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ON INEQUALITIES FOR ZEROS OF ENTIRE FUNCTIONS

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ABSTRACT. We derive inequalities for zeros of an entire function of finite order in terms of the coefficients of its Taylor series. Our results are new even for polynomials.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Consider the function

$$f(\lambda) = \sum_{k=0}^{\infty} c_k \lambda^k \ (\lambda \in \mathbf{C};\ c_0 = 1)$$

with complex, in general, coefficients $c_k, k = 1, 2, \ldots$. Put

$$M_f(r) := \max_{|z|=r} |f(z)| \ \ (r > 0).$$

Recall that $f(z)$ is an entire function of finite order ρ if

$$\rho := \overline{\lim}_{r\to\infty} \frac{\ln \ln M_f(r)}{\ln r} < \infty.$$

Moreover, the relation

$$\rho = \overline{\lim}_{n\to\infty} \frac{n \ln n}{\ln (1/|c_n|)} \tag{1.1}$$

is true; cf. [4, p. 6]. Everywhere below $\{z_k(f)\}_{k=1}^m$ $(m \le \infty)$ is the set of all the zeros of f taken with their multiplicities. If $m < \infty$, we set $1/z_k(f) = 0$ for $k > m$. For a positive number p, denote

$$S_p(f) := \sum_{k=1}^{\infty} |z_k(f)|^{-p}.$$

Due to the Hadamard theorem, the series $S_p(f)$ converges, provided $p > \rho$; cf. [4, p. 18]. The quantity $\rho_1 = \inf\{p \mid S_p(f) < \infty\}$ is called the "exponent of convergence" of the zeros of f, and ρ_1 has the following properties. For any entire function f, we have $\rho_1 \le \rho$, and that if ρ is finite and non-integral, then $\rho_1 = \rho$; cf. [3, Section I.10]. Depending on the particular function f, $S_\rho(f)$ may be finite or infinite [3, Section I.10].

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In the present paper we derive estimates for $S_p(f)$, which are new even for polynomials. To this end, let us rewrite function f in the form

$$f(\lambda) = \sum_{k=0}^{\infty} \frac{a_k \lambda^k}{(k!)^\gamma} \quad (\gamma > 0;\ a_0 = 1) \tag{1.2}$$

assuming that for a $p \in [1, \infty) \cap (1/\gamma, \infty)$,

$$\sum_{k=1}^{\infty} |a_k|^p < \infty. \tag{1.3}$$

According to (1.1), relations (1.2) and (1.3) imply that function f has order $\rho \le 1/\gamma$. Moreover, for any function f with $f(0) = 1$, whose order is ρ, we can take $\gamma > 1/\rho$ and $p > \max\{1, \rho\}$, such that representation (1.2) holds with condition (1.3).

The aim of this paper is to prove the following.

Theorem 1.1. *Let f be defined by* (1.2)*, and let condition* (1.3) *hold. Then*

$$S_p(f) \le \sum_{k=1}^{\infty} [|a_k|^{p'} + \frac{1}{(k+1)^{\gamma p'}}]^{p/p'} \quad (\frac{1}{p} + \frac{1}{p'} = 1) \text{ if } p > \max\{2, 1/\gamma\} \tag{1.4}$$

and

$$S_p(f) \le \sum_{k=1}^{\infty} |a_k|^p + \zeta(p\gamma) - 1 \text{ if } p \in [1, 2] \cap (1/\gamma, 2], \tag{1.5}$$

where $\zeta(.)$ is the Riemann zeta function.

The proof of this theorem is presented in the next section. Note that the case $p = 2$ was investigated in [2], but the condition $p \ne 2$ requires a new approach.

Due to (1.4) and the Minkowski inequality, we have

Corollary 1.2. *Let f be defined by* (1.2)*, and suppose condition* (1.3) *holds with $p > \max\{2, 1/\gamma\}$. Then*

$$(S_p(f))^{p'/p} \le [\sum_{k=1}^{\infty} |a_k|^p]^{p'/p} + [\ \zeta(p\gamma) - 1]^{p'/p} \quad (1/p + 1/p' = 1). \tag{1.6}$$

Let us assume that under (1.2), the condition

$$0 < b(f) := \overline{\lim}_{k\to\infty} \sqrt[k]{|a_k|} < \infty \tag{1.7}$$

holds, and consider the function

$$h_t(\lambda) \equiv f(t\lambda) = \sum_{k=0}^{\infty} \frac{a_k (t\lambda)^k}{(k!)^\gamma} \ (0 < t < 1/b(f)).$$

Due to (1.7),

$$\overline{\lim}_{k\to\infty} \sqrt[k]{|t^k a_k|^p} \le t^p b^p(f) < 1 \text{ and } \sum_{k=0}^{\infty} |a_k t^k|^p < \infty \ \ (p > 0,\ 0 < t < 1/b(f)).$$

Clearly,

$$S_p(h_t) = \sum_{k=1}^{\infty} t^p |z_k(f)|^{-p} = t^p S_p(f).$$

Now relations (1.5) and (1.6) imply

Corollary 1.3. *Let f be defined by* (1.2), *and suppose condition* (1.7) *holds. Then with $0 < t < 1/b(f)$, we have*

$$t^{p'}(S_p(f))^{p'/p} \le [\sum_{k=1}^{\infty} |t^k a_k|^p]^{p'/p} + [\zeta(p\gamma) - 1]^{p'/p} \text{ if } p > \max\{2, 1/\gamma\} \tag{1.8}$$

and

$$t^p S_p(f) \le \sum_{k=1}^{\infty} |t^k a_k|^p + \zeta(p\gamma) - 1 \text{ if } p \in [1,2] \cap (1/\gamma, 2]. \tag{1.9}$$

2. Proof of Theorem 1.1

For a natural $n \ge 2$, let us consider the polynomial

$$q_n(\lambda) = \sum_{k=0}^{n} \frac{a_k \lambda^{n-k}}{(k!)^\gamma} \ \ (a_0 = 1;\ \gamma > 0).$$

Put $\tau_k = (k+1)^{-\gamma}$ $(k = 1, ..., n-1)$ and $\tau_n = 0$.

Lemma 2.1. *The zeros $z_k(q_n)$ $(k = 1, ..., n)$ of polynomial q_n satisfy the inequalities*

$$\sum_{k=1}^{n} |z_k(q_n)|^p \le \sum_{k=1}^{n} (|a_k|^{p'} + \tau_k^{p'})^{p/p'} \ \ (1/p + 1/p' = 1) \text{ if } 2 < p < \infty \tag{2.1}$$

and

$$\sum_{k=1}^{n} |z_k(q_n)|^p \le \sum_{k=1}^{n} |a_k|^p + \tau_k^p \quad \text{if } p \in [1,2]. \tag{2.2}$$

Proof. Introduce the $(n \times n)$-matrix

$$B_n = \begin{pmatrix} -a_1 & -a_2 & -a_3 & \dots & -a_{n-1} & -a_n \\ 1/2^\gamma & 0 & 0 & \dots & 0 & 0 \\ 0 & 1/3^\gamma & 0 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & 1/n^\gamma & 0 \end{pmatrix}.$$

The direct calculations show that $q_n(\lambda) = \det(\lambda I_n - B_n)$ $(\lambda \in \mathbf{C})$, where I_n is the unit matrix. So $z_k(q_n) = \lambda_k(B_n)$. Here and below $\lambda_k(T)$ $(k = 1, ..., n)$ mean the eigenvalues of an $(n \times n)$-matrix T with their multiplicities. As is well known, for any matrix $T = (t_{jk})_{j,k=1}^n$, we have

$$\sum_{k=1}^{n} |\lambda_k(T)|^p \le \sum_{k=1}^{n} [\sum_{j=1}^{n} |t_{jk}|^{p'}]^{p/p'} \ \ (2 < p < \infty) \tag{2.3}$$

(cf. [5, p. 323, Section O.P.21]) and

$$\sum_{k=1}^{n} |\lambda_k(T)|^p \le \sum_{k=1}^{n} \sum_{j=1}^{n} |t_{jk}|^p \ \ (1 \le p \le 2) \tag{2.4}$$

(cf. [1, p. 82, Theorem 4.7]). If $T = B_n$, then $t_{1k} = -a_k$, $t_{k+1,k} = 1/(k+1)^\gamma$,

$$t_{jk} = 0 \ \ (j \ne k+1;\ j = 2, ..., n-1);\ t_{n+1,n} = 0.$$

So $t_{k+1,k} = \tau_k \ \ (k \le n)$. Hence,

$$\sum_{k=1}^n [\sum_{j=1}^n |t_{jk}|^{p'}]^{p/p'} = \sum_{k=1}^n [\ |a_k|^{p'} + |t_{k+1,k}|^{p'}]^{p/p'}$$
$$= \sum_{k=1}^n [\ |a_k|^{p'} + \tau_k^{p'}]^{p/p'} \ \ (2 < p < \infty)$$

and

$$\sum_{k=1}^n \sum_{j=1}^n |t_{jk}|^p = \sum_{k=1}^n \ |a_k|^p + |t_{k+1,k}|^p = \sum_{k=1}^n \ |a_k|^p + \tau_k^p \ \ (1 \le p \le 2).$$

Therefore (2.3) and (2.4) imply (2.1) and (2.2), respectively, as claimed. □

Corollary 2.2. *The zeros* $z_k(q_n) \ \ (k = 1, ..., n)$ *of the polynomial* q_n *satisfy the inequalities*

$$\sum_{k=1}^n |z_k(q_n)|^p \le \sum_{k=1}^n (|a_k|^{p'} + \frac{1}{(k+1)^{p'\gamma}})^{p/p'} \ \text{ if } 2 < p < \infty$$

and

$$\sum_{k=1}^n |z_k(q_n)|^p \le \sum_{k=1}^n |a_k|^p + \frac{1}{(k+1)^{p\gamma}} \ \text{ if } p \in [1,2].$$

Proof of Theorem 1.1. Consider the polynomial

$$f_n(\lambda) = \sum_{k=0}^n \frac{a_k \lambda^k}{(k!)^\gamma}.$$

Clearly, $\lambda^n f_n(1/\lambda) = q_n(\lambda)$. So $z_k(q_n) = 1/z_k(f_n)$. Taking into account that the roots are continuously dependent on the coefficients, we have the required result, letting in the previous corollary as $n \to \infty$. □

3. Examples

Let us consider the function

$$f_s(\lambda) := \frac{\sin \pi \ \lambda}{\pi\lambda} = \sum_{k=0}^\infty \frac{(\pi\lambda)^{2k}(-1)^k}{(2k+1)!}. \tag{3.1}$$

The zeros of f_s are $\pm 1, \pm 2, ...$ We thus have

$$S_2(f_s) = \sum_{k=1}^\infty |z_k(f_s)|^{-2} = 2\sum_{k=1}^\infty k^{-2} = 2\zeta(2). \tag{3.2}$$

Take $\gamma = 1, p = 2$. Then according to (1.7), $b(f_s) = \pi$. Thanks to (1.9), with $0 < t < 1/\pi$,

$$t^2 S_2(f_s) \le \sum_{k=1}^\infty (t\pi)^{4k} + \zeta(2) - 1 \le \frac{1}{1 - (t\pi)^4} + \zeta(2) - 1.$$

Take

$$t_0 = \frac{1}{\pi \sqrt[4]{3}}.$$

Then

$$\inf_{0<t<1/\pi} t^{-2}(1-(\pi t)^4)^{-1} = t_0^{-2}(1-(\pi t_0)^4)^{-1} = \pi^2 3\sqrt{3}/2.$$

We thus arrive at the inequality

$$S_2(f_s) \le \sqrt{3}\pi^2[3/2+\zeta(2)-1]. \tag{3.3}$$

Now take $\gamma = 3/4$ and $p = 2$. According to (3.1), we can write

$$f_s(\lambda) = \sum_{k=0}^{\infty} \frac{a_{2k}\lambda^{2k}}{[(2k+1)!]^{3/4}}$$

with

$$a_{2k} = \frac{\pi^{2k}(-1)^k}{[(2k+1)!]^{1/4}},\ a_{2k+1} = 0;\ k = 0, 1, 2, ...$$

Thus, due to relation (1.5),

$$S_2(f_s) \le \sum_{k=1}^{\infty} |a_k|^2 + \zeta(3/2) - 1 = \sum_{k=1}^{\infty} \frac{\pi^{4k}}{[(2k+1)!]^{1/2}} + \zeta(3/2) - 1. \tag{3.4}$$

Clearly, inequality (3.3) is sharper than (3.4). If γ increases toward 1/2, then sharpness decreases, since $\zeta(1) = \infty$.

The above illustrates how our bounds may be used to obtain an upper estimate for $S_p(f)$. As may be expected, there is a gap between the exact sum $S_2(f)$ in (3.2) and the estimate (3.3). The question of whether there is an entire function f for which equality holds in (1.4) and (1.5) is left open. The problem of providing an error estimate in using the bounds given in this paper is also left open. It may be noted that in view of the proof of Theorem 1.1, answers to those questions would seem to require first an estimate of the sharpness of (2.3) and (2.4) for the eigenvalues of matrices of the form B_n introduced in the proof of Theorem 1.1.

Finally, note that Theorem 1.1 yields no bounds on the zeros in the case $0 < \rho \le p \le 1$.

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