PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 1, Pages 21–26
S 0002-9939(04)07509-4
Article electronically published on July 26, 2004

BLOCKS OF CENTRAL p-GROUP EXTENSIONS

SHIGEO KOSHITANI AND NAOKO KUNUGI

(Communicated by Jonathan I. Hall)

ABSTRACT. Let G and G' be finite groups that have a common central p-subgroup Z for a prime number p, and let $\overline{A}$ and $\overline{A'}$ respectively be p-blocks of G/Z and G'/Z induced by p-blocks A and A' respectively of G and G', both of which have the same defect group. We prove that if $\overline{A}$ and $\overline{A'}$ are Morita equivalent via a certain special $(\overline{A}, \overline{A'})$-bimodule, then such a Morita equivalence lifts to a Morita equivalence between A and A'.

0. Introduction

Let G and G' be finite groups, and let p be a prime. Let $(\mathcal{O}, K, k)$ be a splitting p-modular system for all subgroups of G and G'; that is, $\mathcal{O}$ is a complete discrete valuation ring of rank one with its quotient field K of characteristic zero and with its residue field k of characteristic p, and both K and k are splitting fields for all subgroups of G and G'.

Let A and A', respectively, be block algebras of $\mathcal{O}G$ and $\mathcal{O}G'$ such that A and A' have a common defect group P. Suppose, moreover, that P has a subgroup Z satisfying $Z \subseteq Z(G) \cap Z(G')$, where $Z(G)$ is the center of G. Then, it is well known that the algebra $\overline{A}$, which is the image of A via an epimorphism $\mathcal{O}G \twoheadrightarrow \mathcal{O}\overline{G}$ induced by the canonical epimorphism $G \twoheadrightarrow \overline{G} = G/Z$, is again a block algebra of $\mathcal{O}\overline{G}$ with defect group $\overline{P} = P/Z$; see [3, Chap. 5, Theorems 8.10 and 8.11], for instance. Similarly, we get a block algebra $\overline{A'}$ of $\mathcal{O}\overline{G'}$ with the same defect group $\overline{P}$, where $\overline{G'} = G'/Z$. Then, one may ask the following natural question:

Question. If $\overline{A}$ and $\overline{A'}$ have common properties, then can we lift them to A and A'?

There are several results concerning this question, [6, Corollary 1.12], [2, Theorem 7] and [9, Appendix A.4.]. For example, Puig in [6] proves that, under a certain hypothesis, A and A' have isomorphic source algebras as interior P-algebras (we say that A and A' are *Puig equivalent* when these two block algebras are in this situation) if $\overline{A}$ and $\overline{A'}$ have isomorphic source algebras as interior $\overline{P}$-algebras. In their recent paper [11, 3.5] Usami and Nakabayashi show that, under a certain condition, if A and A' are the principal block algebras and if $\overline{A}$ and $\overline{A'}$ are Morita equivalent, then so are A and A'.

Received by the editors April 25, 2003 and, in revised form, September 8, 2003.
2000 *Mathematics Subject Classification.* Primary 20C20, 20C05, 20C11.
Key words and phrases. p-block, Morita equivalence, central extension.

The purpose of this note is to generalize the result by Usami and Nakabayashi to the case that A and A' are arbitrary block algebras. It is announced in a survey article of Rouquier [9, lines 11–12 of p. 143] that, if P is abelian and if A and A' are the principal block algebras, then this is always the case; namely, A and A' are Morita (respectively Puig) equivalent whenever so are $\overline{A}$ and $\overline{A'}$ (note that in [9] he means that the same thing can be proved for non-principal block algebras; however it appears that the detailed proof has not yet shown up).

Throughout this note we use the following notation. First, we mean by a module a finitely generated right module unless stated otherwise. Let R and R' be rings. We denote by 1_R the unit element of R. For an (R, R')-bimodule M we sometimes write ${}_RM_{R'}$ to emphasize it. Similarly, for a left R-module N and a right R'-module N' we write also ${}_RN$ and $N'_{R'}$. We denote by $\mathrm{IBr}(G)$ the set of all non-isomorphic simple (irreducible) kG-modules. For a block algebra A of $\mathcal{O}G$, we set $\mathrm{IBr}(A) = \{S \in \mathrm{IBr}(G) \mid S \text{ belongs to } A\}$. Let X be a kG-module. We write $P(X)$ or $P_G(X)$ for the projective cover of X. For a projective indecomposable kG-module P', we denote by $[P'|X]$ the multiplicity of P' in the direct summands of X. For a (kG, kG')-bimodule M, we can consider M as a right $k[G\times G']$-module as usual, namely, via $m{\cdot}(g, g') = g^{-1}mg'$ for $m \in M$, $g \in G$ and $g' \in G'$. We write k_G for the trivial kG-module. For a kG-module X' we write $X'|X$ when X' is isomorphic to a direct summand of X as a kG-module. Let H be a subgroup of G, and let Y be a kH-module. Then, we let $Y{\uparrow}^G = Y{\uparrow}_H^G$ be the induced module of Y from H to G, namely, $Y{\uparrow}^G = Y \otimes_{kH} kG$, and let $X{\downarrow}_H = X{\downarrow}_H^G$ be the restriction of X from G to H. We write $Z(G)$ for the center of G, and ΔG for the diagonal copy of G, namely, $\Delta G = \{(g, g) \in G \times G \mid g \in G\} \cong G$. For other notation and terminology, see the books of Alperin [1], Nagao-Tsushima [3] and Thévenaz [10].

1. Main theorem

Theorem. *Let G and G' be finite groups such that G and G' have a common subgroup H satisfying $H \supseteq P \supseteq Z$ for a p-subgroup P of H and a central p-subgroup Z of G and G'; that is, $G = C_G(Z)$ and $G' = C_{G'}(Z)$. Let A and A', respectively, be block algebras of $\mathcal{O}G$ and $\mathcal{O}G'$ such that P is a defect group of A and A'. Set $\overline{G} = G/Z$, $\overline{G'} = G'/Z$, $\overline{P} = P/Z$ and $\overline{H} = H/Z$, and let $\pi : \mathcal{O}G \twoheadrightarrow \mathcal{O}\overline{G}$ and $\pi' : \mathcal{O}G' \twoheadrightarrow \mathcal{O}\overline{G'}$ be the canonical $\mathcal{O}$-algebra-epimorphisms induced by the canonical group-epimorphisms $G \twoheadrightarrow \overline{G}$ and $G' \twoheadrightarrow \overline{G'}$, respectively. Write $\overline{A} = \pi(A)$ and $\overline{A'} = \pi'(A')$. Then, it is well known that $\overline{A}$ and $\overline{A'}$, respectively, are again block algebras of $\mathcal{O}\overline{G}$ and $\mathcal{O}\overline{G'}$ such that $\overline{P}$ is a defect group of $\overline{A}$ and $\overline{A'}$ (see [3, Chap. 5, Theorems 8.10 and 8.11]).*

Now, assume that

$${}_{\overline{A}}(\overline{A} \otimes_{\mathcal{O}\overline{H}} \overline{A'})_{\overline{A'}} \;=\; \bigoplus_{i=1}^{m} X_i$$

is a decomposition of indecomposable right $\mathcal{O}[\overline{G}\times\overline{G'}]$-modules such that $X_1, \ldots, X_s$ are projective indecomposables and $X_{s+1}, \ldots, X_m$ are non-projective indecomposables, and set

$$\overline{M} \;=\; X_{s+1} \oplus \cdots \oplus X_m.$$

Similarly, assume that

$${}_A(A \otimes_{\mathcal{O}H} A')_{A'} \;=\; \bigoplus_{j=1}^{n} Y_j$$

is a decomposition of indecomposable right $\mathcal{O}[G \times G']$-modules such that $Y_1, \dots, Y_t$ are indecomposables with vertex ΔZ and $Y_{t+1}, \dots, Y_n$ are indecomposables whose vertices are not ΔZ, and set

$$M \;=\; Y_{t+1} \oplus \cdots \oplus Y_n .$$

If the $(\overline{A}, \overline{A'})$-bimodule $\overline{M}$ realizes a Morita equivalence between $\overline{A}$ and $\overline{A'}$ (so that $\overline{M}_{\overline{G}\times\overline{G'}}$ is an indecomposable right $\mathcal{O}[\overline{G} \times \overline{G'}]$-module), then the (A, A')-bimodule M realizes a Morita equivalence between A and A'.

Proof. First, we prove this over k instead of over $\mathcal{O}$. Therefore, all block algebras A, A', $\overline{A}$, $\overline{A'}$ and modules M and $\overline{M}$ are over k instead of $\mathcal{O}$ for a while. It is well known that we may consider $\mathrm{IBr}(A) = \mathrm{IBr}(\overline{A})$ and $\mathrm{IBr}(A') = \mathrm{IBr}(\overline{A'})$. We can write

$$(1) \qquad \left(\overline{A} \otimes_{k\overline{H}} \overline{A'}\right)_{\overline{G}\times\overline{G'}} \;=\; \overline{M}_{\overline{G}\times\overline{G'}} \;\oplus\; \left(\bigoplus_{\substack{S \in \mathrm{IBr}(\overline{A}) \\ S' \in \mathrm{IBr}(\overline{A'})}} m(S,S') \times P_{\overline{G}\times\overline{G'}}(S \otimes_k S') \right)$$

for non-negative integers $m(S,S')$ since $1_{\overline{A}} {\cdot} X_i {\cdot} 1_{\overline{A'}} \;=\; X_i$ for any i. Since

$$\overline{A} \otimes_{k\overline{H}} \overline{A'} \;\Big|\; k\overline{G} \otimes_{k\overline{H}} k\overline{G'} \;=\; k_{\Delta\overline{H}} \uparrow^{\overline{G}\times\overline{G'}},$$

we obtain that, for each S and S',

$$(2) \qquad m(S,S') \;=\; \left[P_{\overline{G}\times\overline{G'}}(S \otimes_k S') \;\Big|\; k_{\Delta\overline{H}} \uparrow^{\overline{G}\times\overline{G'}} \right].$$

Note that

$$\{P_{(G\times G')/\Delta Z}(S \otimes_k S') \mid S \in \mathrm{IBr}(A),\ S' \in \mathrm{IBr}(A')\}$$

is the set of all trivial source (p-permutation) $k[G \times G']$-modules with vertex ΔZ in a block algebra $A \otimes_k A'$ of $k[G \times G']$; see [3, Chap. 4, Problem 10]. Hence, by the definition of M, we can write

$$(3) \qquad \begin{aligned} &\left(A \otimes_{kH} A'\right)_{G\times G'} \\ &= M_{G\times G'} \;\oplus\; \left(\bigoplus_{\substack{S \in \mathrm{IBr}(A) \\ S' \in \mathrm{IBr}(A')}} n(S,S') \times P_{(G\times G')/\Delta Z}(S \otimes_k S') \right) \end{aligned}$$

for non-negative integers $n(S,S')$. Since

$$\begin{aligned} (A \otimes_{kH} A')_{G\times G'} \;\Big|\; &{}_{kG}(kG \otimes_{kH} kG')_{kG'} \\ &\cong k_{\Delta H} \uparrow^{G\times G'} \\ &\cong k_{\Delta H/\Delta Z} \uparrow^{(G\times G')/\Delta Z}, \end{aligned}$$

as right $k[G \times G']$-modules, we know that, for each S and S',

$$(4) \qquad n(S,S') \;=\; \left[P_{(G\times G')/\Delta Z}(S \otimes_k S') \;\Big|\; k_{\Delta H/\Delta Z} \uparrow^{(G\times G')/\Delta Z} \right].$$

Now, set

$$\overline{N} \;=\; k\overline{G} \otimes_{kG} A \otimes_{kH} A' \otimes_{kG'} k\overline{G'}.$$

Then, we get by (3) that

$$\begin{aligned}\overline{N} &= (k\overline{G} \otimes_{kG} M \otimes_{kG'} k\overline{G'}) \\ &\oplus \left(\bigoplus_{\substack{S\in \mathrm{IBr}(A) \\ S'\in \mathrm{IBr}(A')}} n(S,S') \times \left(k\overline{G} \otimes_{kG} P_{(G\times G')/\Delta Z}(S \otimes_k S') \otimes_{kG'} k\overline{G'} \right) \right).\end{aligned}$$

On the other hand,

$$\begin{aligned}\overline{N} &= (k\overline{G} \otimes_{kG} A) \otimes_{kH} (A' \otimes_{kG'} k\overline{G'}) \\ &= ({}_{k\overline{G}}\overline{A}_{kG}) \otimes_{kH} ({}_{kG'}\overline{A'}_{k\overline{G'}}) \qquad \text{since} \quad k\overline{G} \otimes_{kG} A = \pi(A) = \bar{A} \\ &= {}_{k\overline{G}}(\overline{A} \otimes_{k\overline{H}} \overline{A'})_{k\overline{G'}} \\ &= \overline{M}_{\overline{G}\times\overline{G'}} \oplus \left(\bigoplus_{\substack{S\in \mathrm{IBr}(\overline{A}) \\ S'\in \mathrm{IBr}(\overline{A'})}} m(S,S') \times P_{\overline{G}\times\overline{G'}}(S \otimes_k S') \right) \qquad \text{by} \quad (1).\end{aligned}$$

Take any $S \in \mathrm{IBr}(A) = \mathrm{IBr}(\overline{A})$ and $S' \in \mathrm{IBr}(A') = \mathrm{IBr}(\overline{A'})$. Then,

$$\begin{aligned}n(S,S') &= \left[P_{(G\times G')/\Delta Z}(S \otimes_k S') \,\middle|\, k_{\Delta H/\Delta Z} \uparrow^{(G\times G')/\Delta Z} \right] \qquad \text{by} \quad (4) \\ &= \left[P(k_{\Delta H/\Delta Z}) \,\middle|\, (S \otimes_k S') \downarrow^{(G\times G')/\Delta Z}_{\Delta H/\Delta Z} \right] \qquad \text{by} \quad [7, \text{Theorem } 3] \\ &= \left[P(k_{\Delta \overline{H}}) \,\middle|\, (S \otimes_k S') \downarrow_{\Delta \overline{H}} \right] \qquad \text{since } \Delta H/\Delta Z \cong \Delta \overline{H} \\ &= \left[P(k_{\Delta \overline{H}}) \,\middle|\, (S \otimes_k S') \downarrow^{\overline{G}\times\overline{G'}}_{\Delta \overline{H}} \right] \\ &= \left[P_{\overline{G}\times\overline{G'}}(S \otimes_k S') \,\middle|\, k_{\Delta \overline{H}} \uparrow^{(\overline{G}\times\overline{G'})} \right] \qquad \text{by} \quad [7, \text{Theorem } 3] \\ &= m(S,S') \qquad \text{by} \quad (2).\end{aligned}$$

Moreover, it follows that

$$\begin{aligned}P_{\overline{G}\times\overline{G'}}(S \otimes_k S') &= P_{[(G\times G')/\Delta Z]/[(Z\times Z)/\Delta Z]}(S \otimes_k S') \\ &= P_{(G\times G')/\Delta Z}(S \otimes_k S') \otimes_{k[(G\times G')/\Delta Z]} k[\overline{G} \times \overline{G'}] \\ &\qquad \text{by} \quad [12, 2.1. \text{ Proposition (a)}] \\ &= P_{(G\times G')/\Delta Z}(S \otimes_k S') \otimes_{k[G\times G']} k[\overline{G} \times \overline{G'}] \\ &\qquad \text{since there are canonical epimorphisms} \\ &\qquad G \times G' \twoheadrightarrow (G \times G')/\Delta Z \twoheadrightarrow \overline{G} \times \overline{G'} \\ &= k\overline{G} \otimes_{kG} P_{(G\times G')/\Delta Z}(S \otimes_k S') \otimes_{kG'} k\overline{G'}.\end{aligned}$$

Therefore, it follows from a theorem of Krull-Schmidt that

$$\overline{M}_{\overline{G}\times\overline{G'}} = k\overline{G} \otimes_{kG} M_{G\times G'} \otimes_{kG'} k\overline{G'}. \tag{5}$$

Now, by the hypothesis, $\overline{A}$ and $\overline{A'}$ are Morita equivalent via ${}_{\overline{A}}\overline{M}_{\overline{A'}}$. Thus, we get from (5) and a result of Rouquier [8, Lemma 10.2.11 and its proof] that A and A' are Morita equivalent via ${}_AM_{A'}$.

Now, it follows that

$${}_AM_{A'} = M_{G\times G'} \,\Big|\, A \otimes_{kH} A' \,\Big|\, kG \otimes_{kH} kG' = k_{\Delta H} \uparrow^{G\times G'},$$

so that $M_{G\times G'}$ is a trivial source (p-permutation) $k[G\times G']$-module and is ΔH-projective. Thus, $M_{G\times G'}$ has ΔP as a vertex since A and A' have P as defect groups. Therefore, A and A' are Puig equivalent by a theorem of Puig (independently by Scott) [5, Remark 7.5]. That is, source algebras of A and A' are isomorphic as interior P-algebras. Hence, we get by a theorem of Puig [4, Lemma 7.8] (see [10, (38.7)Proposition and (38.8)Proposition]) that the Morita (Puig) equivalence between A and A' lifts from k to $\mathcal{O}$. We are done. □

Corollary. *Keep the notation and assumption as in the theorem. Suppose, moreover, that $G' = H \supseteq N_G(P)$, and let $B = A'$. If*

$$1_{\overline{A}}\cdot\mathcal{O}\overline{G}\cdot 1_{\overline{B}} = f{\downarrow}^{\overline{G}\times\overline{G}}_{\overline{G}\times\overline{H},\Delta\overline{P}}(\overline{A})\ \oplus \text{(projective } \mathcal{O}[\overline{G}\times\overline{H}]\text{-module)}$$

and if $\overline{M} = f{\downarrow}^{\overline{G}\times\overline{G}}_{\overline{G}\times\overline{H},\Delta\overline{P}}(\overline{A})$ realizes a Morita equivalence between $\overline{A}$ and $\overline{B}$, then $M = f{\downarrow}^{G\times G}_{G\times H,\Delta P}(A)$ realizes a Morita equivalence between A and B, where $f{\downarrow}^{\overline{G}\times\overline{G}}_{\overline{G}\times\overline{H},\Delta\overline{P}}$ and $f{\downarrow}^{G\times G}_{G\times H,\Delta P}$ are the Green correspondences with respect to $(\overline{G}\times\overline{G}, \Delta\overline{P}, \overline{G}\times\overline{H})$ and $(G\times G, \Delta P, G\times H)$, respectively; see [3, *Chap.* 4, *p.* 276].

Acknowledgments

This paper was in part prepared while the second author was staying at the School of Mathematics, University of Bristol as a Research Fellow of the Japan Society for Promotion of Science (JSPS). She is grateful to Jeremy Rickard for his kind hospitality and also to the JSPS for the financial support.

The first author was in part supported by the Ministry of Education, Culture, Sports, Science and Technology, Grant-in-Aid for JSPS Fellows 01016, 2001–2002; and the JSPS, Grant-in-Aid for Scientific Research C(2) 14540009, 2002. The second author was in part supported by the Ministry of Education, Culture, Sports, Science and Technology, Grant-in-Aid for JSPS Fellows 07600, 2001-2002.

References

1. J. L. Alperin, *Local representation theory*, Cambridge Univ. Press, Cambridge, 1986. MR87i:20002
2. B. Külshammer, T. Okuyama, A. Watanabe, *A lifting theorem with applications to blocks and source algebras*, J. Algebra **232** (2000), 299–309. MR2001g:20013
3. H. Nagao, Y. Tsushima, *Representations of finite groups*, Academic Press, New York, 1988. MR90h:20008
4. L. Puig, *Nilpotent blocks and their source algebras*, Invent. Math. **93** (1988), 77–116. MR89e:20023
5. L. Puig, *On the local structure of Morita and Rickard equivalences between Brauer blocks*, Birkhäuser, Basel, 1999. MR2001d:20006
6. L. Puig, *Source algebras of p-central group extensions*, J. Algebra **235** (2001), 359–398. MR2001k:20006
7. G. R. Robinson, *On projective summands of induced modules*, J. Algebra **122** (1989), 106–111. MR90c:20011
8. R. Rouquier, *The derived category of blocks with cyclic defect groups*, in *Derived equivalences for group rings*, S. König and A. Zimmermann, Lecture Notes in Mathematics, Vol. 1685, Springer, Berlin, 1998, pp. 199–220. MR2000g:16018
9. R. Rouquier, *Block theory via stable and Rickard equivalences*, in *Modular representation theory of finite groups*, M. J. Collins, B. J. Parshall, L. L. Scott (Eds.), de Gruyter, Berlin, 2001, pp. 101–146. MR2003g:20018
10. J. Thévenaz, *G-Algebras and modular representation theory*, Clarendon Press, Oxford, 1995. MR96j:20017

11. Y. Usami, M. Nakabayashi, *Morita equivalent principal* 3*-blocks of the Chevalley group* $G_2(q)$, Proc. London Math. Soc.**(3) 86** (2003), 397–434. MR2004c:20025
12. W. Willems, *On the projectives of a group algebra*, Math. Zeit. **171** (1980), 163–174. MR81g:20007

Department of Mathematics, Faculty of Science, Chiba University, Yayoi-cho, Inage-ku, Chiba, 263-8522, Japan
E-mail address: koshitan@math.s.chiba-u.ac.jp

Department of Mathematics, Aichi University of Education, Hirosawa, Igaya-cho, Kariya, 448-8542, Japan
E-mail address: nkunugi@auecc.aichi-edu.ac.jp