

## COMPRESSIONS ON PARTIALLY ORDERED ABELIAN GROUPS

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ABSTRACT. If  $A$  is a  $C^*$ -algebra and  $p \in A$  is a self-adjoint idempotent, the mapping  $a \mapsto pap$  is called a compression on  $A$ . We introduce effect-ordered rings as generalizations of unital  $C^*$ -algebras and characterize compressions on these rings. The resulting characterization leads to a generalization of the notion of compression on partially ordered abelian groups with order units.

### 1. INTRODUCTION

Let  $A$  be a  $C^*$ -algebra with unit 1, and let  $p = p^2 = p^*$  be a projection in  $A$ . Then the mapping  $J_p: A \rightarrow A$  defined by  $J_p(a) := pap$  for all  $a \in A$  is called the *compression* determined by  $p$ . If  $d \in A$  and  $a = J_p(d)$ , then  $d$  is called a *dilation* of  $a$  [7]. By invoking various *dilation theorems*, one can dilate suitably structured families in  $A$  to more perspicuous structured families. For instance, by the Naimark (or Nagy-Naimark) dilation theorem [6, Chapter 2], a positive operator-valued measure on a Hilbert space can be dilated to a projection-valued measure.

In what follows, we denote by  $G(A)$  the additive group of self-adjoint elements in the unital  $C^*$ -algebra  $A$ . Then  $G(A)$  can be organized into an archimedean partially ordered abelian group with positive cone  $G(A)^+ := \{aa^* \mid a \in A\} = \{g^2 \mid g \in G(A)\}$ , and the unit 1 is an order unit in  $G(A)$ . We denote by  $P(A)$  the set of all projections  $p \in A$ . Then  $0, 1 \in P(A) \subseteq G(A)$  and  $P(A)$  acquires the structure of an orthomodular poset under the restriction of the partial order on  $G(A)$ . If  $p \in P(A)$ , then the compression  $J_p$  maps  $G(A)$  into itself, and the restriction of  $J_p$  to  $G(A)$ , which we shall continue to denote by the symbol  $J_p$ , has the following properties: (1)  $J_p: G(A) \rightarrow G(A)$  is an order-preserving endomorphism of the group  $G(A)$ , (2)  $p = J_p(1) \leq 1$ , (3) for all  $e \in G(A)^+$ ,  $e \leq p \Rightarrow J_p(e) = e$ , and (4)  $J_p$  is idempotent. Our main result in this article is that, conversely, if  $J: G(A) \rightarrow G(A)$  is an order-preserving endomorphism,  $J(1) \leq 1$ , and for all  $e \in G(A)^+$ ,  $e \leq J(1) \Rightarrow J(e) = e$ , then  $J = J_p$  where  $p = J(1) \in P(A)$ .

Properties (1)–(4) above make sense in any partially ordered abelian group with order unit, thus suggesting a generalized notion of compression that we shall call a *retraction* (Definition 2.1 below). As a consequence of Theorem 4.5 below, *every*

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retraction  $J$  on the partially ordered group  $G(A)$  of self-adjoint elements of a  $C^*$ -algebra with unit 1 has the form  $J = J_p$  where  $p = J(1) \in P(A)$ .

A *partially ordered abelian group* is an additively written abelian group  $G$  with a distinguished subset  $G^+$ , called the *positive cone*, such that  $G^+$  is closed under addition,  $0 \in G^+$ , and  $g, -g \in G^+ \Rightarrow g = 0$ . The positive cone determines a translation-invariant partial order  $\leq$  on  $G$  according to  $g \leq h \Leftrightarrow h - g \in G^+$  for  $g, h \in G$ , and in turn the partial order determines the positive cone according to  $G^+ = \{g \in G \mid 0 \leq g\}$ . If  $H$  is a subgroup of  $G$ , then  $H$  forms a partially ordered abelian group with the *induced* positive cone  $H^+ := H \cap G^+$ .

Let  $G$  be a partially ordered abelian group. An element  $u \in G^+$  is called an *order unit* iff for every  $g \in G$  there is a positive integer  $n$  such that  $g \leq nu$ . If  $G = G^+ - G^+$ , i.e., if  $G^+$  generates  $G$  as a group, then  $G$  is said to be *directed*. If  $G$  has an order unit, then it is directed. A subgroup  $H$  of  $G$  is *order convex* iff  $h_1, h_2 \in H, g \in G, h_1 \leq g \leq h_2 \Rightarrow g \in H$ . The kernel of an order-preserving group homomorphism is order convex. A directed order convex subgroup of  $G$  is called an *ideal*. If, for  $a, b \in G$ , the condition  $na \leq b$  holds for all positive integers  $n$  only if  $a \leq 0$ , then  $G$  is said to be *archimedean*. See [5, Chapters 1 and 2] for more details.

An element  $u \in G^+$  is *generative* iff every element  $g \in G^+$  can be written as a finite sum  $g = e_1 + e_2 + \cdots + e_n$  with  $0 \leq e_i \leq u$  for  $i = 1, 2, \dots, n$  [2, Definition 3.2]. If  $G$  is directed, then a generative element in  $G^+$  is automatically an order unit.

**1.1. Definition.** A *unital group* is a partially ordered abelian group  $G$  together with a specified generative order unit  $u \in G^+$  called the *unit* in  $G$  [3, Definition 2.5].

**1.2. Example.** If  $A$  is a unital  $C^*$ -algebra, then  $G(A)$  is an archimedean unital group with unit 1.  $\square$

## 2. RETRACTIONS ON PARTIALLY ORDERED ABELIAN GROUPS WITH ORDER UNITS

Abstracting from properties (1)-(4) in Section 1 of a compression  $J_p$  on the group  $G(A)$  of self-adjoint elements of a unital  $C^*$ -algebra  $A$ , we formulate the following definition.

**2.1. Definition.** Let  $G$  be a partially ordered abelian group with order unit  $u$ . A mapping  $J : G \rightarrow G$  is called a *retraction* on  $G$  iff:

- (i)  $J : G \rightarrow G$  is an order-preserving group endomorphism.
- (ii)  $J(u) \leq u$ .
- (iii) If  $e \in G$  with  $0 \leq e \leq J(u)$ , then  $J(e) = e$ .
- (iv)  $J$  is idempotent.

If  $J$  is a retraction on  $G$ , then  $J(u)$  is called the *focus* of  $J$ .

**2.2. Lemma.** *If  $G$  is a unital group, then condition (iv) in Definition 2.1 is redundant.*

*Proof.* Suppose  $G$  is a unital group, i.e.,  $u$  is a generative order unit in  $G^+$ , and let  $J : G \rightarrow G$  satisfy conditions (i)-(iii) in Definition 2.1. Let  $g \in G^+$ . Since  $u$  is generative, we can write  $g = \sum_{i=1}^n e_i$  with  $0 \leq e_i \leq u$  for  $i = 1, 2, \dots, n$ . Then, for  $i = 1, 2, \dots, n$ ,  $0 \leq J(e_i) \leq J(u)$ , whence  $J(J(e_i)) = J(e_i)$ , and it follows that  $J(J(g)) = J(g)$ . Because  $G$  has an order unit, it is directed; so  $G = G^+ - G^+$ , and it follows that every element  $g \in G$  satisfies  $J(J(g)) = J(g)$ .  $\square$

For the remainder of this section, we assume that  $J$  is a retraction on the partially ordered abelian group  $G$  with order unit  $u \in G^+$  and that  $p := J(u)$  is the focus of  $J$ . Thus, by parts (i) and (ii) of Definition 2.1, we have  $0 \leq p \leq u$ .

**2.3. Lemma.** *Let  $e, f, k \in G$  with  $0 \leq e, f \leq p \leq u$ . Then:*  
 (i)  $e + p \leq f + u \Leftrightarrow e \leq f$ .      (ii)  $e \leq u - p \Rightarrow e = 0$ .  
 (iii)  $2p \leq u \Leftrightarrow p = 0$ .      (iv)  $e + f \leq u \Rightarrow e + f \leq p$ .  
 (v)  $0 \leq k \leq u - p \Rightarrow J(k) = 0$ .

*Proof.* Since  $0 \leq e, f, p \leq p = J(u)$ , Definition 2.1 (iii) implies that  $J(e) = e$ ,  $J(f) = f$ , and  $J(p) = p$ . (i) If  $e + p \leq f + u$ , then  $e + p = J(e) + J(u) \leq J(f) + J(u) = f + p$ ; so  $e \leq f$ . Conversely, since  $p \leq u$ , we have  $e \leq f \Rightarrow e + p \leq f + u$ . (ii) Follows from (i) with  $f = 0$ . (iii) Follows from (ii) with  $e = p$ . (iv) If  $e + f \leq u$ , then  $e + f = J(e) + J(f) \leq J(u) = p$ . (v)  $0 \leq k \leq u - p \Rightarrow 0 \leq J(k) \leq p - J(p) = p - p = 0$ . □

Suppose that  $p \in P(A)$  is a projection in the unital  $C^*$ -algebra  $A$ , and let  $k \in G(A)$  with  $0 \leq k \leq u$ . By Lemma 3.2 (v),  $k \leq 1 - p \Rightarrow J_p(k) = 0$ , but in this case, we also have the converse  $J_p(k) = 0 \Rightarrow k \leq 1 - p$ . This observation motivates the following.

**2.4. Definition.** The retraction  $J: G \rightarrow G$  with focus  $p = J(u)$  is a *compression* iff for all  $k \in G$  with  $0 \leq k \leq u$ ,  $J(k) = 0 \Rightarrow k \leq u - p$ .

**2.5. Example.** Let  $\mathbb{Z}$  be the additive group of integers with the standard positive cone  $\mathbb{Z}^+$ . Let  $G := \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$  with coordinatewise addition, partially ordered with the nonstandard cone  $G^+ := \{(x, y, z) \in G \mid x, y, z \in \mathbb{Z}^+, x \leq y + z\}$ . Then  $G$  is an archimedean unital group with unit  $u := (1, 1, 1)$ . Define  $J: G \rightarrow G$  by  $J(x, y, z) := (0, 0, z)$  for all  $(x, y, z) \in G$ . Then  $J$  is a retraction with focus  $p = J(u) = (0, 0, 1)$ . But  $k := (0, 1, 0)$  satisfies  $(0, 0, 0) \leq k \leq u$  and  $J(k) = (0, 0, 0)$ ; yet  $k \not\leq u - p = (1, 1, 0)$ , and so  $J$  is not a compression. □

**2.6. Definition.** The retraction  $J: G \rightarrow G$  is *direct* iff  $g \in G^+ \Rightarrow J(g) \leq g$ .

**2.7. Example.** Let  $H$  and  $K$  be partially ordered abelian groups with order units  $v$  and  $w$ , respectively, and let  $G := H \times K$  be organized into a partially ordered abelian group with coordinatewise operations, with positive cone  $G^+ := H^+ \times K^+$ , and with order unit  $u := (v, w)$ . Let  $J, J': G \rightarrow G$  be defined by  $J(h, k) := (h, 0)$  and  $J'(h, k) := (0, k)$  for all  $(h, k) \in G$ . Then both  $J$  and  $J'$  are direct compressions. □

We omit the straightforward proof of the following theorem.

**2.8. Theorem.** *Let  $J: G \rightarrow G$  be a direct retraction, and define  $J': G \rightarrow G$  by  $J'(g) := g - J(g)$  for all  $g \in G$ . Let  $v := J(u)$ ,  $w := J'(u)$ ,  $H := J(G)$ ,  $K := J'(G)$ , and define  $\Phi: H \times K \rightarrow G$  by  $\Phi(h, k) := h + k$  for all  $h \in H, k \in K$ . Then: (i)  $J$  and  $J'$  are direct compressions. (ii) With the induced partial orders,  $H$  and  $K$  are partially ordered abelian groups,  $v$  is an order unit in  $H$ , and  $w$  is an order unit in  $K$ . (iii)  $\Phi: H \times K \rightarrow G$  is an isomorphism of partially ordered abelian groups. (iv)  $H$  and  $K$  are ideals in  $G$ .*

An element  $p \in G$  is said to be *characteristic* iff  $0 \leq p \leq u$ , the greatest lower bound  $p \wedge (u - p)$  exists in  $G$ , and  $p \wedge (u - p) = 0$  [5, Chapter 8].

**2.9. Theorem.** *Suppose  $G$  is an interpolation group [5, Chapter 2]. Then  $G$  is a unital group, the retraction  $J: G \rightarrow G$  is a direct compression, and its focus  $p = J(u)$  is a characteristic element of  $G$ . Conversely, if  $p$  is a characteristic element of  $G$ , there is a uniquely determined retraction  $J: G \rightarrow G$  with  $J(u) = p$ .*

*Proof.* Suppose  $G$  is an interpolation group. By [5, Proposition 2.2 (b)],  $u$  is a generative order unit in  $G$ . By Lemma 2.3 (ii), if  $e \in G$  with  $0 \leq e \leq p, u - p$ , then  $e = 0$ . We claim that the greatest lower bound  $p \wedge (u - p)$  exists in  $G$  and equals 0. Clearly,  $0 \leq p, u - p$ . Suppose  $g \in G$  with  $g \leq p, u - p$ . Then  $g, 0 \leq p, u - p$ ; so by interpolation there exists  $e \in G$  with  $g, 0 \leq e \leq p, u - p$ . Thus,  $e = 0$ , so  $g \leq 0$ , whence  $p \wedge (u - p) = 0$ , and it follows that  $p$  is a characteristic element of  $G$ . The remainder of the proof follows in a straightforward way from the development in [5, pp. 127-130].  $\square$

As a consequence of Theorem 2.9 and [5, Theorem 8.7], the compressions on an interpolation group with order unit can be organized into a Boolean algebra.

### 3. COMPRESSIBLE GROUPS

If  $A$  is a unital  $C^*$ -algebra and  $p \in P(A)$ , then the compressions  $J_p$  and  $J_{1-p}$  on  $G(A)$  are *quasicomplementary* in the following sense (cf. [1]).

**3.1. Definition.** Let  $G$  be a partially ordered abelian group with order unit  $u$ . Then the retractions  $J$  and  $I$  on  $G$  are said to be *quasicomplementary* iff, for all  $g \in G^+$ ,  $J(g) = g \Leftrightarrow I(g) = 0$  and  $J(g) = 0 \Leftrightarrow I(g) = g$ . If  $J$  and  $I$  are quasicomplementary retractions on  $G$ , we say that  $I$  is a *quasicomplement* of  $J$  and that  $J$  is a *quasicomplement* of  $I$ .

**3.2. Lemma.** *Let  $G$  be a unital group with unit  $u$  and unit interval  $E$ . Suppose that  $J$  and  $I$  are quasicomplementary retractions on  $G$ . Then:*

- (i) *If  $p = J(u)$  is the focus of  $J$ , then the focus of  $I$  is  $u - p$ .*
- (ii) *The images  $J(G)$  and  $I(G)$  of  $J$  and  $I$  are ideals in  $G$ .*
- (iii)  *$J$  and  $I$  are compressions.*

*Proof.* (i) We have  $0 \leq u - p$  with  $J(u - p) = 0$ , whence  $I(u) - I(p) = I(u - p) = u - p$ . But,  $0 \leq p$  with  $J(p) = p$ ; so  $I(p) = 0$ , and it follows that  $I(u) = u - p$ .

(ii) Let  $h \in H := J(G)$ . Since  $G$  is directed, there exist  $a, b \in G^+$  with  $h = a - b$ . Since  $J$  is idempotent,  $h = J(h) = J(a) - J(b)$ , and since  $J$  is order preserving,  $J(a), J(b) \in H \cap G^+ = H^+$ . Therefore,  $H$  is directed. Assume that  $g \in G$  and  $h \in H$  with  $0 \leq g \leq h$ . To prove that  $H$  is order convex in  $G$ , it is sufficient to show that  $g \in H$ . Since  $0 \leq h$  and  $J(h) = h$ , we have  $0 \leq I(g) \leq I(h) = 0$ , whence  $I(g) = 0$ . Therefore,  $g = J(g) \in H$  and  $H$  is an ideal in  $G$ . By symmetry,  $I(G)$  is an ideal in  $G$ .

(iii) Let  $p = J(u)$ ; so  $u - p = I(u)$  by (i). Suppose  $e \in E$  with  $J(e) = 0$ . Then  $I(e) = e$  and since  $0 \leq e \leq u$ , we have  $0 \leq e = I(e) \leq I(u) = u - p$ . Thus,  $J$  is a compression, and by symmetry, so is  $I$ .  $\square$

**3.3. Definition.** A *compressible group* is a unital group  $G$  such that every retraction on  $G$  is uniquely determined by its focus and every retraction on  $G$  has a quasicomplement. Let  $G$  be a compressible group with unit  $u$ . An element  $p \in G$  is called a *projection* iff it is the focus  $p = J(u)$  of a retraction  $J$  on  $G$ , and the set of all projections in  $G$  is denoted by  $P(G)$ . If  $p \in P(G)$ , we denote by  $J_p$  the unique retraction on  $G$  with focus  $p$ . Thus,  $J_p(u) = p$ .

**3.4. Lemma.** *Let  $G$  be a compressible group with unit  $u$ , and let  $p \in P(G)$ . Then every retraction on  $G$  is a compression, and the compression  $J_p$  has a unique quasicomplement, namely  $J_{u-p}$ .*

*Proof.* Assume the hypotheses. By Lemma 3.2 (iii), every retraction on  $G$  is a compression. Let  $I$  be a compression on  $G$ , and suppose that  $J_p$  is a quasicomplement of  $I$ . Then  $0 \leq p, u-p \leq u$  and by Lemma 3.2 (i),  $I$  has focus  $u-p$ ; so  $u-p \in P(G)$  and  $I = J_{u-p}$ .  $\square$

The terminology “compressible group” is suggested by the notion that the compressions on such a group are particularly well-behaved. If  $G$  is an interpolation group with order unit as in Theorem 2.9, then  $G$  is a compressible group and  $P(G)$  is a Boolean algebra. For a compressible group  $G$  it can be shown that  $P(G)$  is always an orthomodular poset (OMP) and that every finite set of pairwise compatible elements of  $P(G)$  is jointly compatible, i.e., is contained in a Boolean sub-OMP. In the next section we prove that, if  $A$  is a unital  $C^*$ -algebra, then  $G(A)$  is a compressible group. Therefore, compressible groups constitute a generalization of both interpolation groups with order unit and the additive group of self-adjoint elements of a unital  $C^*$ -algebra.

#### 4. EFFECT-ORDERED RINGS

The terminology in the following definition is suggested by the fact that, in certain approaches to the mathematical foundations of quantum mechanics, the self-adjoint elements between 0 and 1 in a unital  $C^*$ -algebra are called *effects*.

**4.1. Definition.** An *effect-ordered ring* is a ring  $A$  with unit 1 such that (1) under addition,  $A$  forms a partially ordered abelian group with positive cone  $A^+$ , (2)  $1 \in A^+$ , (3) the additive subgroup  $G(A) := A^+ - A^+$  of  $A$  is a unital group with positive cone  $G(A)^+ = A^+$  and unit 1, and (4) for all  $a, b \in A^+$ ,

- (i)  $ab = ba \Rightarrow ab \in A^+$ ,                      (ii)  $aba \in A^+$ ,
- (iii)  $aba = 0 \Rightarrow ab = ba = 0$ , and      (iv)  $(a - b)^2 \in A^+$ .

If  $A$  is an effect-ordered ring, we define  $P(A) := \{p \in G(A) \mid p = p^2\}$ , and for  $p \in P(A)$ , we define  $J_p: G(A) \rightarrow G(A)$  by  $J_p(g) := p g p$  for all  $g \in G(A)$ . If the unital group  $G(A)$  is archimedean, we say that  $A$  is an *archimedean* effect-ordered ring.

If  $A$  is an effect-ordered ring and  $E := \{e \in G(A) \mid 0 \leq e \leq 1\}$  is the set of effects in  $A$ , then  $G(A)^+$  is the set of all finite linear combinations with nonnegative integer coefficients of elements of  $E$ . Thus, not only does the partial order on  $A$  determine the set  $E$ , but conversely, the set  $E$  determines the positive cone  $G(A)^+ = A^+$ , hence the partial order on  $A$ . This accounts for our terminology “effect-ordered ring.”

**4.2. Example.** If  $A$  is a unital  $C^*$ -algebra, then, with  $A^+ := \{aa^* \mid a \in A\}$ ,  $A$  is an archimedean effect-ordered ring,  $G(A)$  is the additive group of self-adjoint elements in  $A$ ,  $P(A)$  is the orthomodular poset of projections in  $A$ , and  $J_p$  is the compression determined by  $p \in P(A)$ .

**4.3. Example.** Let  $\mathbb{Z}$  be the additive group of integers, ordered as usual, let  $X$  be a nonempty set, and let  $\mathcal{B}$  be a field of subsets of  $X$ . Define  $\mathcal{F}(X, \mathcal{B})$  to be the set of all bounded functions  $f: X \rightarrow \mathbb{Z}$  such that  $f^{-1}(z) \in \mathcal{B}$  for every  $z \in \mathbb{Z}$ ,

and organize  $\mathcal{F}(X, \mathcal{B})$  into a ring with pointwise operations. The constant function  $1(x) = 1$  for all  $x \in X$  is a unit for the ring  $\mathcal{F}(X, \mathcal{B})$ , and with the positive cone  $\mathcal{F}(X, \mathcal{B})^+ := \{f \in \mathcal{F}(X, \mathcal{B}) \mid f(X) \subseteq \mathbb{Z}^+\}$ ,  $\mathcal{F}(X, \mathcal{B})$  is an archimedean effect-ordered ring. Furthermore,  $G(\mathcal{F}(X, \mathcal{B})) = \mathcal{F}(X, \mathcal{B})$  is a lattice ordered (hence an interpolation) group and  $P(\mathcal{F}(X, \mathcal{B}))$  is the Boolean algebra of all characteristic set functions  $\chi_B$  of sets  $B \in \mathcal{B}$ .

By the Stone representation theorem, every Boolean algebra can be represented as the field  $\mathcal{B}$  of compact open subsets of a totally-disconnected compact Hausdorff space  $X$ . Hence, as in Example 4.3, every Boolean algebra can be represented as  $P(A)$  for an archimedean effect-ordered ring  $A$ .

Evidently, if  $A$  is an effect-ordered ring, then  $0, 1 \in P(A) \subseteq G(A)^+$  and by Definition 4.1 (4, iv),  $p \in P(A) \Rightarrow 1 - p \in P(A)$ . In [4, Definition 6.1] a weaker version of an effect-ordered ring  $A$ , called an *effect ring*, is defined in which it is not assumed that 1 is a generative order unit in  $G(A)$  and condition (4, iv) in Definition 4.1 is replaced by the weaker condition that, for all  $p \in G(A)^+$ ,  $p = p^2 \Rightarrow 1 - p \in G(A)^+$ . Therefore, the properties of an effect ring developed in [4] hold as well for an effect-ordered ring  $A$ . For instance, by [4, Corollary 6.7],  $P(A)$  is an orthomodular poset with  $p \mapsto 1 - p$  as orthocomplementation.

**4.4. Lemma.** *If  $A$  is an effect-ordered ring and  $p \in P(A)$ , then  $J_p$  is a compression on  $G(A)$ .*

*Proof.* It follows directly from Definition 4.1 that  $J_p$  is additive, order preserving, and  $J_p(1) = p \leq 1$ . Hence, by [4, Theorem 6.6 (iii)],  $J_p$  is a compression on  $G(A)$ .  $\square$

**4.5. Theorem.** *Let  $A$  be an archimedean effect-ordered ring, and let  $J : G(A) \rightarrow G(A)$  be a retraction with  $p := J(1)$ . Then  $p \in P(A)$  and  $J = J_p$ .*

*Proof.* Assume the hypotheses, and let  $p' := 1 - p$ . By Lemma 2.3 (ii), the infimum of  $p$  and  $p'$ , calculated in  $G(A)^+$ , exists and equals 0, whence  $p \in P(A)$  by [4, Theorem 6.8], and it follows that  $p' \in P(A)$ . By Lemma 2.3 (v),  $0 \leq k \leq p' \Rightarrow J(k) = 0$  for all  $k \in G(A)$ . If  $e \in G(A)$  with  $0 \leq e \leq 1$ , then  $0 \leq p'ep' \leq p'$ , whence, for  $e \in G(A)$ ,

$$(1) \quad 0 \leq e \leq 1 \Rightarrow J(p'ep') = 0.$$

Also, if  $e \in G(A)$  with  $0 \leq e \leq 1$ , then  $0 \leq pep \leq p$ , whence

$$(2) \quad 0 \leq e \leq 1 \Rightarrow J(pep) = pep.$$

We claim that

$$(3) \quad g \in G(A) \Rightarrow J(pgp) = pgp = J_p(g).$$

Indeed, since  $G(A)$  is directed, we can write  $g = x - y$  with  $x, y \in G(A)^+$ , and since 1 is generative, we can write  $x = e_1 + e_2 + \cdots + e_n$  with  $0 \leq e_i \leq 1$  for  $i = 1, 2, \dots, n$ . Thus,  $pxp = pe_1p + pe_2p + \cdots + pe_np$  and, by (2),  $J(pe_i p) = pe_i p$  for  $i = 1, 2, \dots, n$ . Therefore,  $J(pxp) = pxp$ . Likewise,  $J(pyp) = pyp$ ; so  $J(pgp) = pgp$ , proving (3). Arguing as we did to prove (3), and using (1), we find that

$$(4) \quad g \in G(A) \Rightarrow J(p'gp') = 0.$$

Now let  $g \in G(A)$ , let  $b := pgp' + p'gp$ , and consider the Peirce decomposition

$$(5) \quad g = pgp + b + p'gp'.$$

Since  $g, pgp, p'gp' \in G(A)$ , it follows from (5) that  $b = g - pgp - p'gp' \in G(A)$ . Evidently,  $b = p'b + bp'$ . Let  $n$  be an arbitrary integer. Then  $b, np' \in G(A)$ , and so by Definition 4.1 (4, iv),

$$(6) \quad 0 \leq (b - np')^2 = b^2 - nb + n^2p'.$$

Applying  $J$  to (6) and using the fact that  $J(p') = 0$ , we find that

$$(7) \quad nJ(b) \leq J(b^2) \text{ for all } n \in \mathbb{Z}.$$

Owing to the hypothesis that  $G(A)$  is archimedean, the fact that (7) holds for positive  $n \in \mathbb{Z}$  implies that  $J(b) \leq 0$ , and the fact that it holds for negative  $n$  implies that  $-J(b) \leq 0$ . Consequently,  $J(b) = 0$ . Therefore, by (5), (3), and (4), we have  $J(g) = pgp = J_p(g)$ .  $\square$

**4.6. Corollary.** *If  $A$  is an archimedean effect-ordered ring, then  $G(A)$  is an archimedean compressible group, every retraction  $J$  on  $G(A)$  is a compression of the form  $J = J_p$  for  $p = J(1) \in P(A)$ , and  $P(A) = \{J(1) \mid J \text{ is a compression on } G(A)\}$ .*

**4.7. Corollary.** *If  $A$  is a unital  $C^*$ -algebra, then the set  $G(A)$  of self-adjoint elements in  $A$  is an archimedean compressible group with unit 1 and the compressions on  $G(A)$  are the mappings  $J_p$  for  $p = p^2 = p^* \in A$ .*

By Corollary 4.7, Definition 2.1 does not overgeneralize the standard notion of a compression on a  $C^*$ -algebra.

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