

ON THE EVALUATION OF GENERALIZED WATSON INTEGRALS

G. S. JOYCE AND I. J. ZUCKER

(Communicated by Jonathan M. Borwein)

ABSTRACT. The triple integrals

$$W_1(z_1) = \frac{1}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \frac{d\theta_1 d\theta_2 d\theta_3}{1 - \frac{z_1}{3}(\cos \theta_1 \cos \theta_2 + \cos \theta_2 \cos \theta_3 + \cos \theta_3 \cos \theta_1)}$$

and

$$W_2(z_2) = \frac{1}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \frac{d\theta_1 d\theta_2 d\theta_3}{1 - \frac{z_2}{3}(\cos \theta_1 + \cos \theta_2 + \cos \theta_3)},$$

where z_1 and z_2 are complex variables in suitably defined cut planes, were first evaluated by Watson in 1939 for the special cases $z_1 = 1$ and $z_2 = 1$, respectively. In the present paper simple direct methods are used to prove that $\{W_j(z_j): j = 1, 2\}$ can be expressed in terms of squares of complete elliptic integrals of the first kind for *general* values of z_1 and z_2 . It is also shown that $W_1(z_1)$ and $W_2(z_2)$ are related by the transformation formula

$$W_2(z_2) = (1 - z_1)^{1/2} W_1(z_1),$$

where

$$z_2^2 = -z_1 \left(\frac{3 + z_1}{1 - z_1} \right).$$

Thus both of Watson's results for $\{W_j(1): j = 1, 2\}$ are contained within a *single* formula for $W_1(z_1)$.

1. INTRODUCTION

In this paper we shall investigate the analytic properties of the two generalized Watson integrals

$$(1.1) \quad W_j(z_j) = \frac{1}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \frac{d\theta_1 d\theta_2 d\theta_3}{1 - z_j \lambda_j(\theta_1, \theta_2, \theta_3)} \quad (j = 1, 2),$$

where

$$(1.2) \quad \lambda_1(\theta_1, \theta_2, \theta_3) = \frac{1}{3}(\cos \theta_1 \cos \theta_2 + \cos \theta_2 \cos \theta_3 + \cos \theta_3 \cos \theta_1),$$

$$(1.3) \quad \lambda_2(\theta_1, \theta_2, \theta_3) = \frac{1}{3}(\cos \theta_1 + \cos \theta_2 + \cos \theta_3)$$

and z_1, z_2 are complex variables. The integral $W_1(z_1)$ defines a single-valued analytic function in the z_1 plane provided that cuts are made along the real axis from $z_1 = -\infty$ to $z_1 = -3$ and from $z_1 = +1$ to $z_1 = +\infty$. A similar property holds for the integral $W_2(z_2)$ provided that the cuts are made along the real axis

Received by the editors March 13, 2003.

2000 *Mathematics Subject Classification.* Primary 33-xx.

©2004 American Mathematical Society

from $z_2 = -\infty$ to $z_2 = -1$ and from $z_2 = +1$ to $z_2 = +\infty$. The sets of points in the z_1 and z_2 cut planes will be denoted by \mathcal{C}_1^- and \mathcal{C}_2^- respectively.

An exact evaluation of the integral $W_1(z_1)$ was first carried out by Watson [W] for the special case $z_1 = 1$. In particular, he found that

$$(1.4) \quad W_1(1) = \frac{3\sqrt{3}}{4} \left[\frac{2}{\pi} K(k_3) \right]^2 = \frac{9}{2^{14/3}\pi^4} \left[\Gamma\left(\frac{1}{3}\right) \right]^6,$$

where $K(k)$ is the complete elliptic integral of the first kind with a modulus k and

$$(1.5) \quad k_3 = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

is the third *singular value* (see Borwein and Borwein [B-B]). In the same paper Watson also proved that

$$(1.6) \quad W_2(1) = 3 \left(18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6} \right) \left[\frac{2}{\pi} K(k_6) \right]^2,$$

where

$$(1.7) \quad k_6 = (2 - \sqrt{3})(\sqrt{3} - \sqrt{2})$$

is the sixth *singular value*. The evaluation of $W_2(1)$ in terms of the gamma function $\Gamma(x)$ can be carried out using an identity for $K(k_6)$ derived by Borwein and Zucker [B-Z]. We find that

$$(1.8) \quad W_2(1) = \frac{(\sqrt{3}-1)}{32\pi^3} \left[\Gamma\left(\frac{1}{24}\right) \Gamma\left(\frac{11}{24}\right) \right]^2.$$

A direct evaluation of the triple integral $W_1(z_1)$ was first carried out for *general* values of z_1 by Iwata [I] in terms of a product of two complete elliptic integrals of the first kind. A similar product formula for $W_2(z_2)$ was later derived by Joyce [J1], [J2] using an *indirect* method which involved differential equations and Heun function transformation theory (see Snow [S]). In more recent work, Joyce [J3], [J4] has shown that the product forms for $\{W_j(z_j) : j = 1, 2\}$ can both be written in terms of *squares* of complete elliptic integrals of the first kind.

Our main aim in this paper is to give a *new* and *direct* procedure for evaluating $W_2(z_2)$. In this method a simple transformation formula is derived which enables one to determine the properties of $W_2(z_2)$ using the known results for $W_1(z_1)$. In Section 2 we shall also present a detailed *review* of the work of Iwata [I] and Joyce [J3], [J4] on the analytic properties of $W_1(z_1)$.

2. ANALYTIC PROPERTIES OF $W_1(z_1)$

In this section we shall review the most important results for the triple integral $W_1(z_1)$.

2.1. Iwata formula for $W_1(z_1)$. We begin by evaluating the integral $W_1(z_1)$ by following the method of Iwata [I]. In the first stage of the analysis we perform the integration over the variable θ_3 in (1.1), with $j = 1$. This procedure gives

$$(2.1) \quad W_1(z_1) = \frac{3}{z_1\pi^2} \int_0^\pi \frac{d\theta_1}{\sin\theta_1} \int_0^\pi \frac{d\theta_2}{[(a + \cos\theta_2)(b - \cos\theta_2)]^{1/2}},$$

where

$$(2.2) \quad a = (3 + z_1 \cos \theta_1)/[z_1(1 - \cos \theta_1)],$$

$$(2.3) \quad b = (3 - z_1 \cos \theta_1)/[z_1(1 + \cos \theta_1)].$$

We can now use the standard result

$$(2.4) \quad \int_0^\pi \frac{d\theta_2}{[(a + \cos \theta_2)(b - \cos \theta_2)]^{1/2}} = \frac{2}{[(a + 1)(b + 1)]^{1/2}} K(k),$$

where

$$(2.5) \quad k^2 = \frac{2(a + b)}{(a + 1)(b + 1)},$$

to write (2.1) in the form

$$(2.6) \quad W_1(z_1) = \frac{3}{3 + z_1} I(A_1, B_1),$$

where

$$(2.7) \quad A_1 \equiv A_1(z_1) = 2z_1(6 + z_1)/(3 + z_1)^2,$$

$$(2.8) \quad B_1 \equiv B_1(z_1) = 2z_1^2/(3 + z_1)^2$$

and

$$(2.9) \quad I(A, B) \equiv \frac{2}{\pi^2} \int_0^\pi K(\sqrt{A + B \cos \psi}) d\psi.$$

In general, the variables (A, B) in the definition (2.9) can be taken to be *independent* complex variables. However, it should be noted that the particular set of points $\{A_1(z_1), B_1(z_1): z_1 \in \mathcal{C}_1^-\}$ is *restricted* to lie on the complex *rational* curve

$$(2.10) \quad (A_1 + B_1)^2 - 8B_1 = 0.$$

Next we apply the Gaussian hypergeometric series

$$(2.11) \quad \frac{2}{\pi} K(k) = \sum_{n=0}^\infty \frac{(\frac{1}{2})_n^2}{(1)_n^2} k^{2n}$$

to the integrand in (2.9). In this manner, we obtain

$$(2.12) \quad I(A, B) = \sum_{n=0}^\infty \frac{(\frac{1}{2})_n^2}{(1)_n^2} \Omega_n(A, B),$$

where

$$(2.13) \quad \Omega_n(A, B) = \frac{1}{\pi} \int_0^\pi (A + B \cos \psi)^n d\psi$$

and $(a)_n$ denotes the Pochhammer symbol. The integral (2.13) can be readily evaluated using the method of residues. Hence, we obtain

$$(2.14) \quad \Omega_n(A, B) = (x_+)^n \sum_{m=0}^n \binom{n}{m}^2 \left(\frac{x_-}{x_+}\right)^m,$$

where

$$(2.15) \quad x_\pm = \frac{1}{2} \left(A \pm \sqrt{A^2 - B^2} \right).$$

If (2.14) is substituted in (2.12) and the order of the resulting two summations is interchanged we find that it is possible to express $I(A, B)$ in the form

$$(2.16) \quad I(A, B) = F_4\left(\frac{1}{2}, \frac{1}{2}; 1, 1; x_+, x_-\right),$$

where $F_4(\alpha, \beta; \gamma, \gamma'; x, y)$ is the fourth Appell hypergeometric function in two variables x and y .

In the final stage of the analysis Iwata simplified (2.16) by using the Bailey [B] identity

$$(2.17) \quad \begin{aligned} &F_4[\alpha, \beta; \gamma, \alpha + \beta - \gamma + 1; x(1 - y), y(1 - x)] \\ &= {}_2F_1(\alpha, \beta; \gamma; x) {}_2F_1(\alpha, \beta; \alpha + \beta - \gamma + 1; y). \end{aligned}$$

This procedure yields the important *general* formula

$$(2.18) \quad I(A, B) = \left(\frac{2}{\pi}\right)^2 K(k_+)K(k_-),$$

where

$$(2.19) \quad k_{\pm}^2 \equiv k_{\pm}^2(A, B) = \frac{1}{2} \pm \frac{1}{2} \sqrt{A^2 - B^2} - \frac{1}{2} \sqrt{(1 - A)^2 - B^2}.$$

If we now substitute (2.18) in (2.6) and apply the relations (2.7) and (2.8) we obtain the required result:

$$(2.20) \quad W_1(z_1) = \frac{3}{3 + z_1} \left(\frac{2}{\pi}\right)^2 K(k_+)K(k_-),$$

where

$$(2.21) \quad k_{\pm}^2 \equiv k_{\pm}^2(z_1) = \frac{1}{2} \pm \frac{2\sqrt{3}z_1}{(3 + z_1)^{3/2}} - \frac{\sqrt{3}(3 - z_1)(1 - z_1)^{1/2}}{2(3 + z_1)^{3/2}}.$$

It can be shown that the formula (2.20) is valid provided that z_1 lies in a certain finite region \mathcal{R}_1 of the cut z_1 plane that includes the point $z_1 = 0$. The points z_1 on the boundary of \mathcal{R}_1 are associated with values of k_+^2 that lie in the interval $\frac{1}{2}(1 + \sqrt{2}) \leq k_+^2 < \infty$. We can extend the range of validity of (2.20) across the boundary curve by constructing the analytic continuation of $K(k_+)$ onto the appropriate adjacent Riemann sheet.

For the special case $z_1 = 1$ we find that k_+ becomes the complementary modulus k'_- and $k_- = k_3$, where k_N denotes the N th singular value for the modulus. From these results and (2.20) we obtain the expected Watson formula (1.4).

2.2. Application of the cubic modular transformation. The aim in this subsection is to show that the Iwata formula (2.20) can be expressed in terms of the *square* of just *one* complete elliptic integral $K(k_-)$. It is found from (2.21) that the algebraic functions $m_{\pm} = k_{\pm}^2(z_1)$ are both solutions of the polynomial equation

$$(2.22) \quad \begin{aligned} Q(m, z_1) \equiv &(3 + z_1)^4 m^4 - 2(3 + z_1)^4 m^3 + 3(3 + z_1)(9 + 21z_1 - 7z_1^2 + z_1^3) m^2 \\ &- 2z_1(3 + z_1)(18 - 15z_1 + z_1^2) m + z_1^4 = 0. \end{aligned}$$

If we determine the resultant of the two polynomials $Q(m_{\pm}, z_1)$, with respect to the variable z_1 , we deduce that the moduli $k_{\pm}(z_1)$ satisfy the cubic modular equation

(Joyce [J4])

$$(2.23) \quad m_+^4 - 4m_-(33 - 96m_- + 64m_-^2)m_+^3 + 6m_-(64 - 127m_- + 64m_-^2)m_+^2 - 4m_-(64 - 96m_- + 33m_-^2)m_+ + m_-^4 = 0,$$

where $m_{\pm} = k_{\pm}^2(z_1)$. This surprising connection enables one to prove that

$$(2.24) \quad K(k_+) = M_3(z_1)K(k_-),$$

where the *multiplier* $M_3(z_1)$ is an *algebraic* function of z_1 (see Borwein and Borwein [B-B]). From the work of Joyce [J3] we also have the explicit formula

$$(2.25) \quad M_3(z_1) = \frac{\sqrt{3}}{(3 + z_1)^{1/2}}(2 - \sqrt{1 - z_1}).$$

The application of (2.24) and (2.25) to the Iwata formula (2.20) gives the alternative product form

$$(2.26) \quad W_1(z_1) = \frac{3\sqrt{3}}{(3 + z_1)^{3/2}}(2 - \sqrt{1 - z_1}) \left[\frac{2}{\pi} K(k_-) \right]^2,$$

where $k_- = k_-(z_1)$ is defined in (2.21). In the derivation of (2.26) it was necessary to assume that the variable z_1 lies in the restricted region \mathcal{R}_1 of the cut plane \mathcal{C}_1^- . It should be stressed, however, that the final result (2.26) can in fact be used to calculate $W_1(z_1)$ for *all* values of $z_1 \in \mathcal{C}_1^-$. We see, therefore, that the application of cubic modular theory leads to a *much improved* formula for $W_1(z_1)$. The evaluation of (2.26) for the case $z_1 = 1$ leads directly to the Watson result (1.4).

Further product forms for $W_1(z_1)$ may be derived by applying various ${}_2F_1$ transformation formulae to the complete elliptic integral $K(k_-)$ in (2.26). For example, it is found that

$$(2.27) \quad W_1(z_1) = \frac{3\sqrt{3}}{(3 + z_1)^{3/2}}(2 - \sqrt{1 - z_1}) \left[{}_2F_1 \left(\frac{1}{4}, \frac{1}{4}; 1; s_1 \right) \right]^2,$$

where

$$(2.28) \quad s_1 \equiv s_1(z_1) = \frac{4z_1}{(3 + z_1)^3} [(3 - z_1) - 3\sqrt{1 - z_1}]^2.$$

When $z_1 = 1$, we see that this result gives

$$(2.29) \quad W_1(1) = \frac{3\sqrt{3}}{4} \left[{}_2F_1 \left(\frac{1}{4}, \frac{1}{4}; 1; \frac{1}{4} \right) \right]^2.$$

2.3. Parametric representations for $W_1(z_1)$. A considerable reduction in the algebraic complexity of the Iwata formulae (2.20) and (2.21) can be achieved by applying the rational transformation

$$(2.30) \quad z_1 \equiv z_1(\xi_1) = \frac{12\xi_1(1 + \xi_1)(1 - 3\xi_1)}{(1 + 3\xi_1^2)^2}.$$

In this manner, we obtain the ξ_1 parametric representation (Joyce [J4])

$$(2.31) \quad W_1(z_1) = \frac{(1 + 3\xi_1^2)^2}{(1 - \xi_1)^2(1 + 3\xi_1)^2} \left(\frac{2}{\pi} \right)^2 K(k_+)K(k_-),$$

where

$$(2.32) \quad k_+^2 \equiv k_+^2(\xi_1) = \frac{16\xi_1}{(1-\xi_1)(1+3\xi_1)^3},$$

$$(2.33) \quad k_-^2 \equiv k_-^2(\xi_1) = \frac{16\xi_1^3}{(1-\xi_1)^3(1+3\xi_1)}.$$

It is possible to derive *rational* parametric formulae for $z_1(\xi_1)$ and $k_{\pm}^2(\xi_1)$, because the equation $Q(m, z_1) = 0$ defines a complex curve that has a genus $g = 0$.

The application of the transformation (2.30) to equation (2.26) gives the further simplified representation

$$(2.34) \quad W_1(z_1) = \frac{(1+3\xi_1^2)^2}{(1-\xi_1)^3(1+3\xi_1)} \left[\frac{2}{\pi} K(k_-) \right]^2,$$

where $k_- = k_-(\xi_1)$ is defined in (2.33). Finally, we note that equation (2.30) can also be used to establish the inverse relation

$$(2.35) \quad \xi_1 \equiv \xi_1(z_1) = (1 + \sqrt{1-z_1})^{-1} \left(-1 + \sqrt{1 + \frac{z_1}{3}} \right).$$

3. ANALYTIC PROPERTIES OF $W_2(z_2)$

Watson [W] evaluated $W_2(1)$ by using a *direct* method which involved an inspired sequence of integrations and changes of variable. Unfortunately, it appears that Watson's analysis has never been generalized to deal with $W_2(z_2)$, when $z_2 \neq 1$. However, it was shown by Joyce [J1], [J2], that it is possible to evaluate $W_2(z_2)$ for general z_2 by following an alternative *indirect* procedure. In this method the third-order differential equations satisfied by $\{W_j(z_j): j = 1, 2\}$ are used to express $W_1(z_1)$ and $W_2(z_2)$ in terms of squares of certain Heun functions. A connection between $W_1(z_1)$ and $W_2(z_2)$ is then established by applying various bilinear and quadratic transformations to the Heun functions (see Snow [S]). This connection enables one to derive an exact product form for $W_2(z_2)$ by making a simple algebraic substitution in the Iwata formula (2.20) for $W_1(z_1)$.

Our main aim in the remainder of this section is to give a *new* and *direct* derivation of the Joyce product form for $W_2(z_2)$. In particular, the connection between $W_1(z_1)$ and $W_2(z_2)$ will be *simplified* and made more *transparent* by avoiding the use of differential equations and Heun functions.

3.1. Connection between $W_1(z_1)$ and $W_2(z_2)$. We begin by performing the integration over the variable θ_3 in (1.1), with $j = 2$. In this manner, it is found that

$$(3.1) \quad W_2(z_2) = \frac{3}{z_2\pi^2} \int_0^\pi d\theta_1 \int_0^\pi \frac{d\theta_2}{[(c - \cos\theta_2)(d - \cos\theta_2)]^{1/2}},$$

where

$$(3.2) \quad c = (3/z_2) - \cos\theta_1 + 1,$$

$$(3.3) \quad d = (3/z_2) - \cos\theta_1 - 1.$$

If we now apply the standard result

$$(3.4) \quad \int_0^\pi \frac{d\theta_2}{[(c - \cos\theta_2)(d - \cos\theta_2)]^{1/2}} = \frac{2}{[(c-1)(d+1)]^{1/2}} K(k),$$

where

$$(3.5) \quad k^2 = \frac{2(c-d)}{(c-1)(d+1)},$$

to (3.1) we obtain the formula (Tikson [T])

$$(3.6) \quad W_2(z_2) = \frac{6}{\pi^2} \int_0^\pi \frac{1}{3 - z_2 \cos \theta_1} K \left(\frac{2z_2}{3 - z_2 \cos \theta_1} \right) d\theta_1.$$

Next the integration variable in (3.6) is changed from θ_1 to ψ using the transformation

$$(3.7) \quad \cos \theta_1 = \frac{3 \cos \psi + z_2}{3 + z_2 \cos \psi}.$$

Hence, we find that

$$(3.8) \quad W_2(z_2) = \frac{3}{(9 - z_2^2)^{1/2}} J(C_2, D_2),$$

where

$$(3.9) \quad C_2 \equiv C_2(z_2) = 6z_2/(9 - z_2^2),$$

$$(3.10) \quad D_2 \equiv D_2(z_2) = 2z_2^2/(9 - z_2^2)$$

and

$$(3.11) \quad J(C, D) \equiv \frac{2}{\pi^2} \int_0^\pi K(C + D \cos \psi) d\psi.$$

In general, the variables (C, D) in the definition (3.11) can be taken to be *independent* complex variables. However, it should be noted that the particular set of points $\{C_2(z_2), D_2(z_2) : z_2 \in \mathcal{C}_2^-\}$ is *restricted* to lie on the complex curve

$$(3.12) \quad C_2^2 - D_2^2 - 2D_2 = 0.$$

Recently, Joyce (to be published) has shown that

$$(3.13) \quad I(A, B) = (1 - A + B)^{-1/2} J(C, D),$$

where

$$(3.14) \quad C^2 = -\frac{(A-B)(1-A-B)}{(1-A+B)^2},$$

$$(3.15) \quad D^2 = \frac{2B}{(1-A+B)^2}$$

and the integrals $I(A, B)$ and $J(C, D)$ are defined in (2.9) and (3.11) respectively. This relation was derived by first applying the hypergeometric series (2.11) to the integrand in (3.11). Hence, one obtains a Kampé de Fériet series representation for $J(C, D)$ in the two variables C^2 and D^2 (see Srivastava and Karlsson [S-K]). It is then possible to establish the connection between $I(A, B)$ and $J(C, D)$ by applying two known transformation formulae to the Kampé de Fériet series. From equation (2.6) and the new general result (3.13) we see that

$$(3.16) \quad W_1(z_1) = \frac{3}{3 - z_1} J(C_1, D_1),$$

where

$$(3.17) \quad C_1^2 \equiv C_1^2(z_1) = -36z_1(1-z_1)(3+z_1)/(3-z_1)^4,$$

$$(3.18) \quad D_1^2 \equiv D_1^2(z_1) = 4z_1^2(3+z_1)^2/(3-z_1)^4.$$

It may be verified using (3.9), (3.10), (3.17) and (3.18) that the two sets of points

$$(3.19) \quad \{C_j^2(z_j), D_j^2(z_j) : z_j \in C_j^-\} \quad (j = 1, 2)$$

both lie on the *same* complex rational curve

$$(3.20) \quad (C_j^2 - D_j^2)^2 - 4D_j^2 = 0,$$

with $j = 1, 2$. It follows, therefore, that a *rational* function $z_2^2 = R(z_1)$ must exist that will give the identities

$$(3.21) \quad C_2^2 \left(\sqrt{R(z_1)} \right) \equiv C_1^2(z_1) \text{ and } D_2^2 \left(\sqrt{R(z_1)} \right) \equiv D_1^2(z_1).$$

We find that this transformation function is

$$(3.22) \quad z_2^2 = R(z_1) = -z_1(3+z_1)/(1-z_1).$$

The application of this result to (3.8) and (3.16) yields the required connection formula

$$(3.23) \quad W_2(z_2) = (1-z_1)^{1/2}W_1(z_1),$$

where $z_2 = z_2(z_1)$ is defined in (3.22).

The relations (3.22) and (3.23) with $z_1 = -1$ lead to a very *simple* derivation of the Watson result (1.6). In particular, we have

$$(3.24) \quad W_2(1) = \sqrt{2}W_1(-1),$$

while from (2.26) it is found that

$$(3.25) \quad W_1(-1) = \frac{6\sqrt{3}}{\pi^2} (\sqrt{2}-1) [K(k_-)]^2,$$

where

$$(3.26) \quad k_-^2 = -\frac{1}{2} (2\sqrt{3} - \sqrt{6} - 1).$$

The application of the standard transformation formula

$$(3.27) \quad K(k) = \frac{1}{\sqrt{1-k^2}} K \left(\sqrt{\frac{-k^2}{1-k^2}} \right)$$

to (3.25) then gives

$$(3.28) \quad W_1(-1) = \frac{6\sqrt{3}}{\pi^2} (\sqrt{2}+1) (\sqrt{3}-1)^2 (\sqrt{3}-\sqrt{2}) [K(k_6)]^2,$$

where the singular value k_6 is defined in (1.7). Finally, the substitution of (3.28) in (3.24) yields the expected Watson formula (1.6) for $W_2(1)$.

3.2. **Product forms for $W_2(z_2)$.** A general product form for $W_2(z_2)$ can now be derived by first substituting the formula (2.26) in (3.23). It is then possible to eliminate the variable z_1 by applying the inverse transformation

$$(3.29) \quad z_1 = z_1(z_2) = -\frac{1}{2}(3 - z_2^2) + \frac{1}{2}\sqrt{(1 - z_2^2)(9 - z_2^2)}.$$

After some algebraic manipulations it is found that

$$(3.30) \quad W_2(z_2) = \frac{6\sqrt{6} \left(\sqrt{9 - z_2^2} + \sqrt{1 - z_2^2}\right)^{1/2} \left(4 - \sqrt{9 - z_2^2} + \sqrt{1 - z_2^2}\right)}{\pi^2 \left(\sqrt{9 - z_2^2} + 3\sqrt{1 - z_2^2}\right)^{3/2}} [K(k_-)]^2,$$

where

$$(3.31) \quad k_-^2 \equiv k_-^2(z_2) = \frac{1}{2} - \frac{\sqrt{3}\sqrt{9 - z_2^2} \left(\sqrt{9 - z_2^2} - \sqrt{1 - z_2^2}\right)^{1/2}}{\left(\sqrt{9 - z_2^2} + 3\sqrt{1 - z_2^2}\right)^{3/2}} + \frac{4\sqrt{6}z_2^2 \left(\sqrt{9 - z_2^2} + \sqrt{1 - z_2^2}\right)^{1/2}}{\left(\sqrt{9 - z_2^2} + 3\sqrt{1 - z_2^2}\right)^{5/2}}.$$

By investigating the conformal mapping $z_2 \mapsto k_-^2(z_2)$ it can be shown that the formula (3.30) is valid for *all* values of $z_2 \in C_2^-$.

The rather complicated structure of (3.30) and (3.31) can be simplified by applying various ${}_2F_1$ transformation formulae to the complete elliptic integral $K(k_-)$ in (3.30). For example, Delves and Joyce [D-J] have shown that

$$(3.32) \quad W_2(z_2) = \frac{3 \left(2 - \sqrt{1 - z_2^2}\right)}{3 + z_2^2} \left[{}_2F_1\left(\frac{1}{8}, \frac{3}{8}; 1; s_2\right)\right]^2,$$

where

$$(3.33) \quad s_2 \equiv s_2(z_2) = \frac{16z_2^2}{9(3 + z_2^2)^4} \left[(9 - 5z_2^2) - (9 - z_2^2)\sqrt{1 - z_2^2}\right]^2.$$

This result can also be used to calculate $W_2(z_2)$ at *any* point $z_2 \in C_2^-$. When $z_2 = 1$ we find that (3.32) gives

$$(3.34) \quad W_2(1) = \left(\frac{3}{2}\right) \left[{}_2F_1\left(\frac{1}{8}, \frac{3}{8}; 1; \frac{1}{9}\right)\right]^2.$$

Finally, we note that the Clausen product theorem (see [B-B], p. 188) can be used to write (3.34) in the alternative form

$$(3.35) \quad W_2(1) = \left(\frac{3}{2}\right) {}_3F_2\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1; \frac{1}{9}\right).$$

From this result and (1.8) we can derive the striking identity

$$(3.36) \quad \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{1}{(48)^{2n}} = \frac{(\sqrt{3} - 1)}{48\pi^3} \left[\Gamma\left(\frac{1}{24}\right) \Gamma\left(\frac{11}{24}\right)\right]^2.$$

3.3. Parametric representations for $W_2(z_2)$. The algebraic complexity of the formulae (3.30) and (3.31) can also be much reduced by applying the rational transformation [J3], [J4]

$$(3.37) \quad z_2^2 = 36\xi_2^2(1 - \xi_2^2)(1 - 9\xi_2^2)/(1 - 9\xi_2^4)^2.$$

In this manner, we obtain the ξ_2 parametric representation

$$(3.38) \quad W_2(z_2) = \frac{(1 - 9\xi_2^4)}{(1 - \xi_2^2)^{3/2}(1 - 9\xi_2^2)^{1/2}} \left[\frac{2}{\pi} K(k_-) \right]^2,$$

where

$$(3.39) \quad k_-^2 \equiv k_-^2(\xi_2) = \frac{1}{2} - \frac{(1 - 6\xi_2^2 - 3\xi_2^4)}{2(1 - \xi_2^2)^{3/2}(1 - 9\xi_2^2)^{1/2}}.$$

Next, the quadratic transformation formula

$$(3.40) \quad {}_2F_1 \left[\frac{1}{2}, \frac{1}{2}; 1; \frac{1}{2} - \frac{(2 - \tilde{k}^2)}{4\tilde{k}'} \right] = (\tilde{k}')^{1/2} {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1; \tilde{k}^2 \right),$$

with

$$(3.41) \quad \tilde{k}^2 \equiv \tilde{k}^2(\xi_2) = \frac{16\xi_2^3}{(1 - \xi_2)^3(1 + 3\xi_2)},$$

is applied to (3.38). This procedure yields the further simple representation

$$(3.42) \quad W_2(z_2) = \frac{(1 - 9\xi_2^4)}{(1 - \xi_2)^3(1 + 3\xi_2)} \left[\frac{2}{\pi} K(\tilde{k}) \right]^2,$$

where $\tilde{k} = \tilde{k}(\xi_2)$ is defined in (3.41). It is also possible to express the parameter ξ_2 in terms of the variable z_2 using the inverse relation

$$(3.43) \quad \xi_2 \equiv \xi_2(z_2) = \left(1 + \sqrt{1 - z_2^2} \right)^{-1/2} \left(1 - \sqrt{1 - \frac{z_2^2}{9}} \right)^{1/2}.$$

Recently, Joyce [J5] has shown that the parametric formula (3.42) plays a crucial role in the exact evaluation of the general lattice Green function

$$(3.44) \quad W_2(l, m, n; z_2) = \frac{1}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \frac{\cos l\theta_1 \cos m\theta_2 \cos n\theta_3}{1 - z_2 \lambda_2(\theta_1, \theta_2, \theta_3)} d\theta_1 d\theta_2 d\theta_3,$$

where $\{l, m, n\}$ denotes a set of integers and $z_2 \in \mathcal{C}_2^-$.

ACKNOWLEDGEMENTS

We wish to thank our colleague Richard Delves for assistance in deriving various transformations involved in the evaluation of $W_2(z_2)$.

REFERENCES

- [B] W. N. Bailey, (1933), *A reducible case of the fourth type of Appell's hypergeometric functions of two variables*, Q. J. Math. Oxford **4**, 305–308.
- [B-B] J. M. Borwein and P. B. Borwein, (1987), *Pi and the AGM*, Wiley, New York. MR1641658 (99h:11147)
- [B-Z] J. M. Borwein and I. J. Zucker, (1992), *Fast evaluation of the gamma function for small rational fractions using complete elliptic integrals of the first kind*, IMA J. Numerical Analysis **12**, 519–526. MR1186733 (93g:65028)
- [D-J] R. T. Delves and G. S. Joyce, (2001), *On the Green function for the anisotropic simple cubic lattice*, Ann. Phys. (N.Y.) **291**, 71–133. MR1848300 (2002j:82047)

- [I] G. Iwata, (1969), *Evaluation of the Watson integral of a face-centered lattice*, Natn. Sci. Rep. Ochanomizu Univ., Tokyo **20**, 13–18. MR0267143 (42:2045)
- [J1] G. S. Joyce, (1972), *Lattice Green function for the simple cubic lattice*, J. Phys. A: Gen. Phys. **5**, L65-L68.
- [J2] G. S. Joyce, (1973), *On the simple cubic lattice Green function*, Phil. Trans. R. Soc. Lond. **A273**, 583–610. MR0591189 (58:28708)
- [J3] G. S. Joyce, (1994), *On the cubic lattice Green functions*, Proc. R. Soc. Lond. **A445**, 463–477. MR1276913 (95j:33041)
- [J4] G. S. Joyce, (1998), *On the cubic modular transformation and the cubic lattice Green functions*, J. Phys. A: Math. Gen. **31**, 5105–5115.
- [J5] G. S. Joyce, (2002), *Exact evaluation of the simple cubic lattice Green function for a general lattice point*, J. Phys. A: Math. Gen. **35**, 9811–9828. MR1947241 (2003m:82012)
- [S] C. Snow, (1952), *Hypergeometric and Legendre functions with applications to integral equations in potential theory*, Natl. Bur. Stds. Appl. Math. Ser., no. 19 (US GPO:Washington DC). MR0048145 (13:988b)
- [S-K] H. M. Srivastava and P. W. Karlsson, (1985), *Multiple Gaussian hypergeometric series*, Ellis Horwood, Chichester, West Sussex. MR0834385 (87f:33015)
- [T] M. Tikson, (1953), *Tabulation of an integral arising in the theory of cooperative phenomena*, J. Res. Natl. Bur. Stds. **50**, 177–178. MR0055780 (14:1125h)
- [W] G. N. Watson, (1939), *Three triple integrals*, Q. J. Math. Oxford **10**, 266–276. MR0001257 (1:205b)

WHEATSTONE PHYSICS LABORATORY, KING'S COLLEGE, UNIVERSITY OF LONDON, STRAND,
LONDON WC2R 2LS, UNITED KINGDOM
E-mail address: `gsj@maxwell.ph.kcl.ac.uk`

WHEATSTONE PHYSICS LABORATORY, KING'S COLLEGE, UNIVERSITY OF LONDON, STRAND,
LONDON WC2R 2LS, UNITED KINGDOM
E-mail address: `jz@maxwell.ph.kcl.ac.uk`