

A COUNTEREXAMPLE TO A CONJECTURE ON FLAT BILINEAR FORMS

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ABSTRACT. We provide a counterexample to a conjecture on the dimension of the nullity of a flat symmetric bilinear form.

The algebraic theory of flat bilinear forms was developed by J. D. Moore after the seminal work of E. Cartan on exterior orthogonal quadratic forms as a tool to treat the “rigidity problem” for submanifolds; see [5] and the references therein. An \mathbb{R} -bilinear form $\beta: \mathbb{V}^n \times \mathbb{V}^n \rightarrow \mathbb{W}^{p,q}$ into a vector space endowed with an indefinite inner product of type (p, q) is said to be *flat* if

$$\langle \beta(X, Y), \beta(Z, W) \rangle - \langle \beta(X, W), \beta(Z, Y) \rangle = 0 \quad \text{for all } X, Y, Z, W \in \mathbb{V}^n.$$

One main goal of the theory is to estimate the dimension of the nullity space

$$N(\beta) = \{X \in \mathbb{V}^n : \beta(X, Y) = 0 : Y \in \mathbb{V}^n\}$$

of a given β that is assumed to be onto, i.e., $\mathbf{W}^{p,q} = \text{span}\{\beta(X, Y) : X, Y \in \mathbb{V}^n\}$.

In [1] the following result for a symmetric bilinear form was proved.

Theorem 1. *Let $\beta: \mathbb{V}^n \times \mathbb{V}^n \rightarrow \mathbb{W}^{q,q}$ be a flat symmetric bilinear form. If $q \leq 5$ and β is onto, then $\dim N(\beta) \geq n - 2q$.*

A proof of the preceding result for $q = 2$ is contained in the argument by Cartan in [2]. It was conjectured around 1984 by the first author of this paper that the same estimate holds for arbitrary dimension q . A positive answer to the conjecture would have important consequences. For instance, the isometric and conformal rigidity results in [1] would hold after dropping the restriction on the codimension. Moreover, an extension of the results in [3] and [4] for arbitrary codimension with the same bounds would be possible. However, we give next a counterexample that shows that the conjecture is already false for $q = 6$ and that there is no linear estimate.

Theorem 2. *For a given $\tau \in \mathbb{N}$ with $\tau \geq 3$ set $2p = \tau(\tau + 1)$. Then there is an onto flat symmetric bilinear form $\beta: \mathbb{V}^n \times \mathbb{V}^n \rightarrow \mathbb{W}^{p,p}$ such that $\dim N(\beta) = n - 2p - \binom{\tau}{3}$.*

Proof. Denote $L = \{1, 2, \dots, \tau\}$, $I = (L \times L)/S(2)$ and $J = (L \times L \times L)/S(3)$, where $S(n)$ is the group of permutations of n elements. Then, $\#I = p$ and $\#J =$

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$m := \binom{\tau+2}{3}$. For $a \in L$, $k = [(i, j)] \in I$ and $s = [(u, v, w)] \in J$, we say that $a \in s$ if $a \in \{u, v, w\}$, and define $*$: $L \times I \rightarrow J$ by $a * k = [(a, i, j)]$. Then either $a \notin s$ or there is a unique $k \in I$ such that $a * k = s$.

Let $\mathbb{V}^n = \mathbb{R}^\tau \oplus \mathbb{R}^m$, and take bases $\{y_1, \dots, y_\tau\}$ and $\{n_s : s \in J\}$ of \mathbb{R}^τ and \mathbb{R}^m , respectively. Let $B_1 = \{e_r : r \in I\}$ and $B_2 = \{\hat{e}_r : r \in I\}$ be two bases of \mathbb{R}^p , and consider on $\mathbb{W}^{p,p} = \mathbb{R}^{2p}$ the metric of type (p, p) given by $\langle e_r, e_s \rangle = \langle \hat{e}_r, \hat{e}_s \rangle = 0$, $\langle e_r, \hat{e}_s \rangle = \delta_{r,s}$ for all $r, s \in I$. The (ordered) union of the bases B_1 and B_2 is called a pseudo-orthonormal basis of $\mathbb{W}^{p,p}$. Define a symmetric bilinear map β as follows:

$$\begin{aligned}\beta(n_s, n_r) &= 0, & r, s \in J, \\ \beta(y_i, y_j) &= \hat{e}_{[(i,j)]}, & i, j \in L, \\ \beta(y_i, n_s) &= 0, & \text{if } i \notin s, \\ \beta(y_i, n_s) &= e_k, & \text{if } i \in s \text{ and } i * k = s.\end{aligned}$$

To prove that $N(\beta) = 0$, take $x = y + n \in N(\beta)$ with $y = \sum_{j=1}^\tau a_j y_j$ and $n = \sum_{s \in J} b_s n_s$. Then $a_i = \langle \beta(x, y_i), e_{[(i,i)]} \rangle = 0$, and $b_s = \langle \beta(x, y_u), \hat{e}_{[(v,w)]} \rangle = 0$ for $s = [(u, v, w)]$. To see that β is flat just observe that $\langle \beta(y_i, y_j), \beta(y_t, n_s) \rangle = \delta_{s, [(i,j,t)]}$ is symmetric in $i, j, t \in L$. \square

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