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BLOCKS OF CENTRAL p-GROUP EXTENSIONS

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Abstract. Let G and G' be finite groups that have a common central p-subgroup Z for a prime number p, and let $\overline{A}$ and $\overline{A'}$ respectively be p-blocks of G/Z and G'/Z induced by p-blocks A and A' respectively of G and G', both of which have the same defect group. We prove that if $\overline{A}$ and $\overline{A'}$ are Morita equivalent via a certain special $(\overline{A}, \overline{A'})$-bimodule, then such a Morita equivalence lifts to a Morita equivalence between A and A'.

0. Introduction

Let G and G' be finite groups, and let p be a prime. Let $(\mathcal{O}, K, k)$ be a splitting p-modular system for all subgroups of G and G'; that is, $\mathcal{O}$ is a complete discrete valuation ring of rank one with its quotient field K of characteristic zero and with its residue field k of characteristic p, and both K and k are splitting fields for all subgroups of G and G'.

Let A and A', respectively, be block algebras of $\mathcal{O}G$ and $\mathcal{O}G'$ such that A and A' have a common defect group P. Suppose, moreover, that P has a subgroup Z satisfying $Z \subseteq Z(G) \cap Z(G')$, where $Z(G)$ is the center of G. Then, it is well known that the algebra $\overline{A}$, which is the image of A via an epimorphism $\mathcal{O}G \twoheadrightarrow \mathcal{O}\overline{G}$ induced by the canonical epimorphism $G \twoheadrightarrow \overline{G} = G/Z$, is again a block algebra of $\mathcal{O}\overline{G}$ with defect group $\overline{P} = P/Z$; see [3, Chap. 5, Theorems 8.10 and 8.11], for instance. Similarly, we get a block algebra $\overline{A'}$ of $\mathcal{O}\overline{G'}$ with the same defect group $\overline{P}$, where $\overline{G'} = G'/Z$. Then, one may ask the following natural question:

Question. If $\overline{A}$ and $\overline{A'}$ have common properties, then can we lift them to A and A'?

There are several results concerning this question, [6, Corollary 1.12], [2, Theorem 7] and [9, Appendix A.4.]. For example, Puig in [6] proves that, under a certain hypothesis, A and A' have isomorphic source algebras as interior P-algebras (we say that A and A' are *Puig equivalent* when these two block algebras are in this situation) if $\overline{A}$ and $\overline{A'}$ have isomorphic source algebras as interior $\overline{P}$-algebras. In their recent paper [11, 3.5] Usami and Nakabayashi show that, under a certain condition, if A and A' are the principal block algebras and if $\overline{A}$ and $\overline{A'}$ are Morita equivalent, then so are A and A'.

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The purpose of this note is to generalize the result by Usami and Nakabayashi to the case that A and A' are arbitrary block algebras. It is announced in a survey article of Rouquier [9, lines 11–12 of p. 143] that, if P is abelian and if A and A' are the principal block algebras, then this is always the case; namely, A and A' are Morita (respectively Puig) equivalent whenever so are $\overline{A}$ and $\overline{A'}$ (note that in [9] he means that the same thing can be proved for non-principal block algebras; however it appears that the detailed proof has not yet shown up).

Throughout this note we use the following notation. First, we mean by a module a finitely generated right module unless stated otherwise. Let R and R' be rings. We denote by 1_R the unit element of R. For an (R, R')-bimodule M we sometimes write ${}_RM_{R'}$ to emphasize it. Similarly, for a left R-module N and a right R'-module N' we write also ${}_RN$ and $N'_{R'}$. We denote by $\mathrm{IBr}(G)$ the set of all non-isomorphic simple (irreducible) kG-modules. For a block algebra A of $\mathcal{O}G$, we set $\mathrm{IBr}(A) = \{S \in \mathrm{IBr}(G) \mid S \text{ belongs to } A\}$. Let X be a kG-module. We write $P(X)$ or $P_G(X)$ for the projective cover of X. For a projective indecomposable kG-module P', we denote by $[P'|X]$ the multiplicity of P' in the direct summands of X. For a (kG, kG')-bimodule M, we can consider M as a right $k[G \times G']$-module as usual, namely, via $m \cdot (g, g') = g^{-1}mg'$ for $m \in M$, $g \in G$ and $g' \in G'$. We write k_G for the trivial kG-module. For a kG-module X' we write $X'|X$ when X' is isomorphic to a direct summand of X as a kG-module. Let H be a subgroup of G, and let Y be a kH-module. Then, we let $Y{\uparrow}^G = Y{\uparrow}_H^G$ be the induced module of Y from H to G, namely, $Y{\uparrow}^G = Y \otimes_{kH} kG$, and let $X{\downarrow}_H = X{\downarrow}_H^G$ be the restriction of X from G to H. We write $Z(G)$ for the center of G, and ΔG for the diagonal copy of G, namely, $\Delta G = \{(g, g) \in G \times G \mid g \in G\} \cong G$. For other notation and terminology, see the books of Alperin [1], Nagao-Tsushima [3] and Thévenaz [10].

1. Main theorem

Theorem. *Let G and G' be finite groups such that G and G' have a common subgroup H satisfying $H \supseteq P \supseteq Z$ for a p-subgroup P of H and a central p-subgroup Z of G and G'; that is, $G = C_G(Z)$ and $G' = C_{G'}(Z)$. Let A and A', respectively, be block algebras of $\mathcal{O}G$ and $\mathcal{O}G'$ such that P is a defect group of A and A'. Set $\overline{G} = G/Z$, $\overline{G'} = G'/Z$, $\overline{P} = P/Z$ and $\overline{H} = H/Z$, and let $\pi : \mathcal{O}G \twoheadrightarrow \mathcal{O}\overline{G}$ and $\pi' : \mathcal{O}G' \twoheadrightarrow \mathcal{O}\overline{G'}$ be the canonical $\mathcal{O}$-algebra-epimorphisms induced by the canonical group-epimorphisms $G \twoheadrightarrow \overline{G}$ and $G' \twoheadrightarrow \overline{G'}$, respectively. Write $\overline{A} = \pi(A)$ and $\overline{A'} = \pi'(A')$. Then, it is well known that $\overline{A}$ and $\overline{A'}$, respectively, are again block algebras of $\mathcal{O}\overline{G}$ and $\mathcal{O}\overline{G'}$ such that $\overline{P}$ is a defect group of $\overline{A}$ and $\overline{A'}$ (see* [3, *Chap.* 5, *Theorems* 8.10 *and* 8.11]*).*

Now, assume that

$$ {}_{\overline{A}}(\overline{A} \otimes_{\mathcal{O}\overline{H}} \overline{A'})_{\overline{A'}} \;=\; \bigoplus_{i=1}^{m} X_i $$

is a decomposition of indecomposable right $\mathcal{O}[\overline{G} \times \overline{G'}]$-modules such that $X_1, \ldots, X_s$ are projective indecomposables and $X_{s+1}, \ldots, X_m$ are non-projective indecomposables, and set

$$ \overline{M} \;=\; X_{s+1} \oplus \cdots \oplus X_m. $$

Similarly, assume that

$$ {}_A(A \otimes_{\mathcal{O}H} A')_{A'} \;=\; \bigoplus_{j=1}^{n} Y_j $$

is a decomposition of indecomposable right $\mathcal{O}[G \times G']$-modules such that $Y_1, \ldots, Y_t$ are indecomposables with vertex ΔZ and $Y_{t+1}, \ldots, Y_n$ are indecomposables whose vertices are not ΔZ, and set

$$M \;=\; Y_{t+1} \oplus \cdots \oplus Y_n.$$

If the $(\overline{A}, \overline{A'})$-bimodule $\overline{M}$ realizes a Morita equivalence between $\overline{A}$ and $\overline{A'}$ (so that $\overline{M}_{\overline{G} \times \overline{G'}}$ is an indecomposable right $\mathcal{O}[\overline{G} \times \overline{G'}]$-module), then the (A, A')-bimodule M realizes a Morita equivalence between A and A'.

Proof. First, we prove this over k instead of over $\mathcal{O}$. Therefore, all block algebras A, A', $\overline{A}$, $\overline{A'}$ and modules M and $\overline{M}$ are over k instead of $\mathcal{O}$ for a while. It is well known that we may consider $\mathrm{IBr}(A) = \mathrm{IBr}(\overline{A})$ and $\mathrm{IBr}(A') = \mathrm{IBr}(\overline{A'})$. We can write

$$\left(\overline{A} \otimes_{k\overline{H}} \overline{A'}\right)_{\overline{G} \times \overline{G'}} \;=\; \overline{M}_{\overline{G} \times \overline{G'}} \;\oplus\; \left(\bigoplus_{\substack{S \in \mathrm{IBr}(\overline{A}) \\ S' \in \mathrm{IBr}(\overline{A'})}} m(S, S') \times P_{\overline{G} \times \overline{G'}}(S \otimes_k S') \right) \tag{1}$$

for non-negative integers $m(S, S')$ since $1_{\overline{A}} \cdot X_i \cdot 1_{\overline{A'}} \;=\; X_i$ for any i. Since

$$\overline{A} \otimes_{k\overline{H}} \overline{A'} \;\Big|\; k\overline{G} \otimes_{k\overline{H}} k\overline{G'} \;=\; k_{\Delta \overline{H}} \uparrow^{\overline{G} \times \overline{G'}},$$

we obtain that, for each S and S',

$$m(S, S') \;=\; \left[P_{\overline{G} \times \overline{G'}}(S \otimes_k S') \;\Big|\; k_{\Delta \overline{H}} \uparrow^{\overline{G} \times \overline{G'}} \right]. \tag{2}$$

Note that

$$\{P_{(G \times G')/\Delta Z}(S \otimes_k S') \mid S \in \mathrm{IBr}(A),\; S' \in \mathrm{IBr}(A')\}$$

is the set of all trivial source (p-permutation) $k[G \times G']$-modules with vertex ΔZ in a block algebra $A \otimes_k A'$ of $k[G \times G']$; see [3, Chap. 4, Problem 10]. Hence, by the definition of M, we can write

$$\begin{aligned} &\left(A \otimes_{kH} A'\right)_{G \times G'} \\ &= M_{G \times G'} \;\oplus\; \left(\bigoplus_{\substack{S \in \mathrm{IBr}(A) \\ S' \in \mathrm{IBr}(A')}} n(S, S') \times P_{(G \times G')/\Delta Z}(S \otimes_k S') \right) \end{aligned} \tag{3}$$

for non-negative integers $n(S, S')$. Since

$$\begin{aligned} (A \otimes_{kH} A')_{G \times G'} \;\Big|\; &{}_{kG}(kG \otimes_{kH} kG')_{kG'} \\ &\cong k_{\Delta H} \uparrow^{G \times G'} \\ &\cong k_{\Delta H/\Delta Z} \uparrow^{(G \times G')/\Delta Z}, \end{aligned}$$

as right $k[G \times G']$-modules, we know that, for each S and S',

$$n(S, S') \;=\; \left[P_{(G \times G')/\Delta Z}(S \otimes_k S') \;\Big|\; k_{\Delta H/\Delta Z} \uparrow^{(G \times G')/\Delta Z} \right]. \tag{4}$$

Now, set

$$\overline{N} \;=\; k\overline{G} \otimes_{kG} A \otimes_{kH} A' \otimes_{kG'} k\overline{G'}.$$

Then, we get by (3) that

$$\begin{aligned}\overline{N} &= (k\overline{G} \otimes_{kG} M \otimes_{kG'} k\overline{G'}) \\ &\oplus \left(\bigoplus_{\substack{S \in \mathrm{IBr}(A) \\ S' \in \mathrm{IBr}(A')}} n(S,S') \times \left(k\overline{G} \otimes_{kG} P_{(G\times G')/\Delta Z}(S \otimes_k S') \otimes_{kG'} k\overline{G'} \right) \right).\end{aligned}$$

On the other hand,

$$\begin{aligned}\overline{N} &= (k\overline{G} \otimes_{kG} A) \otimes_{kH} (A' \otimes_{kG'} k\overline{G'}) \\ &= ({}_{k\overline{G}}\overline{A}_{kG}) \otimes_{kH} ({}_{kG'}\overline{A'}_{k\overline{G'}}) \qquad \text{since} \quad k\overline{G} \otimes_{kG} A = \pi(A) = \bar{A} \\ &= {}_{k\overline{G}}(\overline{A} \otimes_{k\overline{H}} \overline{A'})_{k\overline{G'}} \\ &= \overline{M}_{\overline{G}\times\overline{G'}} \oplus \left(\bigoplus_{\substack{S \in \mathrm{IBr}(\overline{A}) \\ S' \in \mathrm{IBr}(\overline{A'})}} m(S,S') \times P_{\overline{G}\times\overline{G'}}(S \otimes_k S') \right) \qquad \text{by} \quad (1).\end{aligned}$$

Take any $S \in \mathrm{IBr}(A) = \mathrm{IBr}(\overline{A})$ and $S' \in \mathrm{IBr}(A') = \mathrm{IBr}(\overline{A'})$. Then,

$$\begin{aligned}n(S,S') &= \left[P_{(G\times G')/\Delta Z}(S \otimes_k S') \;\middle|\; k_{\Delta H/\Delta Z} \uparrow^{(G\times G')/\Delta Z} \right] \qquad \text{by} \quad (4) \\ &= \left[P(k_{\Delta H/\Delta Z}) \;\middle|\; (S \otimes_k S') \downarrow^{(G\times G')/\Delta Z}_{\Delta H/\Delta Z} \right] \qquad \text{by} \quad [7, \text{Theorem } 3] \\ &= \left[P(k_{\Delta\overline{H}}) \;\middle|\; (S \otimes_k S') \downarrow_{\Delta\overline{H}} \right] \qquad \text{since } \Delta H/\Delta Z \cong \Delta\overline{H} \\ &= \left[P(k_{\Delta\overline{H}}) \;\middle|\; (S \otimes_k S') \downarrow^{\overline{G}\times\overline{G'}}_{\Delta\overline{H}} \right] \\ &= \left[P_{\overline{G}\times\overline{G'}}(S \otimes_k S') \;\middle|\; k_{\Delta\overline{H}} \uparrow^{(\overline{G}\times\overline{G'})} \right] \qquad \text{by} \quad [7, \text{Theorem } 3] \\ &= m(S,S') \qquad \text{by} \quad (2).\end{aligned}$$

Moreover, it follows that

$$\begin{aligned}P_{\overline{G}\times\overline{G'}}(S \otimes_k S') &= P_{[(G\times G')/\Delta Z]/[(Z\times Z)/\Delta Z]}(S \otimes_k S') \\ &= P_{(G\times G')/\Delta Z}(S \otimes_k S') \otimes_{k[(G\times G')/\Delta Z]} k[\overline{G} \times \overline{G'}] \\ &\qquad \text{by} \quad [12, 2.1. \text{ Proposition (a)}] \\ &= P_{(G\times G')/\Delta Z}(S \otimes_k S') \otimes_{k[G\times G']} k[\overline{G} \times \overline{G'}] \\ &\qquad \text{since there are canonical epimorphisms} \\ &\qquad G \times G' \twoheadrightarrow (G\times G')/\Delta Z \twoheadrightarrow \overline{G} \times \overline{G'} \\ &= k\overline{G} \otimes_{kG} P_{(G\times G')/\Delta Z}(S \otimes_k S') \otimes_{kG'} k\overline{G'}.\end{aligned}$$

Therefore, it follows from a theorem of Krull-Schmidt that

$$\overline{M}_{\overline{G}\times\overline{G'}} = k\overline{G} \otimes_{kG} M_{G\times G'} \otimes_{kG'} k\overline{G'}. \tag{5}$$

Now, by the hypothesis, $\overline{A}$ and $\overline{A'}$ are Morita equivalent via ${}_{\overline{A}}\overline{M}_{\overline{A'}}$. Thus, we get from (5) and a result of Rouquier [8, Lemma 10.2.11 and its proof] that A and A' are Morita equivalent via ${}_AM_{A'}$.

Now, it follows that

$${}_AM_{A'} = M_{G\times G'} \;\Big|\; A \otimes_{kH} A' \;\Big|\; kG \otimes_{kH} kG' = k_{\Delta H} \uparrow^{G\times G'},$$

so that $M_{G\times G'}$ is a trivial source (p-permutation) $k[G\times G']$-module and is ΔH-projective. Thus, $M_{G\times G'}$ has ΔP as a vertex since A and A' have P as defect groups. Therefore, A and A' are Puig equivalent by a theorem of Puig (independently by Scott) [5, Remark 7.5]. That is, source algebras of A and A' are isomorphic as interior P-algebras. Hence, we get by a theorem of Puig [4, Lemma 7.8] (see [10, (38.7)Proposition and (38.8)Proposition]) that the Morita (Puig) equivalence between A and A' lifts from k to $\mathcal{O}$. We are done. $\square$

Corollary. *Keep the notation and assumption as in the theorem. Suppose, moreover, that $G' = H \supseteq N_G(P)$, and let $B = A'$. If*

$$1_{\overline{A}}\cdot\mathcal{O}\overline{G}\cdot 1_{\overline{B}} = f{\downarrow}^{\overline{G}\times\overline{G}}_{\overline{G}\times\overline{H},\Delta\overline{P}}(\overline{A}) \ \oplus \textit{(projective } \mathcal{O}[\overline{G}\times\overline{H}]\textit{-module)}$$

and if $\overline{M} = f{\downarrow}^{\overline{G}\times\overline{G}}_{\overline{G}\times\overline{H},\Delta\overline{P}}(\overline{A})$ realizes a Morita equivalence between $\overline{A}$ and $\overline{B}$, then $M = f{\downarrow}^{G\times G}_{G\times H,\Delta P}(A)$ realizes a Morita equivalence between A and B, where $f{\downarrow}^{\overline{G}\times\overline{G}}_{\overline{G}\times\overline{H},\Delta\overline{P}}$ and $f{\downarrow}^{G\times G}_{G\times H,\Delta P}$ are the Green correspondences with respect to $(\overline{G}\times\overline{G}, \Delta\overline{P}, \overline{G}\times\overline{H})$ and $(G\times G, \Delta P, G\times H)$, respectively; see [3, *Chap.* 4, *p.* 276].

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