

ON RESTRICTED WEAK TYPE $(1, 1)$:
THE CONTINUOUS CASE

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ABSTRACT. Let \mathbb{T} denote the unit circle. An example of a sublinear translation-invariant operator T acting on $L^1(\mathbb{T})$ is given such that T is of restricted weak type $(1, 1)$ but not of weak type $(1, 1)$.

Let \mathbb{T} denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Let T be a sublinear translation-invariant operator acting on $L^1(\mathbb{T})$. If for some finite constant C we have that

$$|\{x \in \mathbb{T} : |Tf(x)| > \alpha\}| \leq C \frac{\|f\|_{L^1(\mathbb{T})}}{\alpha}$$

for all $\alpha > 0$ and $f \in L^1(\mathbb{T})$, we say that T is of *weak type* $(1, 1)$. If for some finite constant C we have that

$$|\{x \in \mathbb{T} : |T\chi_E(x)| > \alpha\}| \leq C \frac{\|\chi_E\|_{L^1(\mathbb{T})}}{\alpha}$$

for all $\alpha > 0$ and measurable subsets E of \mathbb{T} , we say T is of *restricted weak type* $(1, 1)$.

In 1974, K. H. Moon [5] proved the following theorem regarding operators of restricted weak type $(1, 1)$:

Theorem 1 (Moon). *Let K_n ($n = 1, 2, \dots$) be linear convolution operators acting on $L^1(\mathbb{T})$, each of the form $K_n f = f * g_n$ for some $g_n \in L^1(\mathbb{T})$, and let $Mf(x) = \sup_n |K_n f(x)|$. Then M is of restricted weak type $(1, 1)$ if and only if M is of weak type $(1, 1)$.*

A natural question is to the degree that Moon's theorem admits further generalization. In particular, we may ask the following:

If T is a translation-invariant sublinear operator acting on $L^1(\mathbb{T})$, which is of restricted weak type $(1, 1)$, must T be of weak type $(1, 1)$?

In this paper we shall show that the answer to the above question is negative. In particular, we will see how a related negative result in the discrete setting due to Akcoglu, Baxter, Bellow, and Jones [1] may be utilized to construct a sublinear

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translation-invariant operator T^* acting on $L^1(\mathbb{T})$ that is of restricted weak type $(1, 1)$ but not of weak type $(1, 1)$.

We now discuss the result of Akcoglu, Baxter, Bellow and Jones. Let $\ell^1(\mathbb{Z})$ denote the space of integers equipped with the counting measure. For any subset A of \mathbb{Z} we denote the number of elements in A by $|A|$. (The symbol $|A|$ will also be used to denote the standard Haar measure of a set A in \mathbb{T} . By context it will be clear if A is being considered as a subset of \mathbb{Z} or \mathbb{T} .)

Let M be an operator acting on $\ell^1(\mathbb{Z})$. Analogously to the $L^1(\mathbb{T})$ case, if for some finite constant C we have

$$|\{x \in \mathbb{Z} : |M\phi(x)| > \alpha\}| \leq C \frac{\|\phi\|_{\ell^1(\mathbb{Z})}}{\alpha}$$

for all $\alpha > 0$ and all $\phi \in \ell^1(\mathbb{Z})$, then we say that M is of weak type $(1, 1)$. If for some constant C we have

$$|\{x \in \mathbb{Z} : |M\chi_E(x)| > \alpha\}| \leq C \frac{\|\chi_E\|_{\ell^1(\mathbb{Z})}}{\alpha}$$

for all $\alpha > 0$ and all finite subsets E of \mathbb{Z} , then we say that M is of restricted weak type $(1, 1)$.

A probability measure on \mathbb{Z} with finite support is defined to be a function ω on \mathbb{Z} such that $\omega(k) \geq 0$ for each k , and there exists an integer M such that $\omega(k) = 0$ if $|k| > M$, and $\sum_{k=-M}^M \omega(k) = 1$.

If f and g are functions in $\ell^1(\mathbb{Z})$, the convolution of f and g will be notated and defined by

$$f \star g(j) = \sum_{k=-\infty}^{\infty} f(j - k)g(k).$$

We note that the convolution of functions f and g in $L^1(\mathbb{T})$ is defined and notated here in the standard way by

$$f * g(e^{i\theta}) = \frac{1}{2\pi} \int_{|\phi| < \pi} f(e^{i\phi}) g(e^{i(\theta - \phi)}) d\phi.$$

The result of Akcoglu, Baxter, Bellow and Jones of interest to us here is the following.

Theorem 2 ([1]). *There exists a sequence $\{\omega_k\}$ of probability measures on \mathbb{Z} such that the associated operators $\{\tilde{T}_k\}$ and \tilde{T}^* defined on $\ell^1(\mathbb{Z})$ by $\tilde{T}_k\phi(j) = \phi \star \omega_k(j)$, and $\tilde{T}^*\phi(j) = \sup_k |\tilde{T}_k\phi(j)|$, have the following properties:*

- For each k the measure ω_k has finite support.
- The maximal operator \tilde{T}^* is of restricted weak type $(1, 1)$.
- The maximal operator \tilde{T}^* is not of weak type $(1, 1)$.

We will now show how the above theorem may be used to prove the following.

Theorem 3. *There exists a sublinear translation-invariant operator T acting on $L^1(\mathbb{T})$ that is of restricted weak type $(1, 1)$ but not of weak type $(1, 1)$.*

The proof of Theorem 3 will rely on the transference principle formulated by A. P. Calderón in his fundamental paper [3]. The idea underlining this principle is that the “bad behavior” of an operator on \mathbb{Z} may be transferred to similar bad behavior of an associated operator on \mathbb{T} . The details specific to the case considered

here are contained in the following lemma, which by Theorem 2 will be sufficient to establish our result.

Lemma 1. *Let $\{\omega_k\}$ be a sequence of probability measures on \mathbb{Z} , each with finite support. Define the associated operators $\{\tilde{T}_k\}$ and \tilde{T}^* on $\ell^1(\mathbb{Z})$ by $\tilde{T}_k\phi(j) = \phi \star \omega_k(j)$ and $\tilde{T}^*\phi(j) = \sup_k |\tilde{T}_k\phi(j)|$, and the operators $\{T_n\}$ and T^* on $L^1(\mathbb{T})$ by*

$$T_n f(e^{i\theta}) = \sum_{k=-\infty}^{\infty} \omega_n(k) f(e^{i(\theta+k)})$$

and

$$T^* f(e^{i\theta}) = \sup_n |T_n f(e^{i\theta})|.$$

If \tilde{T}^* is of restricted weak type (1, 1) but not of weak type (1, 1), then we also have that T^* is of restricted weak type (1, 1) but not of weak type (1, 1).

Proof. We first define the operators T_N^* and \tilde{T}_N^* on $L^1(\mathbb{T})$ and $\ell^1(\mathbb{Z})$ respectively by

$$T_N^* f(e^{i\theta}) = \sup_{1 \leq n \leq N} |T_n f(e^{i\theta})|$$

and

$$\tilde{T}_N^* \phi(j) = \sup_{1 \leq n \leq N} |\tilde{T}_n \phi(j)|.$$

We now show that if \tilde{T}^* is of restricted weak type (1, 1), then T^* is as well. Now, if \tilde{T}^* is of restricted weak type (1, 1), then there exists a constant C such that

$$\left| \left\{ j \in \mathbb{Z} : \tilde{T}_N^* \chi_S(j) > \alpha \right\} \right| \leq \frac{C}{\alpha} |\{j : j \in S\}|$$

holds for every positive integer N , subset S of \mathbb{Z} , and $\alpha > 0$.

We will show that this implies that T^* is of restricted weak type (1, 1). That is, we will show that

$$\left| \left\{ e^{i\theta} \in \mathbb{T} : T_N^* \chi_E(e^{i\theta}) > \alpha \right\} \right| \leq \frac{C}{\alpha} |E|$$

holds for every positive integer N , measurable subset E of \mathbb{T} , and $\alpha > 0$. Letting N go to infinity will then imply that the same inequality holds for T^* .

Fix $E \subset \mathbb{T}$, and let $F = \{e^{i\theta} \in \mathbb{T} : T_N^* \chi_E(e^{i\theta}) > \alpha\}$. Note that $|F| = |\{e^{i\theta} \in \mathbb{T} : e^{i(\theta+k)} \in F\}|$ for any choice of $k \in \mathbb{Z}$. For each positive integer j , we let M_j be a positive integer such that the probability measure ω_j is supported in $[-M_j, M_j]$. We assume, without loss of generality, that $M_j < M_{j+1}$ for each positive integer j . Let L be an arbitrary positive integer. Also, let \tilde{E}_θ be a subset of \mathbb{Z} with the

property that $\chi_{\tilde{E}_\theta}(k) = \chi_E(e^{i(\theta+k)})\chi_{[-M_N, L+M_N]}(k)$. We have

$$\begin{aligned} |F| &= \frac{1}{L} \sum_{k=0}^{L-1} \left| \{e^{i\theta} : e^{i(\theta+k)} \in F\} \right| \\ &= \frac{1}{L} \sum_{k=0}^{L-1} \int_{\mathbb{T}} \chi_F(e^{i(\theta+k)}) d\theta \\ &= \frac{1}{L} \int_{\mathbb{T}} \sum_{k=0}^{L-1} \chi_F(e^{i(\theta+k)}) d\theta \\ &= \frac{1}{L} \int_{\mathbb{T}} \left| \{k \in [0, L-1] : e^{i(\theta+k)} \in F\} \right| d\theta \\ &= \frac{1}{L} \int_{\mathbb{T}} \left| \left\{ k \in [0, L-1] : \sup_{1 \leq n \leq N} \left| \sum_j \omega_n(j) \chi_E(e^{i(\theta+k+j)}) \right| > \alpha \right\} \right| d\theta \\ &\leq \frac{1}{L} \int_{\mathbb{T}} \left| \left\{ k \in \mathbb{Z} : \sup_{1 \leq n \leq N} \left| \sum_j \omega_n(j) \chi_{\tilde{E}_\theta}(k+j) \right| > \alpha \right\} \right| d\theta \\ &\leq \frac{1}{L} \int_{\mathbb{T}} \frac{C}{\alpha} |\tilde{E}_\theta| d\theta \\ &\leq \frac{1}{L} \int_{\mathbb{T}} \frac{C}{\alpha} \sum_{k=-M_N}^{L+M_N} \chi_E(e^{i(\theta+k)}) d\theta \\ &\leq \frac{L+2M_N}{L} \times \frac{C}{\alpha} |E|. \end{aligned}$$

Since L can be taken as large as we like, we see that we have

$$|\{e^{i\theta} \in \mathbb{T} : T_N^* \chi_E(x) > \alpha\}| \leq \frac{C}{\alpha} |E|,$$

as required.

It remains to show that if \tilde{T}^* is not of weak type $(1, 1)$, then neither is T^* .

If \tilde{T}^* is not of weak type $(1, 1)$, we know that given $C > 0$ we can find $\phi \in \ell^1(\mathbb{Z})$ such that ϕ has finite support and an $\alpha > 0$ such that

$$|\{k \in \mathbb{Z} : \tilde{T}^* \phi(k) > \alpha\}| \geq \frac{C}{\alpha} \|\phi\|_{\ell^1}.$$

Assume (without loss of generality) that ϕ has support in $[0, L]$ for some choice of L .

We know that the points $e^i, e^{2i}, \dots, e^{iL}$ will all be disjoint, and consequently there is an arc $B = (e^{ia}, e^{ib})$ in \mathbb{T} such that the arcs

$$(e^{ia}, e^{ib}), (e^{i(a+1)}, e^{i(b+1)}), \dots, (e^{i(a+L)}, e^{i(b+L)})$$

are all disjoint.

Define $f \in L^1(\mathbb{T})$ by $f(e^{i(\theta+k)}) = \phi(k)$ if $0 \leq k \leq L$ and $e^{i\theta} \in (e^{ia}, e^{ib})$. Let f be zero otherwise. Note that $\|f\|_1 = |B| \|\phi\|_{\ell^1}$. We then have

$$\begin{aligned} |\{e^{i\theta} \in \mathbb{T} : T^*f(e^{i\theta}) > \alpha\}| &\geq |B| \times |\{k \in \mathbb{Z} : \tilde{T}^*\phi(k) > \alpha\}| \\ &\geq |B| \times \frac{C}{\alpha} \|\phi\|_{\ell^1} \\ &= \frac{C}{\alpha} \|f\|_1. \end{aligned}$$

Since C can be taken as large as desired, we see that T^* cannot be of weak type $(1, 1)$. \square

We note that Lemma 1 (and hence Theorem 3) can be generalized to the following setting.

Lemma 2. *Let (X, Σ, m) denote a complete non-atomic probability space and let $\tau : X \rightarrow X$ be a measurable invertible ergodic measure-preserving transformation. Let $\{w_k\}$, $\{\tilde{T}_k\}$ and \tilde{T}^* be as in Lemma 1. Define the operators $\{T_n\}$ and T^* on $L^1(X, \Sigma, m)$ by $T_n f(x) = \sum_{k=-\infty}^{\infty} w_n(k) f(\tau^k x)$ and $T^* f(x) = \sup_n |T_n f(x)|$. If the maximal operator \tilde{T}^* is of restricted weak type $(1, 1)$ but not of weak type $(1, 1)$, then the same is true for the associated maximal operator T^* on $L^1(X)$.*

The proof of this result follows the same argument as the proof of Lemma 1. Indeed, Lemma 1 is a special case of this more general result, with $x = e^{i\theta}$ and τ given by $\tau(x) = \tau(e^{i\theta}) = e^{i(\theta+1)}$. For the proof of the first part of this result, all we need to do is replace $e^{i\theta}$ by x , $e^{i(\theta+k)}$ by $\tau^k x$, and copy the argument. For the second half, we replace the disjoint arcs (e^{ia}, e^{ib}) , $(e^{i(a+1)}, e^{i(b+1)})$, \dots , $(e^{i(a+L)}, e^{i(b+L)})$ by disjoint sets of positive measure $B, \tau B, \dots, \tau^L B$. The existence of such a set B is guaranteed by Rohlin’s Lemma (see, e.g., [2]). The argument now follows as before.

We make a couple of closing remarks. Let T^* be the operator defined in the statement of Lemma 1 associated to the sequence $\{\omega_n\}$ of probability measures referred to in the statement of Theorem 2. Note that the operator T^* is clearly bounded on $L^\infty(\mathbb{T})$. Since it is of restricted weak type $(1, 1)$, by the extension of the Marcinkiewicz interpolation theorem to the case of restricted weak-type endpoints (see, for example, [6]), we see that T^* has L^p bounds on the order of magnitude of $\frac{1}{p-1}$ for $1 < p < 2$. Hence, by the Yano extrapolation theorem [7], T^* maps $L \log L(\mathbb{T})$ boundedly into $L^1(\mathbb{T})$. So T^* provides an example of a sublinear translation-invariant operator acting on $L^1(\mathbb{T})$ that is of restricted weak type $(1, 1)$, maps $L \log L(\mathbb{T})$ boundedly into $L^1(\mathbb{T})$, but is not of weak type $(1, 1)$. This appears to be the first example of such an operator provided in the literature. We do note that in [4], letting Q denote the unit square $[0, 1] \times [0, 1]$ in \mathbb{R}^2 , Robert Fefferman provided an example of a sublinear translation-invariant operator M_Ω^* that was bounded on $L^p(Q)$ for $1 < p \leq \infty$, that mapped $L \log L(Q)$ boundedly into $L^1(Q)$, but that was not of weak type $(1, 1)$. This operator M_Ω^* was, however, not of restricted weak type $(1, 1)$.

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