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PROJECTIVE SURFACES WITH MANY SKEW LINES

SŁAWOMIR RAMS

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Abstract. We give an example of a smooth surface $\mathrm{S}_d \subset \mathbb{P}_3(\mathbb{C})$ of degree d that contains $d \cdot (d-2) + 2$ pairwise disjoint lines. In particular, our example shows that the degree in Miyaoka's bound is sharp.

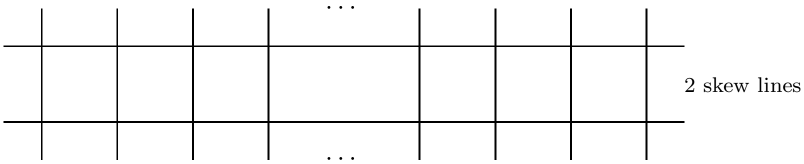
Up to now the maximal number of pairwise disjoint lines on a smooth surface of degree $d \geq 5$ in $\mathbb{P}_3(\mathbb{C})$ was unknown. According to [4, p. 162] this number does not exceed

$$2 \cdot d \cdot (d-2).$$

Quartic surfaces with 16 skew lines are studied in [1], but it is not clear to what extent Miyaoka's bound is sharp for $d \geq 5$. Here we give an example of a smooth surface $\mathrm{S}_d \subset \mathbb{P}_3(\mathbb{C})$ of degree d that contains

$$d \cdot (d-2) + 2$$

pairwise disjoint lines. All lines on S_d form the following configuration:



$d \cdot (d-2) + 2$ pairwise disjoint lines

Our example is inspired by the classical Klein quartic curve.

The equation of a quintic with 19 skew lines is given in [5, Example 2.3]. For $d \geq 6$ the surface S_d contains the largest number of skew lines found on a smooth surface of degree d in $\mathbb{P}_3(\mathbb{C})$. Let us mention that the Fermat surface F_d, i.e. the surface with $3d^2$ lines (the largest number known so far for $d \neq 4, 6, 8, 12, 20$), contains no family of $3d$ pairwise disjoint lines. The latter results from the description of configuration of lines on F_d that can be found in [3].

Example. We define S_d to be the surface given by the polynomial

$$s_d := x_0^{d-1} \cdot x_1 + x_1^{d-1} \cdot x_2 + x_2^{d-1} \cdot x_3 + x_3^{d-1} \cdot x_0 \,,$$

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where $d \geq 6$. One can easily check that S_d is smooth. Let L_1 (resp. L_2) be the line $x_0 = x_2 = 0$ (resp. $x_1 = x_3 = 0$). We claim that

(a) S_d contains $d \cdot (d-2) + 2$ skew lines, each of which meets L_1 and L_2,

(b) the only lines on S_d are L_1, L_2 and the above-mentioned skew lines.

Proof of (a). Fix $r_0, r_1 \in \mathbb{C}$. The line $(r_0\lambda_0 : r_1\lambda_1 : \lambda_0 : \lambda_1)$ lies on S_d iff the polynomial

$$(r_0^{d-1} r_1 + 1)\, \lambda_0^{d-1}\, \lambda_1 + (r_1^{d-1} + r_0)\, \lambda_0\, \lambda_1^{d-1}$$

vanishes identically. So the parameters r_0, r_1 satisfy the conditions

$$r_0 = (-r_1^{d-1}) \quad \text{and} \quad r_1^{(d-1)^2+1} = (-1)^d\,. \tag{1}$$

Let $L(r_1)$ be the line on S_d that corresponds to $r_0 = (-r_1^{d-1})$. We are to show that, for $r_1 \neq r_1'$, the lines $L(r_1)$, $L(r_1')$ are disjoint. Suppose that $L(r_1)$, $L(r_1')$ meet in the point $(y_0 : y_1 : y_2 : y_3)$. If $y_3 \neq 0$, then the parametrization of the lines in question yields $r_1 = r_1'$ and they coincide. Otherwise we have $y_2 \neq 0$; so we get $r_1^{d-1} = (r_1')^{d-1}$. By (1) we have $r_1 = r_1'$. □

Proof of (b). We claim that L_2 is the only line on S_d that does not meet L_1. Indeed, let C_1 (resp. C_2) be the curve residual to the line L_1 in the intersection of S_d with the plane $x_0 = 0$ (resp. $x_2 = 0$). We have the parametrizations

$$\begin{aligned} \psi_1 : \mathbb{C} \ni a_1 \quad &\rightarrow \quad (0 : a_1 : 1 : -a_1^{d-1}) \in \mathrm{C}_1 \setminus \{(0:0:0:1)\}\,, \\ \psi_2 : \mathbb{C} \ni a_2 \quad &\rightarrow \quad (1 : -a_2^{d-1} : 0 : a_2) \in \mathrm{C}_2 \setminus \{(0:1:0:0)\}\,. \end{aligned}$$

The line through the points $\psi_1(a_1)$, $\psi_2(a_2)$ lies on S_d iff all coefficients of the polynomial $s_d(\lambda_1, \lambda_0 a_1 - \lambda_1 a_2^{d-1}, \lambda_0, \lambda_1 a_2 - \lambda_0 a_1^{d-1})$ vanish. Write down the coefficients of the terms $\lambda_0^{d-1}\lambda_1, \ldots, \lambda_0^{d-4}\lambda_1^4$ to see that a_1, a_2 satisfy the conditions

$$-\,(d-1) a_1^{d-2} a_2^{d-1} + a_2 + (-1)^{d-1} a_1^{(d-1)^2} = 0\,, \tag{2}$$

$$\frac{d-2}{2} a_1^{d-3} a_2^{2(d-1)} = (-1)^{d-1} a_1^{(d-1)(d-2)}\, a_2\,, \tag{3}$$

$$\frac{d-3}{3} a_1^{d-4} a_2^{3(d-1)} = (-1)^{d-1} a_1^{(d-1)(d-3)} a_2^2\,, \tag{4}$$

$$\frac{d-4}{4} a_1^{d-5} a_2^{4(d-1)} = (-1)^{d-1} a_1^{(d-1)(d-4)} a_2^3\,. \tag{5}$$

By the equation (2) we have $a_1 = 0$ iff $a_2 = 0$. This solution corresponds to the line L_2. Dividing (3) by (4) and (4) by (5) one gets that $a_1 = a_2 = 0$ is the unique solution. Thus L_2 is the only line on S_d that does not meet L_1.

The symmetry $(x_0 : x_1 : x_2 : x_3) \rightarrow (x_3 : x_0 : x_1 : x_2)$ interchanges the lines L_1, L_2. So the other lines on S_d meet both L_1 and L_2. One can check (see the proof of (a)) that there are precisely $d \cdot (d-2) + 2$ such lines. □

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INSTITUTE OF MATHEMATICS UJ, UL. REYMONTA 4, 30-059 KRAKÓW, POLAND

Current address: Mathematisches Institut, FAU Erlangen-Nürnberg, Bismarckstrasse 1 1/2, D-91054 Erlangen, Germany

E-mail address: `rams@mi.uni-erlangen.de, rams@im.uj.edu.pl`