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ON A CERTAIN CLASS OF MODULAR FUNCTIONS

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ABSTRACT. We give a characterization of those meromorphic modular functions on a subgroup of finite index of the full modular group whose divisors are supported at the cusps, in terms of the growth of the exponents of their infinite product expansions.

1. Introduction and statement of results

Let f be a meromorphic modular function of integral weight k on a subgroup Γ of finite index in $\Gamma(1) := SL_2(\mathbf{Z})$, i.e., f is a meromorphic function on the complex upper half-plane $\mathcal{H}$, satisfies the usual transformation formula in weight k, and is meromorphic at the cusps. We will suppose that f is not identically zero. Like any complex-valued, periodic and meromorphic function on $\mathcal{H} \cup \{\infty\}$ that is not identically zero, f then has a product expansion

$$f(z) = cq_M^h \prod_{n\geq 1} (1 - q_M^n)^{c(n)}$$

where the infinite product is convergent in $|q_M| < \epsilon$ for some $\epsilon > 0$. Here c is a nonzero constant, M is the least positive integer with $\left(\begin{smallmatrix} 1 & M \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$, $q_M := e^{2\pi i z/M}$ for $z \in \mathcal{H}$, h is the order of f at infinity and the $c(n)\,(n \in \mathbf{N})$ are uniquely determined complex numbers [1, 2]. As usual, we understand that the complex powers are defined by the principal branch of the complex logarithm.

The purpose of this note is to give a characterization of those forms f that have no zeros or poles on $\mathcal{H}$, in terms of the coefficients $c(n)$. More precisely, we shall prove

Theorem 1. *Suppose that f has no zeros or poles on $\mathcal{H}$. Then this assertion is equivalent to the following assertions, respectively:*

i) *if Γ is of finite index in $\Gamma(1)$, then $c(n) <<_f \log\log n \cdot \log n \quad (n > 2)$ where the constant implied in $<<_f$ only depends on f;*

ii) *if Γ is a congruence subgroup of $\Gamma(1)$, then $c(n) <<_f (\log\log n)^2 \quad (n > 2)$ where the constant implied in $<<_f$ only depends on f.*

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In case f is on $\Gamma_0(N) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma(1) \,|\, N|c\}$ for some $N \in \mathbf{N}$, one can do much better (at least if N is squarefree) and prove

Theorem 2. *Suppose that $\Gamma = \Gamma_0(N)$ and N is squarefree. Then f has no zeros or poles on $\mathcal{H}$ if and only if $c(n)\,(n \in \mathbf{N})$ depends only on the greatest common divisor (n, N).*

We think that the assertion of Theorem 2 or a slightly weaker statement eventually would be true for arbitrary N, but it seems that this cannot be proved by the method employed here.

Recall that a meromorphic modular function of weight zero with poles and zeros only at the cusps is called a modular unit. Thus if Δ is the usual discriminant function of weight 12 on $\Gamma(1)$, then f^{12}/Δ^k is a modular unit. It was proved by Kubert and Kubert-Lang (see [5, chap. 4] and the references given there) that on the principal congruence subgroup $\Gamma(N)$ of level N the group of modular units (at least up to 2-torsion) is generated by the so-called Siegel units, which are defined in terms of N-division values of the Weierstrass σ-function. It might be interesting to investigate if Theorem 1, ii) or Theorem 2 could also be proved by the methods applied there.

2. Proof of Theorem 1

Put

$$\theta = q_M \frac{d}{dq_M} = \frac{M}{2\pi i} \frac{d}{dz}.$$

Then we have

$$\frac{\theta f}{f} = h - \sum_{n \geq 1} \left(\sum_{d|n} d c(d) \right) q_M^n \qquad (|q_M| < \epsilon) \tag{1}$$

(see [1] and [2]).

Therefore, if $c(n) <<_f \log\log n \cdot \log n \quad (n > 2)$, the right-hand side of (1) is convergent for all $z \in \mathcal{H}$, and hence by standard facts from complex analysis f has no zeros or poles on $\mathcal{H}$.

Now assume the latter condition. Let

$$P(z) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n \qquad (z \in \mathcal{H},\, q = q_1)$$

be the nearly modular Eisenstein series of weight 2 on $\Gamma(1)$, where $\sigma_1(n) = \sum_{d|n} d$.

We denote by $|_k$ the usual slash operator in weight k, i.e., for a function $g : \mathcal{H} \to \mathbf{C}$ and $\gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in GL_2^+(\mathbf{R})$ we put

$$(g|_k\gamma)(z) := (ad - bc)^{k/2}(cz + d)^{-k} g\left(\frac{az + b}{cz + d}\right) \qquad (z \in \mathcal{H}).$$

Recall that for $\gamma \in \Gamma(1)$, we have

$$(P|_2\gamma)(z) = P(z) + \frac{12c}{2\pi i(cz + d)} \qquad (z \in \mathcal{H}), \tag{2}$$

and hence

$$\partial f := 12 \cdot \frac{1}{M} \theta f - kPf$$

is a meromorphic modular function of weight $k+2$ on Γ (compare [6, part IV, sect. 5]). Therefore, in view of our assumption, $\frac{\partial f}{f}$ is a meromorphic modular function of weight 2 on Γ, holomorphic on $\mathcal{H}$.

For all $\gamma \in \Gamma(1)$, we have by (2),

$$\begin{aligned}(\frac{\partial f}{f})|_2\gamma &= 12\frac{\frac{1}{M}(\theta f)|_{k+2}\gamma}{f|_k\gamma} - kP|_2\gamma \\ &= 12\frac{\frac{1}{M}\theta(f|_k\gamma)}{f|_k\gamma} - kP.\end{aligned}$$

Since f, by assumption, is meromorphic at all cusps, it follows that $\frac{\partial f}{f}$ is holomorphic at all cusps, i.e., $\frac{\partial f}{f}$ is a (holomorphic) modular form of weight 2 on Γ.

Therefore (cf., e.g., [7, chap. III, sect. 4]) the Fourier coefficients of $\frac{\partial f}{f}$ are $<<_f n\log n \quad (n>1)$.

Now recall that

$$\sigma_1(n) \le 2n\log\log n$$

for n large. Indeed, as is well known (cf., e.g., [3, Thm. 323]),

$$\limsup_{n\to\infty} \frac{\sigma_1(n)}{n\log\log n} = e^{\gamma}$$

where $\gamma = .57721\ldots$ is Euler's constant.

Therefore we infer from (1) that

$$\sum_{d|n} dc(d) <<_f n\log n \quad (n>1),$$

and hence, inverting, we find that for $n>2$ one has

$$\begin{aligned}c(n) &<<_f \frac{1}{n}\sum_{d|n} d\log d \\ &\le \frac{\log n}{n}\cdot\sigma_1(n) \\ &<< \log\log n\cdot\log n.\end{aligned}$$

This proves i).

To prove ii), assume that f is on the principal congruence subgroup $\Gamma(N)$. Then by [4] the weight 2 modular form $\frac{\partial f}{f}$ is the sum of a cusp form and a linear combination of Eisenstein series, and the Fourier coefficients of the former are $<<_f n$ and those of the latter for $n \ge 1$ have the shape

$$\sum_{d|n,\frac{n}{d}\equiv\alpha \pmod N} |d| e^{2\pi i d\beta/N},$$

where $\alpha,\beta \in \mathbf{Z}/N\mathbf{Z}$. Hence we find that the Fourier coefficients of $\frac{\partial f}{f}$ are $<<_f \sigma_1(n)$, and so, by (1),

$$\sum_{d|n} dc(d) <<_f \sigma_1(n),$$

which implies that

$$c(n) <<_f \frac{1}{n}\sum_{d|n}\sigma_1(d).$$

Arguing in a similar way as before, this proves ii).

3. Proof of Theorem 2

We need only prove that if f has no zeros or poles on $\mathcal{H}$, then the numbers $c(n)\,(n \in \mathbf{N})$ have the asserted property, since the converse statement has already been proved in sect. 2.

Denote by $M_k(N)$ the space of meromorphic modular functions of weight k on $\Gamma_0(N)$. Since N is squarefree, the cusps of $\Gamma_0(N)$ are represented by the numbers $\frac{1}{t}$ with t running over the positive divisors of N.

For $t|N$ we let W_t^N be the Atkin-Lehner involution on $M_k(N)$ defined by

$$g|W_t^N := g|_k \begin{pmatrix} t & \alpha \\ N & t\beta \end{pmatrix}$$

where $\alpha, \beta \in \mathbf{Z}$ with $t^2\beta - N\alpha = t$. The matrix $\left(\begin{smallmatrix} t & \alpha \\ N & t\beta \end{smallmatrix}\right)$ maps the cusp ∞ to the cusp $\frac{1}{N/t}$, and hence (up to normalization) the Fourier expansion of $g|W_t^N$ at ∞ is the Fourier expansion of g at $\frac{1}{N/t}$.

Furthermore, for $d|N$ and g a complex-valued function on $\mathcal{H}$ we set $(g|V_d)(z) := g(dz)\,(z \in \mathcal{H})$.

Then the formulas

$$g|W_t^N|W_{t'}^N = g|W_{tt'}^N \qquad (g \in M_k(N),\, (t,t') = 1) \tag{3}$$

and

$$g|V_d|W_t^N = g|W_t^t|V_d \qquad (g \in M_k(t),\, d|\tfrac{N}{t}) \tag{4}$$

are well known and easily verified.

Let h_t be the order of f^{12} at $\frac{1}{N/t}$. Let

$$\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24} \qquad (z \in \mathcal{H})$$

be the usual discriminant function of weight 12 on $\Gamma(1)$.

We claim that there are integers $a_t\,(t|N)$ and a nonzero integer a such that

$$F := \prod_{d|N}(\Delta|V_d)^{a_d}$$

has order ah_t at the cusp $\frac{1}{N/t}$ for all $t|N$.

Indeed, if $d, t|N$ and $e := (d,t)$, then by (3) and (4) we find that

$$\begin{aligned}\Delta|V_d|W_t^N &= \Delta|V_e|V_{d/e}|W_t^N \\ &= \Delta|V_e|W_t^t|V_{d/e}\end{aligned}$$

and

$$\begin{aligned}\Delta|V_e|W_t^t &= \Delta|V_e|W_e^t|W_{t/e}^t \\ &= \Delta|V_e|W_e^e|W_{t/e}^t \\ &= e^{-6}\Delta|W_{t/e}^t \\ &= e^{-6}\Delta|W_{t/e}^{t/e} \\ &= t^6e^{-12}\Delta|V_{t/e},\end{aligned}$$

and hence

$$\Delta|V_d|W_t^N = (\frac{t}{e^2})^6 \Delta|V_{dt/e^2}.$$

Therefore

$$F|W_t^N = \prod_{d|N} (\frac{t}{e^2})^{6a_d} \cdot \prod_{d|N} (\Delta|V_{dt/e^2})^{a_d}$$

and we find that the order of F at $\frac{1}{N/t}$ is equal to

$$\sum_{d|N} a_d \cdot \frac{dt}{e^2}.$$

(As a matter of taste, if one wants to formally avoid the use of Atkin-Lehner involutions, to compute the order of $\Delta|V_d$ at each cusp one can of course argue entirely in terms of local coordinates at the cusps; cf., e.g., [8, Introduction].)

We want to show that the $(\sigma_0(N), \sigma_0(N))$-matrix

$$A_N := (\frac{dt}{e^2})_{d|N, t|N}$$

(where $\sigma_0(N)$ is the number of positive divisors of N) is invertible, which then proves the above claim. We do this by induction on the number of prime divisors of N. If $N = 1$, there is nothing to prove. Assume that $N = pM$ and $(p, M) = 1$. Then, as is immediate from the definitions,

$$\begin{aligned} A_N &= \begin{pmatrix} A_M & pA_M \\ pA_M & A_M \end{pmatrix} \\ &= \begin{pmatrix} E & pE \\ pE & E \end{pmatrix} \begin{pmatrix} A_M & 0 \\ 0 & A_M \end{pmatrix} \end{aligned}$$

where E is the unit matrix of size $\sigma_0(M)$; hence A_N is invertible by the induction hypothesis.

By the valence formula (cf., e.g., [9, Thm. 4.1.4.]) applied to F and by what we proved above, since Δ does not vanish on $\mathcal{H}$, we have

$$a \sum_{t|N} h_t = \frac{k_1}{12} [\Gamma(1) : \Gamma_0(N)]$$

where k_1 is the weight of F. On the other hand, by the valence formula applied to f^{12}, since f has no zeros or poles on $\mathcal{H}$ by our assumption, we have

$$\sum_{t|N} h_t = k[\Gamma(1) : \Gamma_0(N)].$$

Therefore $k_1 = 12ak$.

The function $\frac{f^{12a}}{F}$ has weight zero and has no zeros or poles on $\mathcal{H}$ or at the cusps, by what we proved above, and therefore is constant. Since a is not zero, this proves Theorem 2.

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