PROJECTIVE SURFACES WITH MANY SKW LINES

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Abstract. We give an example of a smooth surface $S_d \subset \mathbb{P}_3(\mathbb{C})$ of degree $d$ that contains $d \cdot (d - 2) + 2$ pairwise disjoint lines. In particular, our example shows that the degree in Miyaoka's bound is sharp.

Up to now the maximal number of pairwise disjoint lines on a smooth surface of degree $d \geq 5$ in $\mathbb{P}_3(\mathbb{C})$ was unknown. According to [4, p. 162] this number does not exceed

$$2 \cdot d \cdot (d - 2).$$

Quartic surfaces with 16 skew lines are studied in [1], but it is not clear to what extent Miyaoka’s bound is sharp for $d \geq 5$. Here we give an example of a smooth surface $S_d \subset \mathbb{P}_3(\mathbb{C})$ of degree $d$ that contains

$$d \cdot (d - 2) + 2$$

pairwise disjoint lines. All lines on $S_d$ form the following configuration:

\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
\multicolumn{4}{|c|}{2 skew lines} \\
\hline
\cdots & \cdots & \cdots & \cdots \\
\hline
\end{tabular}
\end{center}

Our example is inspired by the classical Klein quartic curve.

The equation of a quintic with 19 skew lines is given in [5, Example 2.3]. For $d \geq 6$ the surface $S_d$ contains the largest number of skew lines found on a smooth surface of degree $d$ in $\mathbb{P}_3(\mathbb{C})$. Let us mention that the Fermat surface $F_d$, i.e. the surface with $3d^2$ lines (the largest number known so far for $d \neq 4, 6, 8, 12, 20$), contains no family of $3d$ pairwise disjoint lines. The latter results from the description of configuration of lines on $F_d$ that can be found in [3].

Example. We define $S_d$ to be the surface given by the polynomial

$$s_d := x_0^{d-1} \cdot x_1 + x_1^{d-1} \cdot x_2 + x_2^{d-1} \cdot x_3 + x_3^{d-1} \cdot x_0,$$
where \( d \geq 6 \). One can easily check that \( S_d \) is smooth. Let \( L_1 \) (resp. \( L_2 \)) be the line \( x_0 = x_2 = 0 \) (resp. \( x_1 = x_3 = 0 \)). We claim that

(a) \( S_d \) contains \( d \cdot (d - 2) + 2 \) skew lines, each of which meets \( L_1 \) and \( L_2 \),

(b) the only lines on \( S_d \) are \( L_1, L_2 \) and the above-mentioned skew lines.

Proof of (a). Fix \( r_0, r_1 \in \mathbb{C} \). The line \((r_0 \lambda_0 : r_1 \lambda_1 : \lambda_0 : \lambda_1)\) lies on \( S_d \) iff the polynomial

\[
(r_0^{d-1} r_1 + 1) \lambda_0^{d-1} \lambda_1 + (r_1^{d-1} + r_0) \lambda_0 \lambda_1^{d-1}
\]

vanishes identically. So the parameters \( r_0, r_1 \) satisfy the conditions

\[
(1) \quad r_0 = (-r_1^{d-1}) \quad \text{and} \quad r_1^{(d-1)^2+1} = (-1)^d.
\]

Let \( L(r_1) \) be the line on \( S_d \) that corresponds to \( r_0 = (-r_1^{d-1}) \). We are to show that, for \( r_1 \neq r_1' \), the lines \( L(r_1), L(r_1') \) are disjoint. Suppose that \( L(r_1), L(r_1') \) meet in the point \((y_0 : y_1 : y_2 : y_3)\). If \( y_3 \neq 0 \), then the parametrization of the lines in question yields \( r_1 = r_1' \) and they coincide. Otherwise we have \( y_2 \neq 0 \); so we get \( r_1^{d-1} = (r_1')^{d-1} \). By (1) we have \( r_1 = r_1' \).

\[ \square \]

Proof of (b). We claim that \( L_2 \) is the only line on \( S_d \) that does not meet \( L_1 \). Indeed, let \( C_1 \) (resp. \( C_2 \)) be the curve residual to the line \( L_1 \) in the intersection of \( S_d \) with the plane \( x_0 = 0 \) (resp. \( x_2 = 0 \)). We have the parametrizations

\[
\psi_1 : C \ni a_1 \rightarrow (0 : a_1 : 1 : -a_1^{d-1}) 
\in C_1 \setminus \{(0 : 0 : 0 : 1)\},
\]

\[
\psi_2 : C \ni a_2 \rightarrow (1 : -a_2^{d-1} : 0 : a_2) 
\in C_2 \setminus \{(0 : 1 : 0 : 0)\}.
\]

The line through the points \( \psi_1(a_1), \psi_2(a_2) \) lies on \( S_d \) iff all coefficients of the polynomial \( s_d(\lambda_1, \lambda_0 a_1 - \lambda_1 a_2^{d-1}, \lambda_0, \lambda_1 a_2 - \lambda_0 a_1^{d-1}) \) vanish. Write down the coefficients of the terms \( \lambda_0^{d-1} \lambda_1, \ldots, \lambda_0^{d-4} \lambda_1^4 \) to see that \( a_1, a_2 \) satisfy the conditions

\[
(2) \quad - (d - 1)a_1^{d-2}a_2^{-d+1} + a_2 + (-1)^{d-1}a_1^{(d-1)^2} = 0,
\]

\[
(3) \quad \frac{d - 2}{2}a_1^{a_1 - 3}a_2^{2(d-1)} = (-1)^{d-1}a_1^{(d-1)(d-2)}a_2,
\]

\[
(4) \quad \frac{d - 3}{3}a_1^{d-4}a_2^{d(d-1)} = (-1)^{d-1}a_1^{(d-1)(d-3)}a_2^2,
\]

\[
(5) \quad \frac{d - 4}{4}a_1^{d-5}a_2^{4(d-1)} = (-1)^{d-1}a_1^{(d-1)(d-4)}a_2^3.
\]

By the equation (2) we have \( a_1 = 0 \) iff \( a_2 = 0 \). This solution corresponds to the line \( L_2 \). Dividing (3) by (4) and (4) by (5) one gets that \( a_1 = a_2 = 0 \) is the unique solution. Thus \( L_2 \) is the only line on \( S_d \) that does not meet \( L_1 \).

The symmetry \((x_0 : x_1 : x_2 : x_3) \rightarrow (x_3 : x_0 : x_1 : x_2)\) interchanges the lines \( L_1, L_2 \). So the other lines on \( S_d \) meet both \( L_1 \) and \( L_2 \). One can check (see the proof of (a)) that there are precisely \( d \cdot (d - 2) + 2 \) such lines.

\[ \square \]

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References


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