\sigma, z) + M_2(\sigma, z).
\end{aligned}$$

For $M_1(\sigma, z)$, we apply the inequality (2.10) and separate the result into two terms:

$$\begin{aligned}
|M_1(\sigma, z)| \le& C\int_{\mathbf{R}} \frac{\sigma\alpha^4(\eta - p_0)^2(\eta^2 + 1)}{\{1 + C\sigma\alpha^4|(\eta - p_0)^2(\eta^2 + 1)|\}^2}\langle\alpha\eta\rangle^{-2\rho}\alpha d\eta\\
\le& C\int_{|\eta - p_0| < 1/4} \frac{\sigma\alpha^4(\eta - p_0)^2}{\{1 + C\sigma\alpha^4(\eta - p_0)^2\}^2}\langle\alpha\eta\rangle^{-2\rho}\alpha d\eta\\
&+ C\int_{|\eta - p_0| > 1/4} \frac{\sigma\alpha^4(\eta^2 + 1)^2}{\{1 + C\sigma\alpha^4(\eta^2 + 1)^2\}^2}\langle\alpha\eta\rangle^{-2\rho}\alpha d\eta\\
\equiv& M_{1,1}(\sigma, z) + M_{1,2}(\sigma, z).
\end{aligned} \tag{2.13}$$

Recalling $\rho > 1$ and $4\sigma\alpha^3 = z$, the first term in the right-hand side of (2.13) is estimated as follows:

$$\begin{aligned}
M_{1,1}(\sigma, z) \le& C\langle\alpha\rangle^{-2\rho}\alpha\int_{|\eta| < 1/4} \frac{\sigma\alpha^4\eta^2}{(1 + c\sigma\alpha^4\eta^2)^2}d\eta\\
=& C\sigma^{-1/2}\langle\alpha\rangle^{-2\rho}\alpha^{-1}\\
\le& C\sigma^{-1/2 + (2\rho+1)/3}|z|^{(-2\rho-1)/3}\\
\le& C\langle T\rangle^{-1/2 + (2\rho+1)/3}|z|^{-1-\varepsilon}, \qquad \text{for } \rho > 1.
\end{aligned} \tag{2.14}$$

Similarly for the second term,

$$\begin{aligned}
&M_{1,2}(\sigma,z)\\
&\le C\int_{\mathbf{R}}\frac{\sigma\alpha^4(\eta^2+1)^2}{\{1+C\sigma\alpha^4(\eta^2+1)^2\}^2}\langle\alpha\eta\rangle^{-2\rho}\alpha d\eta\\
&\le C\int_{|\eta|<1}\frac{\sigma\alpha^4}{(1+C\sigma\alpha^4)^2}\langle\alpha\eta\rangle^{-2\rho}\alpha d\eta\\
&\quad + C\int_{|\eta|>1}\frac{\sigma\alpha^4\eta^4}{(1+C\sigma\alpha^4\eta^4)^2}\frac{\alpha}{(\alpha\eta)^{2\rho}}d\eta\\
&\le C\sigma^{-1}\alpha^{-4}+C\sigma^{-1}\alpha^{-3-2\rho}\\
&\le C\langle T\rangle^{2\rho/3}|z|^{-4/3}.
\end{aligned} \tag{2.15}$$

Combining (2.13)-(2.15), we obtain

$$|M_1(\sigma,z)|\le C|z|^{-1-\varepsilon}. \tag{2.16}$$

Next, we estimate $M_2(\sigma,z)$. We apply an integration by parts to have

$$\begin{aligned}
&M_2(\sigma,z)\\
&=\frac{1}{2\pi}\int_{\mathbf{R}}e^{iz\xi-i\sigma\phi(\xi)}(\xi-\alpha p_0)\\
&\qquad\times\frac{d}{d\xi}\left\{\frac{\xi-\alpha p_0}{\{1-4\sigma i(\xi-\alpha p_0)\prod_{j=0}^{2}(\xi-\alpha p_j)\}^2}\frac{d}{d\xi}\langle\xi\rangle^{-2\rho}\right\}d\xi\\
&\le C\int_{\mathbf{R}}\left\{\frac{\alpha(\eta-p_0)}{\{1+C\sigma\alpha^4(\eta-p_0)^2(\eta^2+1)\}^2}\right.\\
&\quad\left.+\frac{\sigma\alpha^5(\eta-p_0)^3(\eta^2+1)}{\{1+C\sigma\alpha^4(\eta-p_0)^2(\eta^2+1)\}^3}\right\}\langle\alpha\eta\rangle^{-2\rho-1}\alpha d\eta\\
&\quad+C\int_{\mathbf{R}}\frac{\alpha^2(\eta-p_0)^2}{\{1+C\sigma\alpha^4(\eta-p_0)^2(\eta^2+1)\}^2}\langle\alpha\eta\rangle^{-2\rho-2}\alpha d\eta\\
&\equiv M_{2,1}(\sigma,z)+M_{2,2}(\sigma,z).
\end{aligned} \tag{2.17}$$

The evaluations of $M_{2,1}(\sigma,z)$ and $M_{2,2}(\sigma,z)$ are similar to the estimates of $M_{1,1}(\sigma,z)$ and $M_{1,2}(\sigma,z)$, and we proceed by decomposing the integral interval into $|\eta-p_0|<1/4$ and $|\eta-p_0|>1/4$. Then we have

$$|M_{2,1}(\sigma,z)|\le C|z|^{-1-2\rho/3}, \tag{2.18}$$

$$|M_{2,2}(\sigma,z)|\le C|z|^{-2}. \tag{2.19}$$

Combining (2.17)-(2.19), we obtain

$$|M_2(\sigma,z)|\le C|z|^{-1-\varepsilon}. \tag{2.20}$$

Combining (2.12), (2.16), (2.20) and (2.11) we have Lemma 2.4. □

3. Crucial nonlinear estimates

In this section, we first state the nonlinear estimates obtained in the paper [14].

Proposition 3.1. *Let* $\nu < 0,\ s > 7/12,\ a < -1/4$ *and* $b > 1/2$. *Then for any* $u_j \in X^{\nu}_{s,b}$ *with supp* $u_j \subset (-T, T),\ T \in (0,1)$, *we have*

$$\|u_1\overline{u}_2u_3\|_{X^{\nu}_{s,a}} \leq C\prod_{j=1}^{3}\|u_j\|_{X^{\nu}_{s,b}}, \tag{3.1}$$

$$\|u_1\overline{u}_2u_3\overline{u}_4u_5\|_{X^{\nu}_{s,a}} \leq C\prod_{j=1}^{5}\|u_j\|_{X^{\nu}_{s,b}}, \tag{3.2}$$

$$\|\partial_x u_1\overline{u}_2\partial_x u_3\|_{X^{\nu}_{s,a}} \leq C\prod_{j=1}^{3}\|u_j\|_{X^{\nu}_{s,b}}, \tag{3.3}$$

$$\|u_1\partial_x\overline{u}_2\partial_x u_3\|_{X^{\nu}_{s,a}} \leq C\prod_{j=1}^{3}\|u_j\|_{X^{\nu}_{s,b}}, \tag{3.4}$$

$$\|u_1\partial_x^2\overline{u}_2u_3\|_{X^{\nu}_{s,a}} \leq C\prod_{j=1}^{3}\|u_j\|_{X^{\nu}_{s,b}}. \tag{3.5}$$

For the proof of Proposition 3.1, see [14]. The next proposition is the crucial estimate in this paper.

Proposition 3.2. *Let* $\nu < 0,\ s > 7/12,\ a < -1/4$ *and* $b > 1/2$. *Then for any* $u_j \in X^{\nu}_{s,b}$ *with supp* $u_j \subset (-T, T),\ T \in (0,1)$, *we have*

$$\|u_1\overline{u}_2\partial_x^2u_3\|_{X_{s,a}} \leq C\prod_{j=1}^{3}\|u_j\|_{X^{\nu}_{s,b}}. \tag{3.6}$$

Remark. Concerning the estimates (3.1)-(3.5), we can show still smaller s. However we do not need those estimates for $s \leq 7/12$ because of the worst term $|u|^2\partial_x^2u$.

Proof of Proposition 3.2. From the definition of $X^{\nu}_{s,b}$ in (1.4) and duality, the inequality (3.6) is reduced to the following estimate: For any $0 \leq f_4 \in L^2_\tau(L^2_\xi)$,

$$\begin{aligned} I \equiv \int_{\Gamma_\tau}\int_{\Gamma_\xi} & \frac{\langle\xi_4\rangle^s|\xi_3|^2}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s}\frac{\prod_{j=1}^{4}f_j(\tau_j,\xi_j)}{\prod_{j=1}^{3}\langle\tau_j+(-1)^j\phi_\nu(\xi_j)\rangle^b\langle\tau_4+\phi_\nu(\xi_4)\rangle^{|a|}} \\ & \leq \prod_{j=1}^{4}\|f_j\|_{L^2_\tau(L^2_\xi)}. \end{aligned} \tag{3.7}$$

Here we set

$$f_j(\tau,\xi) = \langle\xi\rangle^s\langle\tau+(-1)^j\phi_\nu(\xi)\rangle^b|\hat{u}((-1)^j\tau,(-1)^j\xi)| \qquad \text{for } j = 1,2,3,$$

and $\Gamma_\tau,\ \Gamma_\xi$ denote the hyperplanes on $\mathbf{R}^4$:

$$\Gamma_\tau = \{(\tau_1,\tau_2,\tau_3,\tau_4) \in \mathbf{R}^4; \tau_1+\tau_2+\tau_3+\tau_4 = 0\},$$
$$\Gamma_\xi = \{(\xi_1,\xi_2,\xi_3,\xi_4) \in \mathbf{R}^4; \xi_1+\xi_2+\xi_3+\xi_4 = 0\},$$

respectively. We split the domain of integration I into $|\xi_4| \geq 1$ and $|\xi_4| \leq 1$.

The case $|\xi_4| \geq 1$. We only prove (3.7) for the case $7/12 < s < 3/4$. The case $s \geq 3/4$ is shown in the same manner. It will be convenient to define $|\xi_{max}| \geq |\xi_{med}| \geq |\xi_{min}|$ to be the maximum, median and minimum of $|\xi_1|, |\xi_2|, |\xi_3|$, respectively. Then

$$\frac{|\xi_3|^2\langle\xi_4\rangle^s}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s} \leq C\frac{|\xi_3|^2|\xi_4|^{3/4}}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s} \leq C\frac{|\xi_{max}|^2|\xi_4|^{3/4}}{|\xi_{med}|^{1/4}\langle\xi_{min}\rangle^{3s-3/4}|\xi_{max}|^{1/2}}. \tag{3.8}$$

Without loss of generality, we may assume $|\xi_1| = |\xi_{min}|, |\xi_2| = |\xi_{med}|$ and $|\xi_3| = |\xi_{max}|$. Let

$$\hat{\hat{F}}_{j,b}(\tau,\xi) = \frac{f_j(\tau_j,\xi_j)}{\langle\tau_j + (-1)^j\phi_\nu(\xi_j)\rangle^b}, \qquad \text{for } j = 1, \cdots, 4. \tag{3.9}$$

Plugging those inequalities (3.8) into I in (3.7) and applying Lemma 2.1 (2.2), (2.3), (2.4), and Corollary 2.3 (2.7), the integral I restricted to this case is bounded by the Hölder inequality so that

(3.10)

$$\begin{aligned}
&\int_{\Gamma_\tau}\int_{\Gamma_\xi} \frac{f_1(\tau_1,\xi_1)}{\langle\tau_1 - \phi_\nu(\xi_1)\rangle^b\langle\xi_1\rangle^{3s-\frac{3}{4}}} \frac{f_2(\tau_2,\xi_2)}{\langle\tau_2 + \phi_\nu(\xi_2)\rangle^b|\xi_2|^{\frac{1}{4}}} \frac{|\xi_3|^{\frac{3}{2}}f_3(\tau_3,\xi_3)}{\langle\tau_3 - \phi_\nu(\xi_3)\rangle^b} \frac{|\xi_4|^{\frac{3}{4}}f_4(\tau_4,\xi_4)}{\langle\tau_4 + \phi_\nu(\xi_4)\rangle^{|a|}}\\
\leq& C\int_{\mathbf{R}^2} |\langle D_x\rangle^{-3s+\frac{3}{4}}F_{1,b}(t,x)||D_x^{-\frac{1}{4}}F_{2,b}(t,x)||D_x^{\frac{3}{2}}F_{3,b}(t,x)||D_x^{\frac{3}{4}}F_{4,|a|}(t,x)|dtdx\\
\leq& C\|\langle D_x\rangle^{-3s+\frac{3}{4}}F_{1,b}\|_{L_x^2(L_t^\infty)}\|D_x^{-\frac{1}{4}}F_{2,b}\|_{L_x^4(L_t^\infty)}\|D_x^{\frac{3}{2}}F_{3,b}\|_{L_x^\infty(L_t^2)}\|D_x^{\frac{3}{4}}F_{4,|a|}\|_{L_x^4(L_t^2)}\\
\leq& C\prod_{j=1}^{4}\|f_j\|_{L_\tau^2(L_\xi^2)}.
\end{aligned}$$

Here, we used the fact that $-3s + 3/4 < -1$.

The Case $|\xi_4| \leq 1$. This case is simpler than the case $|\xi_4| \geq 1$. By the same manner as the preceding case, we may assume $|\xi_1| \leq |\xi_2| \leq |\xi_3|$. Then, we easily see that

$$\frac{|\xi_3|^2\langle\xi_4\rangle^s}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s} \leq C\frac{|\xi_3|^2}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s} \leq C\frac{|\xi_{max}|^2}{|\xi_{med}|^{1/4}|\xi_{min}|^{1/4}|\xi_{max}|^{1/2}}. \tag{3.11}$$

Combining Lemma 2.1 (2.2), (2.3) and (3.11), the integral I in this case again is estimated by

(3.12)

$$\begin{aligned}
&\int_{\Gamma_\tau}\int_{\Gamma_\xi} \frac{f_1(\tau_1,\xi_1)}{\langle\tau_1 - \phi_\nu(\xi_1)\rangle^b|\xi_1|^{1/4}} \frac{f_2(\tau_2,\xi_2)}{\langle\tau_2 + \phi_\nu(\xi_2)\rangle^b|\xi_2|^{1/4}} \frac{|\xi_3|^{3/2}f_3(\tau_3,\xi_3)}{\langle\tau_3 - \phi_\nu(\xi_3)\rangle^b} f_4(\tau_4,\xi_4)\\
\leq& C\int_{\mathbf{R}^2} |D_x^{-1/4}F_{1,b}(t,x)||D_x^{-1/4}F_{2,b}(t,x)||D_x^{3/2}F_{3,b}(t,x)||F_{4,0}(t,x)|dtdx\\
\leq& C\|D_x^{-1/4}F_{1,b}\|_{L_x^4(L_t^\infty)}\|D_x^{-1/4}F_{2,b}\|_{L_x^4(L_t^\infty)}\|D_x^{3/2}F_{3,b}\|_{L_x^\infty(L_t^2)}\|F_{4,0}\|_{L_x^2(L_t^2)}\\
\leq& C\prod_{j=1}^{4}\|f_j\|_{L_\tau^2(L_\xi^2)}.
\end{aligned}$$

By collecting (3.10) and (3.12), we obtain the desired estimate (3.7). □

Proof of Theorem 1.1. We put $r = \|u_0\|_{H^s}$. Now for $T \in (0,1)$, we define

$$\begin{aligned}\mathcal{B}(r) =& \{u \in \mathcal{S}' \ : \ \|u\|_{X^\nu_{s,b}} \le 2Cr\},\\ \Phi(u) =& \psi(t)W_\nu(t)u_0 - i\psi(t)\int_0^t W_\nu(t-t')\psi_T(t')F(t')dt'.\end{aligned}$$

By similar arguments as in [4], [11] and [12], we have for b, b' with $1/2 < b < b' < 3/4$ and for $u \in \mathcal{B}(r)$,

$$\|\Phi(u)\|_{X^\nu_{s,b}} \le C_0 r + C_1\|\psi_T F\|_{X^\nu_{s,b}} \le C_0 r + C_1 T^{b'-b}\|F\|_{X^\nu_{s,b-1}}. \tag{3.13}$$

Combining Proposition 3.1 with 3.2, the right-hand side of (3.13) is bounded by

$$C_0 r + C_1 T^{b'-b}(\|u\|^3_{X^\nu_{s,b}} + \|u\|^5_{X^\nu_{s,b}}) \le C_0 r + C_1 T^{b'-b}(1+r^2)r^3.$$

Therefore, if we choose $T^{b'-b} \le C_0\{(1+r^2)r^2 C_1\}^{-1}$, then $\Phi(u) \in \mathcal{B}(r)$. Similarly, we can show that Φ is a contraction on $\mathcal{B}(r)$ by choosing $T > 0$ sufficiently small. Therefore Banach's Fixed Point Theorem guarantees the existence of a solution in $\mathcal{B}(r) \subset X^\nu_{s,b}$. Concerning the uniqueness of the solution in the whole of $X^\nu_{s,b}$, we refer to section 4 in [3]. Similar to [3], we introduce the norm:

$$\|u\|_{X_T} = \inf_w \{\|w\|_{X^\nu_{s,b}} : \ w \in X^\nu_{s,b} \text{ such that } u(t) = w(t), \ t \in [-T,T] \text{ in } H^s(\mathbf{R})\}.$$

If $\|u - u'\|_{X_T} = 0$, we have $u(t) = u'(t)$ in $H^s(\mathbf{R})$ for $t \in [-T,T]$. By similar arguments as in [3], we reduce the uniqueness. The persistency of a solution follows directly from the Sobolev embedding $H^b_t(\mathbf{R}; H^s_x(\mathbf{R})) \hookrightarrow C(\mathbf{R}; H^s_x(\mathbf{R}))$. □

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