LOCAL COHOMOLOGY MODULES WITH INFINITE DIMENSIONAL SOCLES

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Abstract. In this paper we prove the following generalization of a result of Hartshorne: Let $T$ be a commutative Noetherian local ring of dimension at least two, $R = T[x_1, \ldots, x_n]$, and $I = (x_1, \ldots, x_n)$. Let $f$ be a homogeneous element of $R$ such that the coefficients of $f$ form a system of parameters for $T$. Then the socle of $H^1_I(R/fR)$ is infinite dimensional.

1. Introduction

The third of Huneke’s four problems in local cohomology \cite{Hu} is to determine when $H^i_I(M)$ is Artinian for a given ideal $I$ of a commutative Noetherian local ring $R$ and finitely generated $R$-module $M$. A $R$-module $N$ is Artinian if and only if $\text{Supp}_R N \subseteq \{m\}$ and $\text{Hom}_R(R/m, N)$ is finitely generated, where $m$ is the maximal ideal of $R$. Thus, Huneke’s problem may be separated into two subproblems:

- When is $\text{Supp}_R H^1_I(M) \subseteq \{m\}$?
- When is $\text{Hom}_R(R/m, H^1_I(M))$ finitely generated?

This article is concerned with the second question. For an $R$-module $N$, one may identify $\text{Hom}_R(R/m, N)$ with the submodule $\{x \in N \mid mx = 0\}$, which is an $R/m$-vector space called the socle of $N$ (denoted $\text{soc}_R N$). It is known that if $R$ is an unramified regular local ring, then the local cohomology modules $H^i_I(R)$ have finite dimensional socles for all $i \geq 0$ and all ideals $I$ of $R$ (\cite{HS}, \cite{Li}, \cite{L2}).

The first example of a local cohomology module with an infinite dimensional socle was given in 1970 by Hartshorne \cite{Ha}: Let $k$ be a field, $R = k[[u, v]][x, y]$, $P = (u, v, x, y)R$, $I = (x, y)R$, and $f = ux + vy$. Then $\text{soc}_{R_P} H^2_{I_P}(R_P/fR_P)$ is infinite dimensional. Of course, since $I$ and $f$ are homogeneous, this is equivalent to saying that $\text{Hom}_R(R/P, H^2_I(R/fR))$ (the "socle of $H^2_I(R/fR)$") is infinite dimensional. Hartshorne proved this by exhibiting an infinite set of linearly independent elements in the "socle of $H^2_I(R)$.

In the last 30 years there have been few results in the literature which explain or generalize Hartshorne’s example. For affine semigroup rings, a remarkable result proved by Helm and Miller \cite{HM} gives necessary and sufficient conditions (on the semigroup) for the ring to possess a local cohomology module (of a finitely generated...
module) having infinite dimensional socle. Beyond that work, however, little has been done.

In this paper we prove the following:

**Theorem 1.1.** Let \( (T,m) \) be a Noetherian local ring of dimension at least two. Let \( R = T[x_1, \ldots, x_n] \) be a polynomial ring in \( n \) variables over \( T \), \( I = (x_1, \ldots, x_n) \), and \( f \in R \) a homogeneous polynomial whose coefficients form a system of parameters for \( T \). Then the *socle* of \( H^i_T(R/fR) \) is infinite dimensional.

Hartshorne’s example is obtained by letting \( T = k[[u,v]], n = 2, \) and \( f = ux + vy \) (homogeneous of degree 1). Note, however, that we do not require the coefficient ring to be regular, or even Cohen-Macaulay. As a further illustration, consider the following:

**Example 1.2.** Let \( R = k[[u^4, u^3v, u^2v^2, uv^3, v^4]][x,y,z] \), \( I = (x,y,z)R \), and \( f = u^4x^2 + v^8yz \). Then the *socle* of \( H^3_T(R/fR) \) is infinite dimensional.

Part of the proof of Theorem 1.1 was inspired by the recent work of Katzman [Ka] where information on the graded pieces of \( H^i_T(R/fR) \) is obtained by examining matrices of a particular form. We apply this technique in the proof of Lemma 2.8.

Throughout, all rings are assumed to be commutative with identity. The reader should consult [Mat] or [BH] for any unexplained terms or notation and [BS] for the basic properties of local cohomology.

\[2. \text{THE MAIN RESULT}\]

Let \( R = \bigoplus R_\ell \) be a Noetherian ring graded by the nonnegative integers. Assume \( R_0 \) is local and let \( P \) be the homogeneous maximal ideal of \( R \). Given a finitely generated graded \( R \)-module \( M \) we define the *socle* of \( M \) by

\[^* \text{soc}_RM = \{x \in M \mid Px = 0\} \]
\[\cong \text{Hom}_R(R/P, M)\].

Clearly, \(^* \text{soc}_RM \cong \text{soc}_{R_0}M_P\). An interesting special case of Huneke’s third problem is the following:

**Question 2.1.** Let \( n := \mu_R(R_+/PR_+) \), the minimal number of generators of \( R_+ \). When is \(^* \text{soc} H^n_{R_+}(R)\) finitely generated?

For \( i \in \mathbb{N} \) it is well known that \( H^i_{R_+}(R) \) is a graded \( R \)-module, each graded piece \( H^i_{R_+}(R)_\ell \) is a finitely generated \( R_0 \)-module, and \( H^i_{R_+}(R)_\ell = 0 \) for all sufficiently large integers \( \ell \) ([BS 15.1.5]). If we know \textit{a priori} that \( H^i_{R_+}(R)_\ell \) has finite length for all \( \ell \) (e.g., if \( \text{Supp}_R H^i_{R_+}(R) \subseteq \{P\} \)), then Question 2.1 is equivalent to:

**Question 2.2.** When is \( \text{Hom}_R(R/R_+, H^n_{R_+}(R)) \) finitely generated?

We give a partial answer to these questions for hypersurfaces. For the remainder of this section we adopt the following notation: Let \( (T,m) \) be a local ring of dimension \( d \) and \( R = T[x_1, \ldots, x_n] \) a polynomial ring in \( n \) variables over \( T \). We endow \( R \) with an \( \mathbb{N} \)-grading by setting \( \deg T = 0 \) and \( \deg x_i = 1 \) for all \( i \). Let \( I = R_+ = (x_1, \ldots, x_n)R \) and let \( P = m + I \) be the homogeneous maximal ideal of \( R \). Let \( f \in R \) be a homogeneous element of degree \( p \) and \( C_f \) the ideal of \( T \) generated by the coefficients of \( f \).
Our main result is the following:

**Theorem 2.3.** Assume \( d \geq 2 \) and the (nonzero) coefficients of \( f \) form a system of parameters for \( T \). Then \( \text{soc}_R H^n_I(R/fR) \) is not finitely generated.

The proof of this theorem will be given in a series of lemmas below. Before proceeding with the proof we make a couple of remarks:

**Remark 2.4.** (a) If \( d \leq 1 \) in Theorem 2.3, then \( \text{soc}_R H^n_I(R/fR) \) is finitely generated. This follows from [DM, Corollary 2] since \( \dim R/I = \dim T \leq 1 \).

(b) The hypothesis that the nonzero coefficients of \( f \) form a system of parameters for \( T \) is stronger than our proof requires. One only needs that \( C_f \) be \( m \)-primary and that there exists a dimension 2 ideal containing all but two of the coefficients of \( f \). (See the proof of Lemma 2.8.)

The following lemma identifies the support of \( H^n_I(R/fR) \) for a homogeneous element \( f \in R \). This lemma also follows from a much more general result recently proved by Katzman and Sharp [KS, Theorem 1.5].

**Lemma 2.5.** Let \( f \in R \) be a homogeneous element. Then

\[
\text{Supp}_R H^n_I(R/fR) = \{ Q \in \text{Spec } R \mid Q \supseteq I + C_f \}.
\]

**Proof.** It is enough to prove that \( H^n_I(R/fR) = 0 \) if and only if \( C_f = T \). As \( H^n_I(R/fR)_k \) is a finitely generated \( T \)-module for all \( k \), we have by Nakayama that \( H^n_I(R/fR) = 0 \) if and only if \( H^n_I(R/fR) \otimes_T T/m = 0 \). Now

\[
H^n_I(R/fR) \otimes_T T/m \cong H^n_I(R/fR) \otimes_T T/m \\
\cong H^n_N(S/fS)
\]

where \( S = (T/m)[x_1, \ldots, x_n] \) is a polynomial ring in \( n \) variables over a field and \( N = (x_1, \ldots, x_n)S \). As \( \dim S = n \), we see that \( H^n_N(S/fS) = 0 \) if and only if the image of \( f \) modulo \( m \) is nonzero. Hence, \( H^n_I(R/fR) = 0 \) if and only if at least one coefficient of \( f \) is a unit, i.e., \( C_f = T \). \( \square \)

We are mainly interested in the case when the coefficients of \( f \) generate an \( m \)-primary ideal:

**Corollary 2.6.** Let \( f \in R \) be homogeneous and suppose \( C_f \) is \( m \)-primary. Then

\[
\text{Supp}_R H^n_I(R/fR) = \{ P \}.
\]

Our next lemma is the key technical result in the proof of Theorem 2.3.

**Lemma 2.7.** Suppose \( u, v \in T \) such that \( \text{ht}(u, v)T = 2 \). For each integer \( n \geq 1 \) let \( M_n \) be the cokernel of \( \phi_n : T^{n+1} \rightarrow T^n \) where \( \phi_n \) is represented by the matrix

\[
A_n = \begin{pmatrix}
u & v & 0 & 0 & \cdots & 0 & 0 \\
0 & u & v & 0 & \cdots & 0 & 0 \\
0 & 0 & u & v & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & u & v
\end{pmatrix}_{n \times (n+1)}
\]

Let \( J = \bigcap_{n \geq 1} \text{ann}_T M_n \). Then \( \dim T/J = \dim T \).
Proof. Let $\hat{T}$ denote the $m$-adic completion of $T$. Then $\text{ht}(u, v)\hat{T} = 2$, $\text{ann}_T M_n = \text{ann}_{\hat{T}}(M_n \otimes_T \hat{T}) \cap T$, and $\dim T/(I \cap T) \geq \dim \hat{T}/I$ for all ideals $I$ of $T$. Thus, we may assume $T$ is complete. Now let $p$ be a prime ideal of $T$ such that $\dim T/p = \dim T$. Since $T$ is catenary, $\text{ht}(u, v)T/p = 2$. Assume the lemma is true for complete domains. Then $\bigcap_{n \geq 1} \text{ann}_{T/p}(M_n \otimes_T T/p) = p/p$. Hence

$$J = \bigcap_{n \geq 1} \text{ann}_T M_n$$

$$\subseteq \bigcap_{n \geq 1} \text{ann}_T(M_n \otimes_T T/p)$$

$$= p,$$

which implies that $\dim T/J \geq \dim T/p = \dim T$. Thus, it suffices to prove the lemma for complete domains.

As $T$ is complete, the integral closure $S$ of $T$ is a finite $T$-module ([Mat, page 263]). Since $\text{ht}(u, v)S = 2$ ([Mat, Theorem 15.6]) and $S$ is normal, $\{u, v\}$ is a regular sequence on $S$. It is easily seen that $I_n(A_n)$, the ideal of $n \times n$ minors of $A_n$, is $(u, v)^n T$. By the main result of [BE] we obtain $\text{ann}_S(M_n \otimes_T S) = (u, v)^n S$.

Hence $\text{ann}_T M_n \subseteq (u, v)^n S \cap T$. As $S$ is a finite $T$-module, there exists an integer $k$ such that $\text{ann}_T M_n \subseteq (u, v)^{n-k} T$ for all $n \geq k$. Therefore, $\bigcap_{n \geq 1} \text{ann}_T M_n = (0)$, which completes the proof. \hfill $\square$

Lemma 2.8. Assume $d \geq 2$ and let $f \in R$ be a homogeneous element of degree $p$ such that the coefficients of $f$ form a system of parameters for $T$. Then $\dim T/\text{ann}_T H^n_f(R/fR) \geq 2$.

Proof. Let $c_1, \ldots, c_d$ be the nonzero coefficients of $f$. Let $T' = T/(c_1, \ldots, c_d)T$ and $R' = T'[x_1, \ldots, x_n] \cong R/(c_1, \ldots, c_d)R \cong R \otimes_T T'$. Since

$$\dim T/\text{ann}_T H^n_f(R/fR) \geq \dim T/\text{ann}_T(H^n_f(R/fR) \otimes_T T')$$

$$= \dim T'/\text{ann}_{T'} H^n_{R'}(R'/fR'),$$

we may assume that $\dim T = 2$ and $f$ has exactly two nonzero terms.

For any $w \in R$ there is a surjective map $H^n_f(R/wfR) \to H^n_f(R/fR)$. Hence, $\text{ann}_T H^n_f(R/wfR) \subseteq \text{ann}_T H^n_f(R/fR)$. Thus, we may assume that the terms of $f$ have no (non-unit) common factor. Without loss of generality, we may write $R = T[x_1, \ldots, x_k, y_1, \ldots, y_r]$ and $f = u x_1^{d_1} \cdots x_k^{d_k} + v y_1^{c_1} \cdots y_r^{c_r} = u^{d} + v^{e}$, where $\{u, v\}$ is a system of parameters for $T$. As $f$ is homogeneous, $p = \sum_i d_i = \sum_i c_i$.

Applying the right exact functor $H^n_f(\cdot)$ to $R(-p) \xrightarrow{f} R \to R/fR \to 0$ we obtain the exact sequence

$$H^n_f(R)_{-\ell} \xrightarrow{f} H^n_f(R)_{-\ell} \to H^n_f(R/fR)_{-\ell} \to 0$$

for each $\ell \in \mathbb{Z}$. For each $\ell$, $H^n_f(R)_{-\ell}$ is a free $T$-module with basis

$$\{x^{-\alpha}y^{-\beta} \mid \sum \alpha_i + \beta_j = \ell, \alpha_i > 0, \beta_j > 0 \forall i, j\}$$

(e.g., [BS], Example 12.4.1). Let $q$ be an arbitrary positive integer and let $\ell(q) = q p + k + r$. Define $L_{-\ell(q)}$ to be the free $T$-summand of $H^n_f(R)_{-\ell(q)}$ spanned by the set

$$\{x^{-sd-1}y^{-te-1} \mid s + t = q, s, t \geq 0\}.$$
Then the cokernel of $\delta_q : L_{-\ell(q+1)} \xrightarrow{f} L_{-\ell(q)}$ is a direct summand (as a $T$-module) of $H^0_T(R/fR)_{-\ell(q)}$. For a given $q$ we order the basis elements for $L_{-\ell(q)}$ as follows:

$$x^{-sd-1}y^{-te-1} > x^{-s'd-1}y^{-t'e-1}$$

if and only if $s > s'$. With respect to these ordered bases, the matrix representing $\delta_q$ is

$$
\begin{pmatrix}
u & u & v & 0 & 0 & \cdots & 0 & 0 \\
0 & u & v & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & u & v & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & u & v
\end{pmatrix}_{(q+1) \times (q+2)}
$$

By Lemma 2.7 if $J = \bigcap_{q \geq 1} \text{ann}_T \text{coker} \delta_q$, then $\dim T/J = \dim T = 2$. As coker $\delta_q$ is a direct $T$-summand of $H^0_T(R/fR)$, we have $\text{ann}_T H^0_T(R/fR) \subseteq J$. This completes the proof.

Lemma 2.9. Under the assumptions of Lemma 2.8, $\text{Hom}_R(R/I, H^0_T(R/fR))$ is not finitely generated as an $R$-module. Consequently, $\text{Hom}_R(R/I, H^0_T(R/fR))_k \neq 0$ for infinitely many $k$.

Proof. Suppose $\text{Hom}_R(R/I, H^0_T(R/fR))$ is finitely generated. By Lemma 3.5 of [MV] we have that $I + \text{ann}_R H^0_T(R/fR)$ is $P$-primary. (One should note that the hypothesis in [MV, Lemma 3.5] that the ring be complete is not necessary.) This implies that $\text{ann}_T H^0_T(R/fR) \cap T = \text{ann}_T H^0_T(R/fR)$ is $m$-primary, contradicting Lemma 2.8.

We now give the proof of our main result:

Proof of Theorem 2.3. By Corollary 2.6, $\text{Supp}_R H^0_T(R/fR) = \{ P \}$. Thus, $\text{Hom}_R(R/I, H^0_T(R/fR))_k$ has finite length as a $T$-module for all $k$ and is nonzero for infinitely many $k$ by Lemma 2.9. Consequently,

$$\text{Hom}_R(R/P, H^0_T(R/fR))_k = \text{Hom}_T(T/m, \text{Hom}_R(R/I, H^0_T(R/fR)))$$

is nonzero for infinitely many $k$. Hence

$$\text{soc}_R(H^0_T(R/fR)) = \text{Hom}_R(R/P, H^0_T(R/fR))$$

is not finitely generated.

References


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