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STRONG UNIQUENESS FOR THE PLATE EQUATIONS

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ABSTRACT. In this paper we show the strong uniqueness for the plate equations. By using the idea due to Lebeau we transform the given operator to the elliptic operators to which we apply the Carleman estimates given by Alinhac and Lerner.

1. INTRODUCTION

The strong uniqueness for operators whose order is greater than 2 is not well studied except for the iterated Laplace operators (see, for example, Colombini and Grammatico [2] and the references therein). In this paper we consider the strong uniqueness for the plate equations.

Let U be a neighborhood of $\{0\} \times (-\delta, \delta)$, with some $\delta > 0$, in $\mathbb{R}^n_x \times \mathbb{R}_t$, and let $P(x, \partial_x, D_t)$ be a differential operator with coefficients depending only on the variables x given by

$$P(x, \partial_x, D_t) = \rho(x) D_t^2 - \Delta_x^2 + a(x, \partial_x)\Delta_x + b(x, \partial_x)$$

where $\rho(x)$ is a positive smooth function and $a(x, \partial_x)$ and $b(x, \partial_x)$ differential operators with smooth coefficients of orders 1 and 2, respectively. Here we use the notation $D_t = \frac{1}{i}\partial_t$ and $\Delta_x = \sum_{1 \leq j \leq n} \partial_{x_j}^2$.

We say that a function $u(x, t)$ on a neighborhood of $\{0\} \times (-\delta, \delta)$ in $\mathbb{R}^n_x \times \mathbb{R}_t$ is flat on $\{0\} \times (-\delta, \delta)$ when $u(x, t)$ and its derivatives of any order vanish on $\{0\} \times (-\delta, \delta)$. We show in this paper the following theorem.

Theorem 1.1. *Let $u(x, t)$ be a smooth function that is flat on $\{0\} \times (-\delta, \delta)$ and satisfies*

$$Pu = 0$$

on a neighborhood of $\{0\} \times (-\delta, \delta)$ in $\mathbb{R}^n_x \times \mathbb{R}_t$. Then $u(x, t)$ is identically zero on some neighborhood of $\{0\} \times \{0\}$.

The plan of the proof is the following. First using the idea of Lebeau [4] (see also Robbiano [5] and Tataru [6]) we transform the given operator P to the product of second-order elliptic operators. Then, applying the Carleman estimates given by Alinhac and Lerner [1] to our transformed operator, we draw the conclusion of Theorem 1.1. Since the Carleman estimates due to Alinhac and Lerner [1] are crucial for our argument, we give the outline of its proof in the appendix.

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We remark that in our reasoning it is essential that the coefficients are time independent, and we remark also that for plate equations and related equations there are many works on the unique continuation of solutions (see, for example, Isakov [3]).

In the following, we denote by $C^\infty(\Omega)$ the space of all infinitely differentiable functions on Ω and by $C_0^\infty(\Omega)$ the space that consists of compactly supported functions belonging to $C^\infty(\Omega)$. For $f(x,t)$ and $g(x,t)$, we use the notation (f,g) and $\|f\|$ defined by

$$(f,g) = \int f(x,t)\overline{g(x,t)}\,dxdt,$$
$$\|f\| = \sqrt{(f,f)}.$$

We also use the standard notation of multi-indices. The constants appearing in the formulas may be different line by line.

2. Proof of Theorem 1.1

We assume that all the differential operators appearing in this section have smooth coefficients. Let $\phi(t)$ be a function in ${\gamma_0}^{(\alpha)}((-\delta,\delta))$ with $1 < \alpha < 2$ and $\delta > 0$, that is to say, $\phi(t) \in {C_0}^\infty((-\delta,\delta))$ and there exist positive constants K_1 and K_2 such that for any integer $k \geq 0$ we have

$$|\phi^{(k)}(t)| \leq K_1{K_2}^k(k!)^\alpha \text{ for any } t \in (-\delta,\delta). \tag{2.1}$$

Let

$$\begin{aligned}\Psi(z) &= \cosh(\sqrt{z})\\ &= \sum_{n=0}^{\infty}\frac{z^n}{(2n)!}.\end{aligned}$$

Now we define the operator T_ϕ by

$$\begin{aligned}T_\phi(v) &= \langle \Psi(\lambda^2 D_t)v(t), \phi(t)\rangle\\ &= \langle v(t), \Psi(-\lambda^2 D_t)\phi(t)\rangle\\ &= \frac{1}{2\pi}\int_{\mathbb{R}^3} e^{i(s-t)\tau}v(s)\Psi(-\lambda^2\tau)\phi(t)\,dtd\tau ds.\end{aligned} \tag{2.2}$$

Since we see from (2.1) that the Fourier transform of $\phi(t)$ that is denoted by $\hat{\phi}(\tau)$ satisfies

$$|\hat{\phi}(\tau)| \leq C_1 e^{-C_2|\tau|^{\frac{1}{\alpha}}}$$

with positive constants C_1 and C_2, then we see from

$$|\Psi(-\lambda^2\tau)| \leq e^{|\lambda||\tau|^{\frac{1}{2}}}$$

that the integral (2.2) converges for any $v(t) \in C_0^\infty(\mathbb{R})$. Furthermore since

$$\Psi(-\lambda^2 D_t) = \sum_{n=0}^{\infty}\frac{\lambda^{2n}}{(2n)!}(-D_t)^n$$

and $\phi(t) \in {\gamma_0}^{(\alpha)}((-\delta,\delta))$ with $1<\alpha<2$, we see that

$$\Psi(-\lambda^2 D_t)\phi(t)=\sum_{n=0}^{\infty}\frac{\lambda^{2n}}{(2n)!}(-D_t)^n\phi(t)$$

is an entire function of λ with values in $C_0^\infty((-\delta,\delta))$. Then $T_\phi(v)$ is well-defined for any $v\in C^\infty((-\delta,\delta))$. Since it follows from $\Psi(\lambda^2 D_t)=\cosh(\lambda\sqrt{D_t})$ that

$$\partial_\lambda{}^2\Psi(\lambda^2 D_t)=D_t\Psi(\lambda^2 D_t),$$

then we see that

$$\partial_\lambda{}^2 T_\phi(v)=T_\phi(D_t v).$$

We remark that

$$T_\phi(v)|_{\lambda=0}=\int_{\mathbb{R}}\phi(t)v(t)\,dt. \tag{2.3}$$

Let $u(x,t)\in C^\infty(\{(x,t)\in\mathbb{R}^n_x\times\mathbb{R}_t \mid \ |x|<r_1 \text{ and } |t|<\delta\ \})$ with $r_1,\delta>0$ flat on $\{0\}\times(-\delta,\delta)$ and satisfying $Pu=0$ on $\{(x,t)\in\mathbb{R}^n_x\times\mathbb{R}_t \mid \ |x|<r_1 \text{ and } |t|<\delta\}$. We define $U(x,\lambda)$ by

$$U(x,\lambda)=T_\phi(u(x,\cdot)). \tag{2.4}$$

Then $U(x,\lambda)$ is an entire function of λ with values in $C^\infty(\{x\in\mathbb{R}^n_x \mid \ |x|<r_1\})$ and satisfies

$$\rho(x)\partial_\lambda^4 U-\Delta_x^2 U+a(x,\partial_x)\Delta_x U+b(x,\partial_x)U=0. \tag{2.5}$$

Since $\rho(x)>0$, we see that for a first-order differential operator $A(x,\partial_x)$,

$$\begin{aligned}(2.6)\quad &\big(\rho^{\frac12}(x)\partial_\lambda^2+\Delta_x-A(x,\partial_x)\big)\big(\rho^{\frac12}(x)\partial_\lambda^2-\Delta_x+A(x,\partial_x)\big)\\ &=\rho(x)\partial_\lambda^4-\Delta_x^2+(2A(x,\partial_x)+[\Delta_x,\rho^{\frac12}(x)]\rho^{-\frac12}(x))\Delta_x\\ &+[\Delta_x-A(x,\partial_x),\rho^{\frac12}(x)]\rho^{-\frac12}(x)\big(\rho^{\frac12}(x)\partial_\lambda^2-\Delta_x+A(x,\partial_x)\big)+C(x,\partial_x)\end{aligned}$$

with a second-order differential operator $C(x,\partial_x)$.

Hence by picking

$$A(x,\partial_x)=\frac12\big(a(x,\partial_x)-[\Delta_x,\rho^{\frac12}(x)]\rho^{-\frac12}(x)\big)$$

and by setting

$$V(x,\lambda)=\rho^{\frac12}(x)\partial_\lambda^2 U(x,\lambda)-(\Delta_x-A(x,\partial_x))U(x,\lambda),$$

we see from (2.5) and (2.6) that $U(x,\lambda)$ and $V(x,\lambda)$ satisfy the system of equations

$$\begin{aligned}&\rho^{\frac12}(x)\partial_\lambda^2 U-(\Delta_x-A(x,\partial_x))U=V,\\ &\rho^{\frac12}(x)\partial_\lambda^2 V+(\Delta_x-A(x,\partial_x))V=B_1(x,\partial_x)V+B_2(x,\partial_x)U\end{aligned}$$

where $B_j(x,\partial_x)$ is a differential operator of order j.

Since $U(x,\lambda)$ and $V(x,\lambda)$ are entire functions of λ, we get

$$\begin{aligned}\partial_\lambda^2 U(x,t_1+it_2)&=-\partial_{t_2}^2 U(x,t_1+it_2)=-\partial_{t_1}^2 U(x,t_1+it_2)-2\partial_{t_2}^2 U(x,t_1+it_2),\\ \partial_\lambda^2 V(x,t_1+it_2)&=\partial_{t_1}^2 V(x,t_1+it_2)=2\partial_{t_1}^2 V(x,t_1+it_2)+\partial_{t_2}^2 V(x,t_1+it_2).\end{aligned}$$

Therefore, by putting

$$\begin{aligned}(2.7)\qquad W_1(x,t_1,t_2)&=U(x,t_1+it_2),\\ W_2(x,t_1,t_2)&=V(x,t_1+it_2),\end{aligned}$$

we see that $W_1(x, t_1, t_2)$ and $W_2(x, t_1, t_2)$ satisfy the following elliptic system:

$$\begin{aligned} &\rho^{\frac{1}{2}}(x)\partial_{t_1}^2 W_1 + 2\rho^{\frac{1}{2}}(x)\partial_{t_2}^2 W_1 + (\Delta_x - A(x, \partial_x))W_1 = -W_2, \\ &2\rho^{\frac{1}{2}}(x)\partial_{t_1}^2 W_2 + \rho^{\frac{1}{2}}(x)\partial_{t_2}^2 W_2 + (\Delta_x - A(x, \partial_x))W_2 = B_1(x, \partial_x)W_2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + B_2(x, \partial_x)W_1 \end{aligned}$$

on $\{(x, t_1, t_2) \in \mathbb{R}_x^n \times \mathbb{R}_t^2 \mid |x| < r_1\}$.

Since $u(x, t)$ is flat on $\{0\} \times (-\delta, \delta)$, $U(x, \lambda)$ is flat on $\{0\} \times \mathbb{C}$. Then $W_1(x, t_1, t_2)$ and $W_2(x, t_1, t_2)$ are flat on $\{0\} \times \mathbb{R}_t^2$.

Now we apply the Carleman estimates due to Alinhac and Lerner [1] to our system. First of all we make the change of coordinates

$$x = (1 - |T|^2)X, \quad t = T.$$

Then we see that

$$(1 - |t|^2)\frac{\partial}{\partial x_j} = \frac{\partial}{\partial X_j}$$

and

$$(1 - |t|^2)\frac{\partial}{\partial t_k} = (1 - |T|^2)\frac{\partial}{\partial T_k} + 2T_k \sum_{j=1}^{n} X_j \frac{\partial}{\partial X_j}.$$

Hence we see that $w_1(X, T)$ and $w_2(X, T)$, given by

$$w_1(X, T) = \begin{cases} W_1((1 - |T|^2)X, T) & |T| \le 1, \\ 0 & |T| \ge 1, \end{cases} \tag{2.8}$$

$$w_2(X, T) = \begin{cases} W_2((1 - |T|^2)X, T) & |T| \le 1, \\ 0 & |T| \ge 1, \end{cases}$$

satisfy

$$\begin{aligned} &\rho^{\frac{1}{2}}(0)(1 - |T|^2)^2\big(\partial_{T_1}^2 + 2\partial_{T_2}^2\big)w_1 \\ &\qquad + \Delta_X w_1 + A_1(X, T, \partial_X, \partial_T)w_1 = -(1 - |T|^2)^2 w_2, \end{aligned} \tag{2.9}$$

$$\begin{aligned} &\rho^{\frac{1}{2}}(0)(1 - |T|^2)^2\big(2\partial_{T_1}^2 + \partial_{T_2}^2\big)w_2 \\ &\qquad + \Delta_X w_2 + A_2(X, T, \partial_X, \partial_T)w_2 \\ &\qquad\qquad = C_1(X, T, \partial_X, \partial_T)w_2 + C_2(X, T, \partial_X, \partial_T)w_1 \end{aligned} \tag{2.10}$$

where $A_j(X, T, \partial_X, \partial_T)$ and $C_j(X, T, \partial_X, \partial_T)$ are differential operators given by

$$A_j(X, T, \partial_X, \partial_T) = \sum_{|\alpha|+|\beta|\le 2} a_{j,\alpha,\beta}(X, T)(1 - |T|^2)^{|\beta|}\partial_x^\alpha \partial_T^\beta$$

with the coefficients $a_{j,\alpha,\beta}(X, T)$ that satisfy $a_{j,\alpha,\beta}(0, T) = 0$ if $|\alpha| + |\beta| = 2$, and

$$C_j(X, T, \partial_X, \partial_T) = \sum_{|\alpha|+|\beta|\le j} c_{j,\alpha,\beta}(X, T)(1 - |T|^2)^{|\beta|}\partial_x^\alpha \partial_T^\beta.$$

We denote by Q_1 and Q_2 the operators appearing in (2.9) and (2.10):

$$\begin{aligned} Q_1 &= \rho^{\frac{1}{2}}(0)(1 - |T|^2)^2\big(\partial_{T_1}^2 + 2\partial_{T_2}^2\big) + \Delta_X + A_1(X, T, \partial_X, \partial_T), \\ Q_2 &= \rho^{\frac{1}{2}}(0)(1 - |T|^2)^2\big(2\partial_{T_1}^2 + \partial_{T_2}^2\big) + \Delta_X + A_2(X, T, \partial_X, \partial_T). \end{aligned}$$

Recall the Carleman estimates due to Alinhac-Lerner [1] for Q_1 and Q_2.

Lemma 2.1 (Alinhac-Lerner [1]). *There exist positive constants $r_0 \in (0, 1/10)$ and $\gamma_0 > 0$ such that we have for any $w(X,T) \in C^\infty(\mathbb{R}^n_X \times \mathbb{R}^2_T)$ supported in $\{(X,T) \in \mathbb{R}^n_X \times \mathbb{R}^2_T \mid |X| \le r_0$ and $|T| \le 1\}$ and flat on $X = 0$,*

$$\gamma^{-1}\Big(\sum_{k+|\alpha|+|\beta|=2} \gamma^{2k}\|\Phi^{-\gamma}|\log r|^{-1} r^{|\alpha|+|\beta|}(1-|T|^2)^{|\beta|}\partial_X^\alpha \partial_T^\beta w\|^2\Big) \le C\|r^2\Phi^{-\gamma}Q_j w\|^2 \tag{2.11}$$

for $\gamma > \gamma_0$ and $j = 1, 2$, where $r = |X|$ and $\Phi = |\log r| r$.

We give a sketch of the proof of the estimates above in the appendix.

Taking a function $\chi(X) \in C^\infty(\mathbb{R}^n_X)$ satisfying, with $r_2 = \min\{r_0, r_1/2\}$,

$$\chi(X) = \begin{cases} 0 & |X| \ge r_2, \\ 1 & |X| \le \frac{r_2}{2}, \end{cases}$$

we obtain

$$Q_1\chi(X)w_1 = -(1-|T|^2)^2\chi(X)w_2 + f(X,T),$$
$$Q_2\chi(X)w_2 = C_1(X,T,\partial_X,\partial_T)\chi(X)w_2 + C_2(X,T,\partial_X,\partial_T)\chi(X)w_1 + g(X,T)$$

where $f(X,T) = 0$ and $g(X,T) = 0$ if $|X| \le \frac{r_2}{2}$.

Then we see from (2.11) that

$$\gamma^{-1}\Big(\sum_{k+|\alpha|+|\beta|=2} \gamma^{2k}\|\Phi^{-\gamma}|\log r|^{-1} r^{|\alpha|+|\beta|}(1-|T|^2)^{|\beta|}\partial_X^\alpha \partial_T^\beta \chi(X)w_1\|^2\Big) \le 2C\big(\|r^2\Phi^{-\gamma}\chi(X)w_2\|^2 + \|r^2\Phi^{-\gamma}f(X,T)\|^2\big)$$

and

$$\begin{aligned}\gamma^{-1}\Big(\sum_{k+|\alpha|+|\beta|=2} \gamma^{2k}\|\Phi^{-\gamma}|\log r|^{-1} r^{|\alpha|+|\beta|}(1-|T|^2)^{|\beta|}\partial_X^\alpha \partial_T^\beta \chi(X)w_2\|^2\Big) \\ \le C\Big(\sum_{|\alpha|+|\beta|\le 1} \|\Phi^{-\gamma}r^2(1-|T|^2)^{|\beta|}\partial_X^\alpha\partial_T^\beta\chi(X)w_2\|^2 \\ + \sum_{|\alpha|+|\beta|\le 2} \|\Phi^{-\gamma}r^2(1-|T|^2)^{|\beta|}\partial_X^\alpha\partial_T^\beta\chi(X)w_1\|^2 \\ + \|r^2\Phi^{-\gamma}g(X,T)\|^2\Big),\end{aligned}$$

from which we draw

$$\gamma^5\|\Phi^{-\gamma}|\log r|^{-1}\chi(X)w_1\|^2 + \gamma^3\|\Phi^{-\gamma}|\log r|^{-1}\chi(X)w_2\|^2 \le C(\gamma^2\|r^2\Phi^{-\gamma}f(X,T)\|^2 + \|r^2\Phi^{-\gamma}g(X,T)\|^2)$$

for $\gamma > \gamma_1$ with some $\gamma_1 > 0$. Then we obtain

$$w_1(X,T) = 0 \text{ and } w_2(X,T) = 0 \quad \text{for} \quad |X| \le \frac{r_2}{3}\ .$$

Then $w_1(X, 0) = 0$ for $|X| \le \frac{r_2}{3}$. Since, from (2.7) and (2.8) we get $U(X, 0) = W_1(X, 0) = w_1(X, 0)$, then we see that $U(x, 0) = 0$ for $|x| \le \frac{r_2}{3}$. Hence from (2.3) and (2.4),

$$\int_{\mathbb{R}} u(x,t)\phi(t)\,dt = 0 \text{ for } |x| \le \frac{r_2}{3}. \tag{2.12}$$

Since (2.12) is valid for any $\psi(t) \in \gamma_0^{(\alpha)}((-\delta,\delta))$, then we get

$$u(x,t) = 0 \text{ for } |x| \le \frac{r_2}{3} \text{ and } |t| < \delta.$$

Then the proof of Theorem 1.1 is completed.

Remark 2.1. We note that the arguments above can be applied to the heat and Schrödinger coupled system

$$\rho_1(x)\partial_t u - \Delta_x u = a_1(x,\partial_x)u + a_2(x,\partial_x)v,$$
$$i\rho_2(x)\partial_t v - \Delta_x v = b_1(x,\partial_x)u + b_2(x,\partial_x)v$$

with some positive functions $\rho_j(x)$ $(j = 1,2)$ and first-order differential operators $a_j(x,\partial_x)$ and $b_j(x,\partial_x)$ $(j = 1,2)$.

3. Appendix

In this appendix, we sketch the proof of the Carleman estimates following Alinhac and Lerner [1] (see also Tataru [7], Ch. 3, Sec. 6).

For a nonnegative function $p(x)$, when a function $a(x,t) \in C^\infty(U)$ satisfies, for any α and β,

$$|x|^{|\alpha|}|\partial_x^\alpha \partial_t^\beta a(x,t)| \le C_{\alpha,\beta}\, p(x) \text{ on } U,$$

we say that $a(x,t) \in p(x)\mathcal{O}$, where U is an open subset of $\mathbb{R}^n_x\backslash\{0\} \times \mathbb{R}^l_t$.

Recall that Δ_x is the laplacian $\sum_{1\le j\le n} \partial_{x_j}^2$. Let Σ be an elliptic operator given by

$$\Sigma = \sum_{1\le j,k\le l} \partial_{t_j}(1-|t|^2)^2 a_{j,k}(t)\partial_{t_k}$$

where the coefficients $a_{j,k}(t)$ belonging to $C^\infty(V)$ with some neighborhood V of $\{t \in \mathbb{R}^l \mid |t| \le 1\}$ satisfy $a_{j,k}(t) = a_{k,j}(t)$ and

$$\sum_{1\le j,k\le l} \Re a_{j,k}(t)\eta_j\eta_k \ge C_0|\eta|^2 \qquad \text{for any } \eta \in \mathbb{R}^l \text{ and any } t \in V \tag{3.1}$$

with some positive constant C_0.

Let $\Phi(r)$ and μ be

$$\Phi(r) = (-\log r)^\theta r, \tag{3.2}$$
$$\mu = \frac{r\Phi'(r)}{\Phi(r)} = 1 + \theta(\log r)^{-1}$$

where $r = |x|$ and θ is a positive constant that will be determined later. We note that when $0 < r_0 < 1$ and $0 < \theta \le |\log r_0|/2$ we see that for $r \in (0, r_0]$,

$$\frac{1}{2} \le \mu < 1. \tag{3.3}$$

Since

$$\Phi(r)^{-\gamma}\frac{\partial}{\partial x_k}\Phi(r)^\gamma = \frac{\partial}{\partial x_k} + \gamma\frac{\mu}{r}\frac{x_k}{r},$$

we see that for any $\gamma > 0$,

$$\Phi(r)^{-\gamma}\Delta_x\Phi(r)^\gamma = \Delta_x + \gamma^2\frac{\mu^2}{r^2} + \gamma\left(\frac{\mu}{r}\frac{\partial}{\partial r} + \frac{\partial}{\partial r}\frac{\mu}{r} + \frac{(n-1)\mu}{r^2}\right).$$

Let $B(x, t, \partial_x, \partial_t)$ be a second-order operator with C^∞-coefficients vanishing when $x = 0$:

$$B(x, t, x, \partial_x, \partial_t) = \sum_{|\alpha|+|\beta|=2} b_{\alpha,\beta}(x, t)(1 - |t|^2)^{|\beta|} \partial_x^\alpha \partial_t^\beta$$

where $b_{\alpha,\beta}(0, t) = 0$. Then we see that the operator $B(x, t, \gamma, \partial_x, \partial_t)$ given by

$$B(x, t, \gamma, \partial_x, \partial_t) = \Phi(r)^{-\gamma} B(x, t, \partial_x, \partial_t)\Phi(r)^\gamma$$

can be written in the following way:

$$B(x, t, \gamma, \partial_x, \partial_t) = \sum_{|\alpha|+|\beta|+k\le 2} b_{\alpha,\beta,k}(x, t)(1 - |t|^2)^{|\beta|} \gamma^k \partial_x^\alpha \partial_t^\beta \tag{3.4}$$

where $b_{\alpha,\beta,k}(x, t) \in r^{|\alpha|+|\beta|-1}\mathcal{O}$.

Let L be an operator given by

$$L = \Delta_x + \Sigma + B(x, t, \partial_x, \partial_t).$$

We denote by L_γ, ${L_\gamma}^s$ and ${L_\gamma}^a$ the operators defined by

$$L_\gamma = \Phi(r)^{-\gamma} L\Phi(r)^\gamma,\ {L_\gamma}^s = \frac{1}{2}(L_\gamma + L_\gamma^*) \text{ and } {L_\gamma}^a = \frac{1}{2}(L_\gamma - L_\gamma^*) \tag{3.5}$$

where L_γ^* is a formal adjoint of L_γ, that is to say, ${L_\gamma}^s$ and ${L_\gamma}^a$ are the symmetric and skew-symmetric parts of L_γ, respectively.

Therefore we see that

$$\begin{aligned} {L_\gamma}^s &= \Delta_x + \gamma^2 \frac{\mu}{r^2}^2 + \Sigma^s + B^s(x, t, \gamma, \partial_x, \partial_t), \\ {L_\gamma}^a &= \gamma\left(\frac{\mu}{r}\frac{\partial}{\partial r} + \frac{\partial}{\partial r}\frac{\mu}{r} + \frac{(n-1)\mu}{r^2}\right) + \Sigma^a + B^a(x, t, \gamma, \partial_x, \partial_t), \end{aligned} \tag{3.6}$$

where Σ^s and Σ^a are the symmetric and skew-symmetric parts of Σ, respectively, and similar notation is used for $B(x, t, \gamma, \partial_x, \partial_t)$.

Then we see that for any $u(x, t)$ that is supported in $\{(x, t) \in \mathbb{R}_x^n \times \mathbb{R}_t^l \mid |x| \le \frac{1}{2} \text{ and } |t| \le 1\}$ and flat on the plane $x = 0$,

$$\begin{aligned} \|r\mu^{-\frac{1}{2}} L_\gamma(r\mu^{-\frac{1}{2}}u)\|^2 &= \|r\mu^{-\frac{1}{2}} {L_\gamma}^s(r\mu^{-\frac{1}{2}}u)\|^2 \\ &\quad + \|r\mu^{-\frac{1}{2}} {L_\gamma}^a(r\mu^{-\frac{1}{2}}u)\|^2 + ([r\mu^{-\frac{1}{2}}{L_\gamma}^s r\mu^{-\frac{1}{2}}, r\mu^{-\frac{1}{2}}{L_\gamma}^a r\mu^{-\frac{1}{2}}]u, u). \end{aligned}$$

We note that

$$r\mu^{-\frac{1}{2}}\left(\frac{\mu}{r}\frac{\partial}{\partial r} + \frac{\partial}{\partial r}\frac{\mu}{r} + \frac{(n-1)\mu}{r^2}\right)r\mu^{-\frac{1}{2}} = 2r\frac{\partial}{\partial r} + n,$$

$$[r\Delta_x r + \gamma^2\mu^2, 2r\frac{\partial}{\partial r} + n] = -4\gamma^2 r\mu\mu'$$

and, since $[\Sigma^s, r] = 0$,

$$[r\Sigma^s r, 2r\frac{\partial}{\partial r} + n] = -4r^2\Sigma^s.$$

Therefore, noting that $[\mu^{-1/2}, 2r\frac{\partial}{\partial r} + n] = \dfrac{r\mu'}{\mu}\mu^{-1/2}$, we see that $[r\mu^{-\frac{1}{2}}{L_\gamma}^s r\mu^{-\frac{1}{2}}, \gamma(2r\frac{\partial}{\partial r} + n)]$ is equal to

$$\gamma\frac{r\mu'}{\mu} r\mu^{-1/2}{L_\gamma}^s r\mu^{-1/2} + \gamma r\mu^{-1/2}{L_\gamma}^s r\mu^{-1/2}\frac{r\mu'}{\mu} - 4\gamma^3 r\mu' - 4\gamma r^2\mu^{-1}\Sigma^s + \tilde{B}^s,$$

where

$$\tilde{B}^s = \mu^{-\frac{1}{2}}[rB^s(x,t,\gamma,\partial_x,\partial_t)r, \gamma(2r\frac{\partial}{\partial r}+n)]\mu^{-\frac{1}{2}}$$
$$= \sum_{|\alpha|+|\beta|+k\le 2} b^s{}_{\alpha,\beta,k}(x,t)(1-|t|^2)^{|\beta|}\gamma^{k+1}\partial_x^\alpha\partial_t^\beta \quad (3.7)$$

with $b^s{}_{\alpha,\beta,k}(x,t) \in r^{1+|\beta|+|\alpha|}\mathcal{O}$.

Hence

$$\|r\mu^{-\frac{1}{2}}L_\gamma{}^s(r\mu^{-\frac{1}{2}}u)\|^2 + ([r\mu^{-\frac{1}{2}}L_\gamma{}^s r\mu^{-\frac{1}{2}}, \gamma(2r\frac{\partial}{\partial r}+n)]u,u) \quad (3.8)$$
$$= \|r\mu^{-\frac{1}{2}}L_\gamma{}^s(r\mu^{-\frac{1}{2}}u) + \gamma\frac{r\mu'}{\mu}u\|^2$$
$$+ ((-4\gamma^3 r\mu' - (\gamma\frac{r\mu'}{\mu})^2)u,u) + (-4\gamma r^2\mu^{-1}\Sigma^s u,u) + (\tilde{B}^s u,u).$$

Since $r\mu' = -\theta(\log r)^{-2}$, we see from (3.3) that when θ is in $(0, |\log r_0|/2)$ with $r_0 \in (0,1)$, there exists a constant γ_1, which may depend on r_0, such that for $\gamma \ge \gamma_1$ we have

$$((-4\gamma^3 r\mu' - (\gamma\frac{r\mu'}{\mu})^2)u,u) \ge 2\gamma^3\theta\|(\log r)^{-1}u\|^2 \quad (3.9)$$

for any $u(x,t)$ that vanishes for $|x| > r_0$. On the other hand, from (3.1) and (3.3) we see that there exist constants C_1 and C_2 such that for any $u(x,t)$ that vanishes when $|x| > r_0$ or $|t| > 1$, if θ is in $(0, |\log r_0|/2)$ with $r_0 \in (0,1)$, we obtain

$$(-4\gamma r^2\mu^{-1}\Sigma^s u,u) \ge \gamma(C_1\|r(1-|t|^2)\nabla_t u\|^2 - C_2\|ru\|^2). \quad (3.10)$$

Since we obtain

$$|(\tilde{B}^s u,u)| \le C_3\gamma(\sum_{k+|\alpha|+|\beta|\le 1}\gamma^k\|r^{|\alpha|+|\beta|+\frac{1}{2}}(1-|t|^2)^{|\beta|}\partial_x^\alpha\partial_t^\beta u\|)^2$$

from (3.7), we see from (3.8), (3.9) and (3.10) that

$$\|r\mu^{-\frac{1}{2}}L_\gamma{}^s(r\mu^{-\frac{1}{2}}u)\|^2 + ([r\mu^{-\frac{1}{2}}L_\gamma{}^s r\mu^{-\frac{1}{2}}, \gamma(2r\frac{\partial}{\partial r}+n)]u,u)$$
$$\ge \|r\mu^{-\frac{1}{2}}L_\gamma{}^s(r\mu^{-\frac{1}{2}}u) + \gamma\frac{r\mu'}{\mu}u\|^2$$
$$+ 2\gamma^3\theta\|(\log r)^{-1}u\|^2 + \gamma(C_1\|r(1-|t|^2)\nabla_t u\|^2 - C_2\|ru\|^2)$$
$$- C_3\gamma(\sum_{k+|\alpha|+|\beta|\le 1}\gamma^k\|r^{|\alpha|+|\beta|+\frac{1}{2}}(1-|t|^2)^{|\beta|}\partial_x^\alpha\partial_t^\beta u\|)^2.$$

Now we consider the term $[r\mu^{-\frac{1}{2}}L_\gamma{}^s r\mu^{-\frac{1}{2}}, r\mu^{-\frac{1}{2}}\Sigma^a r\mu^{-\frac{1}{2}}]$. First note that

$$[r\mu^{-\frac{1}{2}}\Delta_x r\mu^{-\frac{1}{2}}, r\mu^{-\frac{1}{2}}\Sigma^a r\mu^{-\frac{1}{2}}] = \big((2r\frac{\partial}{\partial r}+n)(\frac{2}{\mu}-\frac{r\mu'}{\mu^2}) + b(r)\big)r^2\mu^{-1}\Sigma^a \quad (3.11)$$

where $b(r) \in 1\mathcal{O}$. Noting that

$$r\mu^{-\frac{1}{2}}L_\gamma{}^a r\mu^{-\frac{1}{2}} = \gamma(2r\frac{\partial}{\partial r}+n) + r^2\mu^{-1}\Sigma^a + \tilde{B}^a,$$

where

$$\tilde{B}^a = \sum_{|\alpha|+|\beta|+k\le 2} b^a{}_{\alpha,\beta,k}(x,t)(1-|t|^2)^{|\beta|}\gamma^k \partial_x^\alpha \partial_t^\beta \tag{3.12}$$

with $b^a{}_{\alpha,\beta,k}(x,t) \in r^{1+|\alpha|+|\beta|}\mathcal{O}$, we see that

$$\begin{aligned}(3.13)\quad &\|r\mu^{-\frac{1}{2}} L_\gamma{}^a(r\mu^{-\frac{1}{2}}u)\|^2 \\ &\qquad + \Re((2r\frac{\partial}{\partial r}+n)(\frac{2}{\mu}-\frac{r\mu'}{\mu^2})r^2\mu^{-1}\Sigma^a u,u) \\ &= \|\Big(\gamma(2r\frac{\partial}{\partial r}+n) + (1-\frac{1}{2\gamma}(\frac{2}{\mu}-\frac{r\mu'}{\mu^2}))\big(r^2\mu^{-1}\Sigma^a+\tilde{B}^a\big)\Big)u\|^2 \\ &\qquad - \Re((2r\frac{\partial}{\partial r}+n)(\frac{2}{\mu}-\frac{r\mu'}{\mu^2})\tilde{B}^a u,u) \\ &\qquad + \|(r^2\mu^{-1}\Sigma^a+\tilde{B}^a)u\|^2 - \|(1-\frac{1}{2\gamma}(\frac{2}{\mu}-\frac{r\mu'}{\mu^2}))(r^2\mu^{-1}\Sigma^a+\tilde{B}^a)u\|^2.\end{aligned}$$

Since for $r \in (0, r_0]$, $\theta \in (0, |\log r_0|/2)$ and $\gamma \ge \gamma_2$ with some $\gamma_2 > 0$ we have $0 < \frac{1}{2\gamma}(\frac{2}{\mu}-\frac{r\mu'}{\mu^2}) < 1$, we see from (3.12) that the right-hand side of (3.13) is not smaller than

$$-C(\sum_{|\alpha|\le 1}\|r^{|\alpha|+\frac{1}{2}}\partial_x^\alpha u\|)(\sum_{k+|\alpha|+|\beta|\le 2}\gamma^k\|r^{|\alpha|+|\beta|+\frac{1}{2}}(1-|t|^2)^{|\beta|}\partial_x^\alpha\partial_t^\beta u\|).$$

Since $b(r)$ is real-valued and Σ^a is skew-symmetric, we see from $[r, \Sigma^a] = 0$ that $\Re(b(r)r^2\mu^{-1}\Sigma^a u, u) = 0$. Then the left-hand side of (3.13) is equal to

$$\|r\mu^{-\frac{1}{2}}L_\gamma{}^a(r\mu^{-\frac{1}{2}}u)\|^2 + ([r\mu^{-\frac{1}{2}}\Delta_x r\mu^{-\frac{1}{2}}, r\mu^{-\frac{1}{2}}\Sigma^a r\mu^{-\frac{1}{2}}]u,u).$$

Since we see from (3.4) that

$$\begin{aligned}&|([r^2\mu^{-1}\Sigma^s, r^2\mu^{-1}\Sigma^a]u,u)| + |([r\mu^{-\frac{1}{2}}B^s r\mu^{-\frac{1}{2}}, r^2\mu^{-1}\Sigma^a]u,u)| \\ &\qquad \le C(\sum_{|\beta|\le 1}\|r^{|\beta|+\frac{1}{2}}(1-|t|^2)^{|\beta|}\partial_t^\beta u\|) \\ &\qquad\qquad \times (\sum_{k+|\alpha|+|\beta|\le 2}\gamma^k\|r^{|\alpha|+|\beta|+\frac{1}{2}}(1-|t|^2)^{|\beta|}\partial_x^\alpha\partial_t^\beta u\|),\end{aligned}$$

we obtain the following:

$$\begin{aligned}&\|r\mu^{-\frac{1}{2}}L_\gamma{}^a(r\mu^{-\frac{1}{2}}u)\|^2 + ([r\mu^{-\frac{1}{2}}L_\gamma{}^s r\mu^{-\frac{1}{2}}, r\mu^{-\frac{1}{2}}\Sigma^a r\mu^{-\frac{1}{2}}]u,u) \\ &\qquad \ge -C(\sum_{|\alpha|+|\beta|\le 1}\|r^{|\alpha|+|\beta|+\frac{1}{2}}(1-|t|^2)^{|\beta|}\partial_x^\alpha\partial_t^\beta u\|) \\ &\qquad\qquad \times (\sum_{k+|\alpha|+|\beta|\le 2}\gamma^k\|r^{|\alpha|+|\beta|+\frac{1}{2}}(1-|t|^2)^{|\beta|}\partial_x^\alpha\partial_t^\beta u\|).\end{aligned}$$

Finally from (3.4) we see that

$$
\begin{aligned}
&|([r\mu^{-\frac{1}{2}}L_\gamma{}^s r\mu^{-\frac{1}{2}}, r\mu^{-\frac{1}{2}}B^a r\mu^{-\frac{1}{2}}]u,u)| \\
&\qquad \le C\Big(\sum_{|\alpha|+|\beta|+k\le 1} \gamma^k \|r^{|\alpha|+|\beta|+\frac{1}{2}}(1-|t|^2)^{|\beta|}\partial_x^\alpha\partial_t^\beta u\|\Big) \\
&\qquad\qquad \times\Big(\sum_{k+|\alpha|+|\beta|\le 2} \gamma^k \|r^{|\alpha|+|\beta|+\frac{1}{2}}(1-|t|^2)^{|\beta|}\partial_x^\alpha\partial_t^\beta u\|\Big).
\end{aligned}
$$

From (3.6) and the ellipticity (3.1), we obtain

$$
\begin{aligned}
&\|(\log r)^{-1}\big(r\mu^{-\frac{1}{2}}L_\gamma{}^s(r\mu^{-\frac{1}{2}}u) + \gamma\frac{r\mu'}{\mu}u\big)\|^2 + C_1\gamma^4\|(\log r)^{-1}u\|^2 \\
&\qquad \ge C_0 \sum_{|\alpha|+|\beta|+k=2} (\gamma^k\|(\log r)^{-1} r^{|\alpha|+|\beta|}(1-|t|^2)^{|\beta|}\partial_x^\alpha\partial_t^\beta u\|)^2
\end{aligned}
$$

for $\gamma > \gamma_3$ with some $\gamma_3 > 0$ and some positive constants C_0 and C_1. Therefore by taking a small enough $r_0 > 0$ and picking $\theta = 1$, we see that for any $u(x,t)$ supported in $\{(x,t) \in \mathbb{R}_x^n \times \mathbb{R}_t{}^l \mid |x| \le r_0 \text{ and } |t| \le 1\}$ and flat on $x = 0$,

$$
\begin{aligned}
&\|r\mu^{-\frac{1}{2}}L_\gamma(r\mu^{-\frac{1}{2}}u)\|^2 \\
&\qquad \ge C_0 \sum_{|\alpha|+|\beta|+k=2} \gamma^{-1}(\gamma^k\|(\log r)^{-1} r^{|\alpha|+|\beta|}(1-|t|^2)^{|\beta|}\partial_x^\alpha\partial_t^\beta u\|)^2
\end{aligned}
$$

for $\gamma > \gamma_0$ with some positive constants γ_0 and C_0. Then from (3.5) we obtain the following Carleman estimates.

Lemma 3.1 (Alinhac-Lerner [1]). *There exist positive constants r_0, γ_0 and C_0 such that for any $u(x,t)$ supported in $\{(x,t) \in \mathbb{R}_x^n \times \mathbb{R}_t{}^l \mid |x| \le r_0 \text{ and } |t| \le 1\}$ and flat on $x = 0$,*

$$
\begin{aligned}
&\|\Phi(r)^{-\gamma} r^2 Lu\|^2 \\
&\qquad \ge C_0 \sum_{|\alpha|+|\beta|+k=2} \gamma^{-1}(\gamma^k\|\Phi(r)^{-\gamma}(\log r)^{-1} r^{|\alpha|+|\beta|}(1-|t|^2)^{|\beta|}\partial_x^\alpha\partial_t^\beta u\|)^2
\end{aligned}
$$

for $\gamma > \gamma_0$ where $\Phi(r)$ is given by (3.2) with $\theta = 1$.

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