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A LIPSCHITZ ESTIMATE FOR BEREZIN'S OPERATOR CALCULUS

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ABSTRACT. F. A. Berezin introduced a general "symbol calculus" for linear operators on reproducing kernel Hilbert spaces. For the particular Hilbert space of Gaussian square-integrable entire functions on complex n-space, $\mathbf{C}^n$, we obtain Lipschitz estimates for the Berezin symbols of arbitrary bounded operators. Additional properties of the Berezin symbol and extensions to more general reproducing kernel Hilbert spaces are discussed.

1. INTRODUCTION

The Segal-Bargmann Hilbert space $H^2(\mathbf{C}^n, d\mu)$ of Gaussian square-integrable entire functions on complex n-space [Ba] has the Bergman "reproducing kernel" property that

$$f(a) = \int_{\mathbf{C}^n} K(a,z)f(z)d\mu(z) = \langle f(\cdot), K(\cdot, a)\rangle$$

for all f in $H^2(\mathbf{C}^n, d\mu)$ and a in $\mathbf{C}^n$ where $a \cdot z = a_1\bar{z}_1 + \cdots + a_n\bar{z}_n$, $|a|^2 = a \cdot a$, $K(a,z) = e^{a\cdot z/2}$ is the Bergman kernel function and

$$d\mu(z) = (2\pi)^{-n} \exp\{-|z|^2/2\} dv(z)$$

where dv is the Lebesgue volume measure. It is easy to check that $k_a(z) = K(z,a)\{K(a,a)\}^{-\frac{1}{2}}$ is a unit vector in the Hilbert space structure that $H^2(\mathbf{C}^n, d\mu)$ inherits as a subspace of $L^2(\mathbf{C}^n, d\mu)$.

Although we will deal primarily with the space $H^2(\mathbf{C}^n, d\mu)$, we will also consider the analogous Bergman spaces $A^2(\Omega)$ for Ω a bounded domain in $\mathbf{C}^n$ and the Gaussian $d\mu$ replaced by dv. Their kernel functions will also be denoted by $K(a,z)$ as in [K, pp. 39-54]. Note that $K(a,z)$ is always analytic in a and conjugate-analytic in z with

$$\overline{K(a,z)} = K(z,a)$$

and that $K(a,a)$ is positive. The "bounded symmetric domains" Ω give interesting special cases that are "closer" to the model $H^2(\mathbf{C}^n, d\mu)$ [H].

For bounded linear operators X on $H^2(\mathbf{C}^n, d\mu)$, or, more generally, linear operators X such that X and X^* have domains that include all the $\{k_a : a \in \mathbf{C}^n\}$, Berezin [Be$_1$, Be$_2$] considered the mapping $Ber : X \to \tilde{X}$, where $\tilde{X}(a) = \langle Xk_a, k_a\rangle$. The function $\tilde{X}(\cdot)$ is real-analytic on $\mathbf{C}^n$ and is uniquely determined by

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X [Be$_1$]; [F, pp. 43, 139]. It is clear that $|\tilde{X}(\cdot)| \leq \|X\|$, where $\|\cdot\|$ is the usual operator norm. Berezin's map has the above properties for any of the spaces $A^2(\Omega)$. Because Ber is linear and one-to-one, it "encodes" operator-theoretic information into function-theory in a striking but somewhat impenetrable way. In fact, since $k_a \to 0$ weakly as $|a| \to \infty$ for $H^2(\mathbf{C}^n, d\mu)$, it is also clear that Ber maps compact operators on these Hilbert spaces into functions that vanish at infinity (there is an analogous result for $A^2(\Omega)$ with "nice" Ω). Because of these properties, the mapping Ber has found useful applications in dealing with operators "of function-theoretic significance" such as Toeplitz and Hankel operators [AZ, BBCZ, BC$_1$, BC$_2$, G, E].

In the next section, I will prove the promised result that $\tilde{X}$ is Lipschitz for arbitrary bounded operators X on $H^2(\mathbf{C}^n, d\mu)$. I will also consider extensions of this result to more general settings. In the third section of the paper, I will discuss some other properties of the range of Ber.

2. The Lipschitz estimate for *Ber*

Let $P_a(f) = \langle f, k_a\rangle k_a$ be the rank one projection onto the span of k_a. We will need a few technical results about trace-class operators, which we take from [GK]. For X any bounded linear operator on $H^2(\mathbf{C}^n, d\mu)$, XP_a is also rank one and, therefore, trace-class. Moreover, since we can augment k_a to an orthonormal basis for $H^2(\mathbf{C}^n, d\mu)$, it is easy to see that

$$trace(XP_a) = \langle XP_a k_a, k_a\rangle = \tilde{X}(a).$$

Additivity of the trace now shows that

$$\tilde{X}(a) - \tilde{X}(b) = trace[X(P_a - P_b)].$$

We need to calculate $\|P_a - P_b\|_{trace}$, where [GK, pp. 47, 48, 92]

$$\|A\|_{trace} = trace(\sqrt{A^*A})$$

and $\sqrt{A^*A}$ is the unique positive square-root of the positive operator A^*A. This calculation holds for any reproducing kernel space.

Theorem 1. *For a, b in $\mathbf{C}^n$,*

$$\|P_a - P_b\|_{trace} = 2\{1 - |\langle k_a, k_b\rangle|^2\}^{\frac{1}{2}}.$$

Proof. We write $(g \otimes h)(f) = \langle f, h\rangle g$ for a typical rank-one operator. Now

$$k_b = \langle k_b, k_a\rangle k_a + h_{a,b}$$

with $\langle h_{a,b}, k_a\rangle = 0$ and

$$\|h_{a,b}\|^2 = 1 - |\langle k_a, k_b\rangle|^2.$$

Direct calculation shows that

$$P_a - P_b = \|h_{a,b}\|^2 k_a \otimes k_a - \langle k_a, k_b\rangle h_{a,b} \otimes k_a - \langle k_b, k_a\rangle k_a \otimes h_{a,b} - h_{a,b} \otimes h_{a,b}.$$

Now $P_a - P_b$ is self-adjoint and another direct calculation shows that

$$(P_a - P_b)^2 = \|h_{a,b}\|^2 k_a \otimes k_a + h_{a,b} \otimes h_{a,b}.$$

Thus, $P_a - P_b = 0$ if and only if $h_{a,b} = 0$. For $h_{a,b} \neq 0$, $(P_a - P_b)^2$ is diagonal in any orthonormal basis including k_a and $h_{a,b}/\|h_{a,b}\|$ and has two non-zero eigenvalues, both equal to $\|h_{a,b}\|^2$. The positive square root of $(P_a - P_b)^2$ is then diagonal in the same basis and it follows that

$$\|P_a - P_b\|_{trace} = 2\|h_{a,b}\| = 2\{1 - |\langle k_a, k_b\rangle|^2\}^{\frac{1}{2}}.$$

Remark. Since $P_a - P_b = 0$ implies that $|\langle k_a, k_b\rangle| = 1$, it is an easy consequence of the Cauchy-Schwarz Lemma that $K(\cdot, a) = \lambda K(\cdot, b)$ for some complex number λ and an easy argument shows, for $A^2(\Omega)$ with Ω bounded or for $H^2(\mathbf{C}^n, d\mu)$, that $a = b$. Subadditivity of $\|\cdot\|_{trace}$ now implies that $d(a,b) = \|P_a - P_b\|_{trace}$ is a topological metric (distance function) on bounded Ω or $\mathbf{C}^n$. For bounded Ω, it is easy to check, using the "transformation laws for reproducing kernels" [K, p. 44], that $|\langle k_{\varphi(a)}, k_{\varphi(b)}\rangle|^2 = |\langle k_a, k_b\rangle|^2$ for φ any biholomorphic automorphism of Ω and it follows that $\|P_{\varphi(a)} - P_{\varphi(b)}\|_{trace} = \|P_a - P_b\|_{trace}$.

We can now establish the desired Lipschitz estimate.

Theorem 2. *For X any bounded operator on $H^2(\mathbf{C}^n, d\mu)$, $\tilde{X}$ is Lipschitz with*

$$|\tilde{X}(a) - \tilde{X}(b)| \leq \sqrt{2}\ \|X\|\ |a - b|.$$

Proof. As noted earlier,

$$\tilde{X}(a) - \tilde{X}(b) = trace[X(P_a - P_b)].$$

For X bounded and T in trace-class, it is known [GK, pp. 27, 104] that XT is in trace-class with

$$|trace(XT)| \leq \|X\|\ \ \|T\|_{trace}.$$

It follows from Theorem 1 that

$$|\tilde{X}(a) - \tilde{X}(b)| \leq 2\|X\|\ \{1 - |\langle k_a, k_b\rangle|^2\}^{\frac{1}{2}}.$$

By direct calculation, using $K(z,a) = e^{z\cdot a/2}$, we see that

$$1 - |\langle k_a, k_b\rangle|^2 = 1 - e^{-|a-b|^2/2} \leq \frac{|a-b|^2}{2},$$

and the proof is finished.

The proof of Theorem 1 holds in general reproducing kernel spaces. It is known that the Bergman kernel function induces a Riemannian metric on Ω or $\mathbf{C}^n$ and this, in turn, gives rise to a distance function $\beta(\cdot,\cdot)$. For $H^2(\mathbf{C}^n, d\mu)$, we can check that $\beta(a,b) = |a - b|$. Using results of [MPS], we can obtain the exact analogue of Theorem 2 for $A^2(\Omega)$ with Ω any bounded domain in $\mathbf{C}^n$.

Theorem 3. *For X any bounded operator on $A^2(\Omega)$ with Ω any bounded domain in $\mathbf{C}^n$, $\tilde{X}$ is "β-Lipschitz" with*

$$|\tilde{X}(a) - \tilde{X}(b)| \leq \sqrt{2}\ \|X\|\ \beta(a,b).$$

Proof. By Theorem 1 (which holds without change in the proof) and [MPS, p. 73], we have

$$\begin{aligned}\|P_a - P_b\|_{trace} &= 2\{1 - |\langle k_a, k_b\rangle|^2\}^{\frac{1}{2}} \\ &\leq 2\{1 + |\langle k_a, k_b\rangle|\}^{\frac{1}{2}}\{1 - |\langle k_a, k_b\rangle|\}^{\frac{1}{2}} \\ &\leq 2\sqrt{2}\{1 - |\langle k_a, k_b\rangle|\}^{\frac{1}{2}} \\ &\leq \sqrt{2}\beta(a,b).\end{aligned}$$

The equation

$$\tilde{X}(a) - \tilde{X}(b) = trace[X(P_a - P_b)]$$

leads to the desired result, as in the proof of Theorem 2.

Corollary. *For ψ any bounded analytic function on the bounded domain Ω in $\mathbf{C}^n$ and for a, b any two points in Ω, we have*

$$|\psi(a) - \psi(b)| \leq 2\,\|\psi\|_\infty\,\{1 - |\langle k_a, k_b\rangle|^2\}^{\frac{1}{2}}$$
$$\leq \sqrt{2}\,\|\psi\|_\infty\,\beta(a,b).$$

Proof. Take X to be "multiplication by ψ" on $A^2(\Omega)$.

Remark. I thank Miroslav Engliš for calling [MPS] to my attention.

3. The range of *Ber*

We now restrict our attention to the Hilbert space $H^2(\mathbf{C}^n, d\mu)$. The linear map $Ber : X \to \tilde{X}$ is defined and one-to-one on all linear operators X such that X and X^* have domains that include all the $\{k_a : a \in \mathbf{C}^n\}$. We denote the algebra of all bounded (and everywhere defined) linear operators on $H^2(\mathbf{C}^n, d\mu)$ by Op. The range $Ber(Op)$ is not well-understood. By our earlier discussion, it is a linear space of bounded real-analytic Lipschitz functions on $\mathbf{C}^n$. On the positive side, we have

Theorem 4. *$Ber(Op)$ is translation-invariant.*

Proof. For all c in $\mathbf{C}^n$, we consider the unitary operators $(W_c f)(z) = k_c(z) f(z-c)$. It is easy to check that

$$W_c k_a = e^{iIm(a\cdot c)/2} k_{a+c}.$$

It follows that, for any X in Op,

$$(*) \qquad \langle W_c^* X W_c k_a, k_a\rangle = \langle X k_{a+c}, k_{a+c}\rangle,$$

and so the desired result follows.

Remark. Jingbo Xia has pointed out to me that $(*)$ in the proof of Theorem 4 can be used, along with a direct calculation, to provide a strengthened version of Theorem 2: for any X in $Op\{H^2(\mathbf{C}^n, d\mu)\}$, $\tilde{X}$ and its partial derivatives of all orders are **bounded**. This version of the Lipschitz estimate depends upon the special properties of the $\{W_c\}$ which have no general analogue, even for bounded symmetric Ω.

We close with a result which shows that very nice, bounded, smooth Lipschitz functions may fail to be in $Ber(Op)$.

Theorem 5. *There is no X in Op with $\tilde{X}(a) = e^{-2|a|^2}$.*

Proof. For t real, consider the linear operators on $H^2(\mathbf{C}^n, d\mu)$ given by

$$(R_t f)(z) = f(tz).$$

The operator R_t is bounded for $|t| \leq 1$ and unbounded otherwise. By direct computation,

$$R_t k_a = k_{at}\; e^{-|a|^2(1-t^2)/4}$$

and

$$\tilde{R}_t(a) = e^{|a|^2(t-1)/2}.$$

Since $\langle R_t k_a, k_b\rangle = \langle k_a, R_t k_b\rangle$ for all a, b in $\mathbf{C}^n$, it follows that $R_t^* k_a = R_t k_a$.

Now suppose there is a bounded operator X with

$$\tilde{X}(a) = e^{-2|a|^2}.$$

Then $A = X - R_{-3}$ has the property that Ak_a, A^*k_a are all in $H^2(\mathbf{C}^n, d\mu)$ and $\tilde{A}(a) \equiv 0$. It follows that $Ak_a = Xk_a - R_{-3}k_a = 0$ and so

$$Xk_a = k_{-3a}\, e^{2|a|^2}.$$

This contradicts the boundedness of X.

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