

ON A GENERALIZED CORONA PROBLEM ON THE UNIT DISC

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ABSTRACT. Let $g, f_1, \dots, f_n \in H^\infty$. We give a sufficient condition on the size of a function g in order for it to be in the ideal generated by f_1, \dots, f_n . In particular, this improves Cegrell's result on this problem.

INTRODUCTION

Let \mathbb{D} be the unit disc in the complex plane, and let $H^\infty = H^\infty(\mathbb{D})$ be the Banach algebra of bounded analytic functions on \mathbb{D} . Carleson's corona theorem says that the unit disc is dense in the space M_{H^∞} of maximal ideals of H^∞ with the weak-star topology. This result is equivalent to the following fact: if we have functions $f_1, \dots, f_n \in H^\infty$ such that

$$\sum_{j=1}^n |f_j(z)| \geq \delta > 0, \quad \forall z \in \mathbb{D},$$

then there exist solutions $g_1, \dots, g_n \in H^\infty$ of the equation

$$\sum_{j=1}^n f_j g_j = 1.$$

In order to generalize this result, it is natural to ask if it is possible to replace the function 1 by an arbitrary function $g \in H^\infty$; that is, one asks if the condition

$$(1) \quad |g(z)| \leq C \sum_{j=1}^n |f_j(z)|, \quad \forall z \in \mathbb{D},$$

implies that the function g belongs to the ideal I generated by f_1, \dots, f_n . Condition (1) is clearly a necessary condition, but an example given by Rao (see [Ra]) shows that the answer is, in general, negative. Thus the following problem arises naturally.

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Problem A. Let h be a positive continuous function on $[0, \infty)$ increasing in a neighbourhood of zero, and let $g, f_1, \dots, f_n \in H^\infty$. For which functions h does the condition

$$(2) \quad |g(z)| \leq h(|f_1(z)| + \dots + |f_n(z)|), \quad \forall z \in \mathbb{D},$$

imply that the function g is in the ideal generated by f_1, \dots, f_n ?

For functions of the form $h(s) = s^\alpha$, with $\alpha \geq 1$, the problem is completely solved. For $1 \leq \alpha < 2$, a variation of Rao's example shows that the answer is negative, and for $\alpha > 2$, work of Wolff, Cegrell and others gives an affirmative answer (see [Ce1], [Ga]). The problem for $\alpha = 2$ was an old question of Wolff, which remained open for twenty years. However, Treil (see [Tr]) has recently shown (using a connection with the best estimates of the solutions of the corona theorem) that the answer is, in general, negative.

In [Li], Lin gave an affirmative answer for this problem for the function

$$h(s) = s^2(-\log s)^{-(3/2+\varepsilon)}$$

with $\varepsilon > 0$, and in [Ce2] Cegrell established the following strongest known positive case for this problem.

Theorem A (Cegrell). *Let $f_1, \dots, f_n \in H^\infty$ with $|f(z)|^2 = \sum_{j=1}^n |f_j(z)|^2 > 0$, for all $z \in \mathbb{D}$. Then, Problem A has an affirmative answer for*

$$h(s) = \frac{s^2}{(-\log s)^{3/2}(\log(-\log s))^{3/2} \log \log(-\log s)}.$$

Our main result below is an improvement of Cegrell's theorem.

Theorem 1. *Let $k : (0, 1) \rightarrow [0, \infty)$ be a nondecreasing bounded continuous function such that $k(x)/x$ is nonincreasing and*

$$\int_0^1 \frac{k(x)}{x} |\log x| dx < +\infty,$$

and let $H(x) = \sqrt{k(x) \int_0^x \frac{k(s)}{s} ds}$. Furthermore, let $g, f_1, \dots, f_n \in H^\infty$, where $0 < |f|^2 := \sum_{j=1}^n |f_j|^2 < 1$. Then the condition

$$|g| \leq |f|^2 H(|f|^2)$$

implies the existence of solutions $g_1, \dots, g_n \in H^\infty$ of the equation

$$g = f_1 g_1 + \dots + f_n g_n.$$

For example, if we take $k(x) = |\log x|^{-2}(\log |\log x|)^{-3/2}$, we see that Problem A has an affirmative answer for the function

$$h(s) = s^2(-\log s)^{-3/2}(\log(-\log s))^{-1},$$

and this clearly improves Cegrell's result.

For $1 \leq p < \infty$, let H^p be the Hardy space of analytic functions in the unit disc such that

$$\|f\|_p^p = \sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$

It is well known that an analytic function f belongs to H^p if and only if the nontangential maximal function $Mf(e^{i\theta}) = \sup\{|f(z)| : z \in \Gamma(\theta)\}$ belongs to the usual Lebesgue space $L^p(\mathbb{T})$, where

$$\Gamma(\theta) := \{z \in \mathbb{D} : |e^{i\theta} - z| < (1 + \alpha)(1 - |z|)\}$$

is the Stolz angle with vertex at $e^{i\theta}$ and fixed aperture $\alpha > 0$ (the choice of α is irrelevant here), and \mathbb{T} denotes the unit circle. Several H^p versions of the corona theorem have been considered. Concretely, one is interested in conditions on functions $f_1, \dots, f_n \in H^\infty$ such that the equation

$$(3) \quad 1 = f_1g_1 + \dots + f_ng_n$$

has solutions g_1, \dots, g_n in H^p . If $|f|^2 = \sum_{j=1}^n |f_j|^2$ and $|g|^2 = \sum_{j=1}^n |g_j|^2$, it follows from (3) that $1 \leq |f||g|$, and hence $M(|f|^{-1}) \in L^p(\mathbb{T})$ is a necessary condition. We note that when $p = \infty$, this is the usual corona condition. However, for $1 \leq p < \infty$, this condition is far from being sufficient. In [ABN], it is shown that, for any $\varepsilon > 0$, the stronger condition $M(|f|^{-2+\varepsilon}) \in L^p(\mathbb{T})$ is not sufficient. Our next result is the H^p version of Theorem 1.

Theorem 2. *Let k be as in Theorem 1, let $H(x) = \left(k(x) \int_0^x k(s)/s ds\right)^{1/2}$ and let $1 \leq p < \infty$. Given functions $g, f_1, \dots, f_n \in H^\infty$, the condition*

$$M\left(\frac{g}{|f|^2 H(|f|^2)}\right) \in L^p(\mathbb{T})$$

implies the existence of solutions $g_1, \dots, g_n \in H^p$ of the equation

$$g = f_1g_1 + \dots + f_ng_n.$$

Finally, we want to remark that in both theorems, only the behavior of the function k , and hence H , near zero is essential.

1. CARLESON MEASURES AND THE $\bar{\partial}$ -EQUATION

Solutions of the $\bar{\partial}$ -equation with boundary control will be of vital importance in the proofs of the main theorems, and Carleson measures play an important role in obtaining these solutions. We recall that a positive Borel measure μ on \mathbb{D} is called a *Carleson measure* if there exists a constant C such that

$$(4) \quad \int_{\mathbb{D}} |h|^2 d\mu \leq C \|h\|_2^2,$$

for every function h in the Hardy space H^2 . It is well known that Carleson measures are those positive measures μ for which there exists a constant A such that

$$\mu(Q) \leq Al(Q)$$

for every *Carleson square* Q defined by

$$Q = \{re^{i\theta} \in \mathbb{D} : 1 - r < l(Q), |\theta - \theta_0| < l(Q)\}.$$

Denote by $N(\mu) = \sup\{\mu(Q)/l(Q)\}$ the Carleson norm of μ , where the supremum is taken over all Carleson squares Q . The operators ∂ and $\bar{\partial}$ are defined by

$$\partial f = \frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), \quad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right).$$

By the Cauchy-Riemann equations, a function f is analytic if and only if $\bar{\partial}f = 0$. Recall that we can rewrite the Laplacian operator as $\Delta = 4\partial\bar{\partial}$. We need the

following result of T. Wolff on the existence of bounded solutions of the $\bar{\partial}$ -equation (see, for example, [Ga] p. 322).

Lemma 1. *Let $G(z)$ be bounded and C^1 on the disc \mathbb{D} , and assume that the measures $d\mu(z) = |G(z)|^2 \log \frac{1}{|z|} dx dy$ and $d\sigma(z) = |\partial G(z)| \log \frac{1}{|z|} dx dy$ are Carleson measures. Then there exists a function $u \in C(\bar{\mathbb{D}}) \cap C^1(\mathbb{D})$ such that $\bar{\partial}u = G$ and*

$$\|u\|_{L^\infty(\mathbb{T})} \leq C_1 N(\mu)^{1/2} + C_2 N(\sigma).$$

We will also need an L^p -version of the Wolff criteria. The next lemma is a refinement of the version given in [ABN].

Lemma 2. *Let $1 \leq p < \infty$, and let G be a C^1 function in $\bar{\mathbb{D}}$ such that:*

- (a) $G = \varphi\psi$, where $M(\varphi) \in L^p(\mathbb{T})$, and $|\psi(z)|^2 \log \frac{1}{|z|} dx dy$ is a Carleson measure;
- (b) for every function $k \in H^q$, where $1/p + 1/q = 1$, we have

$$\int_{\mathbb{D}} |k(z)| |\partial G(z)| \log \frac{1}{|z|} dx dy \leq B \|k\|_q.$$

Then there exists a function $u \in C(\bar{\mathbb{D}}) \cap C^1(\mathbb{D})$ such that $\bar{\partial}u = G$ and

$$\int_0^{2\pi} |u(e^{i\theta})|^p d\theta \leq C,$$

where C depends only on the L^p -norm of $M(\varphi)$, the constant B and the Carleson norm of the measure of (a).

Proof. Let q be the conjugate exponent of p , $1 < q \leq \infty$. By duality,

$$\inf \{ \|b\|_p : \bar{\partial}b = G \} = \sup \left\{ \left| \frac{1}{2\pi} \int_0^{2\pi} Fk d\theta \right| : k \in H^q, k(0) = 0, \|k\|_q \leq 1 \right\}$$

where F is a a priori solution, say the one given by the Cauchy kernel, which is continuous on $\bar{\mathbb{D}}$. By Green's formula,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} Fk d\theta &= \frac{1}{2\pi} \int_{\mathbb{D}} \Delta(Fk) \log \frac{1}{|z|} dx dy \\ &= \frac{2}{\pi} \int_{\mathbb{D}} k'(z) G(z) \log \frac{1}{|z|} dx dy + \frac{2}{\pi} \int_{\mathbb{D}} k(z) \partial G(z) \log \frac{1}{|z|} dx dy = I_1 + I_2. \end{aligned}$$

It is proved in [ABN] that if $|\psi|^2 \log \frac{1}{|z|}$ is a Carleson measure with Carleson norm K , then

$$\int_{\mathbb{D}} |k'(z)| |\varphi(z)| |\psi(z)| \log \frac{1}{|z|} dx dy \leq C \|k\|_{H^q} \|M\varphi\|_p K$$

where C is an absolute constant. This implies the required bound for I_1 , and the boundness of I_2 follows from condition (b). □

The following lemma can be found in [Ni]. For completeness we will give a proof here.

Lemma 3. *Let $u \in C^2(\bar{\mathbb{D}})$ be a bounded subharmonic function. Then*

$$d\lambda(z) = \Delta u(z) \log \frac{1}{|z|} dx dy$$

is a Carleson measure with Carleson norm bounded by $2\pi e \|u\|_\infty$.

Proof. By considering the function $b(z) = u(z) + \|u\|_\infty$ we can assume that our function u is positive. Let $h \in H^2$. Then, for $t > 0$,

$$\begin{aligned} \Delta(|h|^2 e^{tu}) &= 4|h'|^2 e^{tu} + |h|^2 \Delta e^{tu} + 8 \operatorname{Re}(\bar{\partial}(|h|^2) \partial e^{tu}) \\ &= 4|h'|^2 e^{tu} + |h|^2(4t^2|\partial u|^2 + t \Delta u) e^{tu} + 8te^{tu} \operatorname{Re}(h \bar{\partial} h \partial u) \\ &= t|h|^2 e^{tu} \Delta u + e^{tu}|2\partial h + 2th \partial u|^2 \\ &\geq t|h|^2 e^{tu} \Delta u \geq t|h|^2 \Delta u. \end{aligned}$$

Thus we have

$$\int_{\mathbb{D}} |h(z)|^2 \Delta u(z) \log \frac{1}{|z|} dx dy \leq \inf_{t>0} \frac{1}{t} \int_{\mathbb{D}} \Delta(|h(z)|^2 e^{tu(z)}) \log \frac{1}{|z|} dx dy,$$

which, by Green's formula, is bounded by

$$\inf_{t>0} \frac{1}{t} \int_{\partial \mathbb{D}} |h|^2 e^{tu} \leq \inf_{t>0} \frac{2\pi}{t} \|e^{tu}\|_\infty \|h\|_2^2 = 2\pi e \|u\|_\infty \|h\|_2^2.$$

(The last identity is obtained by computation of the minimum of $t^{-1}\|e^{tu}\|_\infty$, which is attained at the point $t_0 = 1/\|u\|_\infty$.) Hence the measure λ is a Carleson measure with Carleson norm bounded by $2\pi e \|u\|_\infty$. \square

Given functions $f_1, \dots, f_n \in H^\infty$, we write $|f|^2 = \sum_{i=1}^n |f_i|^2$, and $|f'|^2 = \sum_{i=1}^n |f'_i|^2$. The next result is the key for the proof of Theorems 1 and 2.

Lemma 4. *Let $k : (0, 1) \rightarrow [0, \infty)$ be a bounded continuous function such that $\int_0^1 \frac{k(x)}{x} |\log x| dx < \infty$. Let $f_1, \dots, f_n \in H^\infty$ with $0 < |f|^2 < 1$. Then the measures*

- (a) $\frac{|\partial(|f|^2)|^2}{|f|^4} k(|f|^2) \log \frac{1}{|z|} dx dy,$
- (b) $\frac{|f|^2 |f'|^2 - |\partial(|f|^2)|^2}{|f|^4} \left(\int_0^{|f|^2} \frac{k(s)}{s} ds \right) \log \frac{1}{|z|} dx dy$

are Carleson measures with Carleson norm bounded by $C \int_0^1 \frac{k(s)}{s} |\log s| ds$.

We note that when $n = 1$, part (b) is vacuous and part (a) is a known result that is also true for bounded analytic functions vanishing in \mathbb{D} (see [ABN]).

Proof. Consider the function

$$U(z) = \log |f(z)| \int_0^{|f(z)|^2} \frac{k(s)}{s} ds + \int_0^{|f(z)|^2} \frac{k(s)}{s} |\log s| ds.$$

It clearly satisfies

$$0 \leq U(z) \leq 2 \int_0^{|f(z)|^2} \frac{k(s)}{s} |\log s| ds,$$

and a computation gives

$$\frac{1}{4} \Delta U(z) = S(z),$$

where

$$S = \frac{|\partial(|f|^2)|^2}{|f|^4} k(|f|^2) + \frac{|f|^2 |f'|^2 - |\partial(|f|^2)|^2}{|f|^4} \left(\int_0^{|f|^2} \frac{k(s)}{s} ds \right).$$

Now, applying Lemma 3, we obtain the desired conclusion. \square

2. PROOF OF THEOREM 1

The proof follows standard arguments with the use of Lemma 4 as a new ingredient. By a standard normal families argument, we can assume that the functions f_1, \dots, f_n are analytic in a neighborhood of the closed unit disc. For $j = 1, \dots, n$ we define

$$\varphi_j(z) = \frac{\overline{f_j(z)}}{|f(z)|^2}.$$

We see that the functions φ_j belong to $\mathcal{C}^\infty(\mathbb{D})$ and satisfy the equation $\sum_{j=1}^n f_j \varphi_j = 1$. For $j, k = 1, \dots, n$, let

$$G_{jk} = \varphi_j \bar{\partial} \varphi_k.$$

Assume that for $j, k = 1, \dots, n$ we can solve the $\bar{\partial}$ -equations

$$(5) \quad \bar{\partial} u_{jk} = g G_{jk},$$

with $\|u_{jk}\|_{L^\infty(\mathbb{T})} \leq M$. Then, for $j = 1, \dots, n$, the functions

$$g_j = g \varphi_j + \sum_{k=1}^n (u_{jk} - u_{kj}) f_k$$

are bounded, satisfy $\bar{\partial} g_j = 0$ and so are analytic, and satisfy the equation

$$g = \sum_{j=1}^n f_j g_j.$$

It only remains to show that (5) has bounded solutions. To see this, we will use Lemma 1. Fix j, k , and denote G_{jk} by G . A computation gives

$$(6) \quad G_{jk} = \frac{\bar{f}_j}{|f|^6} \sum_{l \neq k} f_l (\overline{f_l f'_k} - f_k \overline{f'_l})$$

and

$$(7) \quad \sum_{j,k=1}^n |f'_k f_j - f_k f'_j|^2 = |f|^2 |f'|^2 - |\partial(|f|^2)|^2.$$

By (6) and (7),

$$(8) \quad |G| \leq 2 \frac{(|f|^2 |f'|^2 - |\partial(|f|^2)|^2)^{1/2}}{|f|^4}.$$

Using (8) and our condition on the size of $|g|$ we see that

$$|gG|^2 \leq \frac{(|f|^2 |f'|^2 - |\partial(|f|^2)|^2)}{|f|^4} \left(\int_0^{|f|^2} \frac{k(s)}{s} ds \right) k(|f|^2),$$

and because k is bounded,

$$|g(z)G(z)|^2 \log \frac{1}{|z|} dx dy$$

is a Carleson measure by Lemma 4.

We have $\partial(gG) = g'G + g\partial G$, and since $|g| \leq |f|^2 H(|f|^2)$, we have that the measure $|g(z)\partial G(z)| \log \frac{1}{|z|} dx dy$ is a Carleson measure by the following result.

Lemma 5. *The measure $|f(z)|^2 H(|f(z)|^2) |\partial G(z)| \log \frac{1}{|z|} dx dy$ is a Carleson measure.*

To prove this, we note that $k(|f|^2) \leq \int_0^{|f|^2} \frac{k(s)}{s} ds$, since the function $k(x)/x$ is nonincreasing. Also, a computation gives

$$(9) \quad |\partial G| \leq 2|G| \frac{|\partial(|f|^2)|}{|f|^2} + \frac{|f|^2|f'|^2 - |\partial(|f|^2)|^2}{|f|^6}.$$

So, by (9) and then (8),

$$\begin{aligned} |f|^2 H(|f|^2) |\partial G| &\leq 2H(|f|^2) |G| |\partial(|f|^2)| + \frac{(|f|^2|f'|^2 - |\partial(|f|^2)|^2)}{|f|^4} \int_0^{|f|^2} \frac{k(s)}{s} ds \\ &\leq 2 \frac{|\partial(|f|^2)|^2}{|f|^4} k(|f|^2) + 2 \frac{|f|^2|f'|^2 - |\partial(|f|^2)|^2}{|f|^4} \left(\int_0^{|f|^2} \frac{k(s)}{s} ds \right), \end{aligned}$$

and the result follows by Lemma 4.

It remains to check that

$$|g'(z)G(z)| \log \frac{1}{|z|} dx dy$$

is a Carleson measure. Let $h \in H^2$. Then

$$\int_{\mathbb{D}} |h(z)|^2 |(g'G)(z)| \log \frac{1}{|z|} dx dy = \int_A + \int_{\mathbb{D} \setminus A} = I_1 + I_2,$$

where $A = \{z : |g(z)| \leq |f(z)|^5\}$. For $z \in A$ we have

$$|(g'G)(z)| \leq \frac{|g'(z)|^2}{|g(z)|} + \frac{|f'(z)|^2}{|f(z)|}.$$

Since for $F \in H^\infty$, the measure $\frac{|F'(z)|^2}{|F(z)|} \log \frac{1}{|z|} dx dy$ is Carleson (see [Ga], p. 327, or apply Lemma 4 with $k(x) = x^{1/2}$), we see that $I_1 \leq C_1 \|h\|_{H^2}^2$, by (4). To estimate I_2 , let

$$B(|f|^2) = \frac{(|f|^2|f'|^2 - |\partial(|f|^2)|^2)}{|f|^4} \left(\int_0^{|f|^2} \frac{k(s)}{s} ds \right).$$

Since k is nondecreasing, we see that

$$\begin{aligned} |g'G| &\leq |g'|^2 |g|^{-2} k(|f|^2) + B(|f|^2) \\ &\leq |g'|^2 |g|^{-2} s(|g|^2) + B(|f|^2) \end{aligned}$$

in $\mathbb{D} \setminus A$, where $s(x) = k(x^{1/5})$. One easily verifies that s satisfies the condition $\int_0^1 \frac{s(x)}{x} |\log x| dx < \infty$. Then $I_2 \leq C_2 \|h\|_{H^2}^2$ by Lemma 4 and (4). Hence the measure

$$|(g'G)(z)| \log \frac{1}{|z|} dx dy$$

is a Carleson measure. By Lemma 1, the proof is complete.

3. PROOF OF THEOREM 2

For $j, k = 1, \dots, n$, let $G_{jk} = \varphi_j \bar{\partial} \varphi_k$, where $\varphi_j = \bar{f}_j |f|^{-2}$. As in the proof of Theorem 1, it is sufficient to solve the $\bar{\partial}$ -equations $\bar{\partial} u_{jk} = g G_{jk}$, with $\|u_{jk}\|_{L^p(\mathbb{T})} \leq M$. For this, we will make use of Lemma 2. Fix j, k , and for ease of notation, denote G_{jk} by G . We can write gG in the form $gG = \phi \psi_1$, where $\phi = g |f|^{-2} (H(|f|^2))^{-1}$ and $\psi_1 = |f|^2 H(|f|^2) G$. By hypothesis, $M(\phi) \in L^p(\mathbb{T})$, and the proof of Theorem 1 shows that

$$|\psi_1(z)|^2 \log \frac{1}{|z|} dx dy$$

is a Carleson measure. So condition (a) of Lemma 2 is satisfied. To check condition (b), let $k \in H^q$, where $1/p + 1/q = 1$. We have $\partial(gG) = g'G + g\partial G$, and we can write $|g\partial G|$ as $|\phi|\psi_2$, where $\psi_2 = |f|^2 H(|f|^2) |\partial G|$. By Lemma 5, the measure

$$d\mu(z) = \psi_2(z) \log \frac{1}{|z|} dx dy$$

is a Carleson measure. Then

$$\begin{aligned} \int_{\mathbb{D}} |k(z)| |(g\partial G)(z)| \log \frac{1}{|z|} dx dy &\leq \left(\int_{\mathbb{D}} |\phi|^p d\mu \right)^{1/p} \left(\int_{\mathbb{D}} |k|^q d\mu \right)^{1/q} \\ &\leq C \|M(\phi)\|_{L^p(\mathbb{T})} \|k\|_{H^q}, \end{aligned}$$

since, if μ is a Carleson measure and $M\psi \in L^p(\mathbb{T})$, then $\int_{\mathbb{D}} |\psi|^p d\mu \leq \|M\psi\|_{L^p(\mathbb{T})}^p$ (see [Ga], p. 32). An argument similar to that in the proof of Theorem 1 shows that

$$\int_{\mathbb{D}} |k(z)| |(g'G)(z)| \log \frac{1}{|z|} dx dy \leq C \|k\|_{H^q},$$

and condition (b) of Lemma 2 is satisfied. This completes the proof.

REFERENCES

- [ABN] E. Amar, J. Bruna and A. Nicolau, *On H^p -solutions of the Bezout equation*, Pacific J. Math. **171**: 2 (1995), 297-307. MR97g:30036
- [Ce1] U. Cegrell, *A generalization of the corona theorem in the unit disc*, Math. Z. **203** (1990), 255-261. MR91h:30059
- [Ce2] U. Cegrell, *Generalisations of the corona theorem in the unit disc*, Proc. Roy. Irish Acad. **94** (1994), 25-30. MR95k:30069
- [Ga] J.B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981. MR83g:30037
- [Li] K.C. Lin, *On the constants in the corona theorem and ideals of H^∞* , Houston J. Math. **19** (1993), 97-106. MR94j:30033
- [Ni] N.K. Nikol'skii, *Treatise on the shift operator*, Grund. der Math. Wissen., **273** (1986). MR87i:47042
- [Ra] K.V. Rao, *On a generalized corona problem*, J. Anal. Math. **18** (1967), 277-278. MR35:1795
- [Tr] S. Treil, *Estimates in the corona theorem and ideals of H^∞ : a problem of T. Wolff*, J. Analyse Math. **87** (2002), 481-495. MR2003k:30077

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