PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 1, Pages 185–190
S 0002-9939(04)07530-6
Article electronically published on June 2, 2004

ON RESTRICTED WEAK TYPE $(1,1)$: THE CONTINUOUS CASE

PAUL A. HAGELSTEIN AND ROGER L. JONES

(Communicated by Andreas Seeger)

Abstract. Let $\mathbb{T}$ denote the unit circle. An example of a sublinear translation-invariant operator T acting on $L^1(\mathbb{T})$ is given such that T is of restricted weak type $(1,1)$ but not of weak type $(1,1)$.

Let $\mathbb{T}$ denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Let T be a sublinear translation-invariant operator acting on $L^1(\mathbb{T})$. If for some finite constant C we have that

$$|\{x \in \mathbb{T} : |Tf(x)| > \alpha\}| \leq C \frac{\|f\|_{L^1(\mathbb{T})}}{\alpha}$$

for all $\alpha > 0$ and $f \in L^1(\mathbb{T})$, we say that T is of *weak type* $(1,1)$. If for some finite constant C we have that

$$|\{x \in \mathbb{T} : |T\chi_E(x)| > \alpha\}| \leq C \frac{\|\chi_E\|_{L^1(\mathbb{T})}}{\alpha}$$

for all $\alpha > 0$ and measurable subsets E of $\mathbb{T}$, we say T is of *restricted weak type* $(1,1)$.

In 1974, K. H. Moon [5] proved the following theorem regarding operators of restricted weak type $(1,1)$:

Theorem 1 (Moon). *Let K_n $(n = 1, 2, \ldots)$ be linear convolution operators acting on $L^1(\mathbb{T})$, each of the form $K_n f = f * g_n$ for some $g_n \in L^1(\mathbb{T})$, and let $Mf(x) = \sup_n |K_n f(x)|$. Then M is of restricted weak type $(1,1)$ if and only if M is of weak type $(1,1)$.*

A natural question is to the degree that Moon's theorem admits further generalization. In particular, we may ask the following:

If T is a translation-invariant sublinear operator acting on $L^1(\mathbb{T})$, which is of restricted weak type $(1,1)$, must T be of weak type $(1,1)$?

In this paper we shall show that the answer to the above question is negative. In particular, we will see how a related negative result in the discrete setting due to Akcoglu, Baxter, Bellow, and Jones [1] may be utilized to construct a sublinear

Received by the editors April 14, 2003 and, in revised form, September 12, 2003.
2000 *Mathematics Subject Classification.* Primary 42B35; Secondary 37A25.
The first author's research was partially supported by the Baylor University Summer Sabbatical Program.
The second author was partially supported by a research leave granted by DePaul University's Research Council.

translation-invariant operator T^* acting on $L^1(\mathbb{T})$ that is of restricted weak type $(1,1)$ but not of weak type $(1,1)$.

We now discuss the result of Akcoglu, Baxter, Bellow and Jones. Let $\ell^1(\mathbb{Z})$ denote the space of integers equipped with the counting measure. For any subset A of $\mathbb{Z}$ we denote the number of elements in A by $|A|$. (The symbol $|A|$ will also be used to denote the standard Haar measure of a set A in $\mathbb{T}$. By context it will be clear if A is being considered as a subset of $\mathbb{Z}$ or $\mathbb{T}$.)

Let M be an operator acting on $\ell^1(\mathbb{Z})$. Analogously to the $L^1(\mathbb{T})$ case, if for some finite constant C we have

$$|\{x \in \mathbb{Z} : |M\phi(x)| > \alpha\}| \leq C \frac{\|\phi\|_{\ell^1(\mathbb{Z})}}{\alpha}$$

for all $\alpha > 0$ and all $\phi \in \ell^1(\mathbb{Z})$, then we say that M is of weak type $(1,1)$. If for some constant C we have

$$|\{x \in \mathbb{Z} : |M\chi_E(x)| > \alpha\}| \leq C \frac{\|\chi_E\|_{\ell^1(\mathbb{Z})}}{\alpha}$$

for all $\alpha > 0$ and all finite subsets E of $\mathbb{Z}$, then we say that M is of restricted weak type $(1,1)$.

A *probability measure on* $\mathbb{Z}$ *with finite support* is defined to be a function ω on $\mathbb{Z}$ such that $\omega(k) \geq 0$ for each k, and there exists an integer M such that $\omega(k) = 0$ if $|k| > M$, and $\sum_{k=-M}^{M} \omega(k) = 1$.

If f and g are functions in $\ell^1(\mathbb{Z})$, the convolution of f and g will be notated and defined by

$$f \star g(j) = \sum_{k=-\infty}^{\infty} f(j-k)g(k).$$

We note that the convolution of functions f and g in $L^1(\mathbb{T})$ is defined and notated here in the standard way by

$$f * g\left(e^{i\theta}\right) = \frac{1}{2\pi} \int_{|\phi|<\pi} f\left(e^{i\phi}\right) g\left(e^{i(\theta-\phi)}\right) d\phi.$$

The result of Akcoglu, Baxter, Bellow and Jones of interest to us here is the following.

Theorem 2 ([1]). *There exists a sequence* $\{\omega_k\}$ *of probability measures on* $\mathbb{Z}$ *such that the associated operators* $\{\tilde{T}_k\}$ *and* $\tilde{T}^*$ *defined on* $\ell^1(\mathbb{Z})$ *by* $\tilde{T}_k\phi(j) = \phi \star \omega_k(j)$, *and* $\tilde{T}^*\phi(j) = \sup_k \left|\tilde{T}_k\phi(j)\right|$, *have the following properties:*

- *For each* k *the measure* ω_k *has finite support.*
- *The maximal operator* $\tilde{T}^*$ *is of restricted weak type* $(1,1)$.
- *The maximal operator* $\tilde{T}^*$ *is not of weak type* $(1,1)$.

We will now show how the above theorem may be used to prove the following.

Theorem 3. *There exists a sublinear translation-invariant operator* T *acting on* $L^1(\mathbb{T})$ *that is of restricted weak type* $(1,1)$ *but not of weak type* $(1,1)$.

The proof of Theorem 3 will rely on the transference principle formulated by A. P. Calderón in his fundamental paper [3]. The idea underlining this principle is that the "bad behavior" of an operator on $\mathbb{Z}$ may be transferred to similar bad behavior of an associated operator on $\mathbb{T}$. The details specific to the case considered here are contained in the following lemma, which by Theorem 2 will be sufficient to establish our result.

Lemma 1. *Let $\{\omega_k\}$ be a sequence of probability measures on $\mathbb{Z}$, each with finite support. Define the associated operators $\{\tilde{T}_k\}$ and $\tilde{T}^*$ on $\ell^1(\mathbb{Z})$ by $\tilde{T}_k\phi(j) = \phi \star \omega_k(j)$ and $\tilde{T}^*\phi(j) = \sup_k \left|\tilde{T}_k\phi(j)\right|$, and the operators $\{T_n\}$ and T^* on $L^1(\mathbb{T})$ by*

$$T_n f\left(e^{i\theta}\right) = \sum_{k=-\infty}^{\infty} \omega_n(k) f\left(e^{i(\theta+k)}\right)$$

and

$$T^* f\left(e^{i\theta}\right) = \sup_n \left|T_n f\left(e^{i\theta}\right)\right|.$$

If $\tilde{T}^*$ is of restricted weak type $(1,1)$ but not of weak type $(1,1)$, then we also have that T^* is of restricted weak type $(1,1)$ but not of weak type $(1,1)$.

Proof. We first define the operators T_N^* and $\tilde{T}_N^*$ on $L^1(\mathbb{T})$ and $\ell^1(\mathbb{Z})$ respectively by

$$T_N^* f\left(e^{i\theta}\right) = \sup_{1\le n\le N} \left|T_n f\left(e^{i\theta}\right)\right|$$

and

$$\tilde{T}_N^*\phi(j) = \sup_{1\le n\le N} \left|\tilde{T}_n\phi(j)\right|.$$

We now show that if $\tilde{T}^*$ is of restricted weak type $(1,1)$, then T^* is as well. Now, if $\tilde{T}^*$ is of restricted weak type $(1,1)$, then there exists a constant C such that

$$\left|\left\{j \in \mathbb{Z} : \tilde{T}_N^*\chi_S(j) > \alpha\right\}\right| \le \frac{C}{\alpha}\left|\{j : j \in S\}\right|$$

holds for every positive integer N, subset S of $\mathbb{Z}$, and $\alpha > 0$.

We will show that this implies that T^* is of restricted weak type $(1,1)$. That is, we will show that

$$\left|\left\{e^{i\theta} \in \mathbb{T} : T_N^*\chi_E\left(e^{i\theta}\right) > \alpha\right\}\right| \le \frac{C}{\alpha}|E|$$

holds for every positive integer N, measurable subset E of $\mathbb{T}$, and $\alpha > 0$. Letting N go to infinity will then imply that the same inequality holds for T^*.

Fix $E \subset \mathbb{T}$, and let $F = \left\{e^{i\theta} \in \mathbb{T} : T_N^*\chi_E\left(e^{i\theta}\right) > \alpha\right\}$. Note that $|F| = |\{e^{i\theta} \in \mathbb{T} : e^{i(\theta+k)} \in F\}|$ for any choice of $k \in \mathbb{Z}$. For each positive integer j, we let M_j be a positive integer such that the probability measure ω_j is supported in $[-M_j, M_j]$. We assume, without loss of generality, that $M_j < M_{j+1}$ for each postive integer j. Let L be an arbitrary positive integer. Also, let $\tilde{E}_\theta$ be a subset of $\mathbb{Z}$ with the property that $\chi_{\tilde{E}_\theta}(k) = \chi_E(e^{i(\theta+k)})\chi_{[-M_N, L+M_N]}(k)$. We have

$$\begin{aligned}
|F| &= \frac{1}{L}\sum_{k=0}^{L-1}\left|\{e^{i\theta} : e^{i(\theta+k)} \in F\}\right| \\
&= \frac{1}{L}\sum_{k=0}^{L-1}\int_{\mathbb{T}} \chi_F(e^{i(\theta+k)})\, d\theta \\
&= \frac{1}{L}\int_{\mathbb{T}}\sum_{k=0}^{L-1} \chi_F(e^{i(\theta+k)})\, d\theta \\
&= \frac{1}{L}\int_{\mathbb{T}}\left|\left\{k \in [0, L-1] : e^{i(\theta+k)} \in F\right\}\right| d\theta \\
&= \frac{1}{L}\int_{\mathbb{T}}\left|\left\{k \in [0, L-1] : \sup_{1\le n\le N}\left|\sum_j \omega_n(j)\chi_E(e^{i(\theta+k+j)})\right| > \alpha\right\}\right| d\theta \\
&\le \frac{1}{L}\int_{\mathbb{T}}\left|\left\{k \in \mathbb{Z} : \sup_{1\le n\le N}\left|\sum_j \omega_n(j)\chi_{\tilde{E}_\theta}(k+j)\right| > \alpha\right\}\right| d\theta \\
&\le \frac{1}{L}\int_{\mathbb{T}}\frac{C}{\alpha}|\tilde{E}_\theta|\, d\theta \\
&\le \frac{1}{L}\int_{\mathbb{T}}\frac{C}{\alpha}\sum_{k=-M_N}^{L+M_N} \chi_E(e^{i(\theta+k)})\, d\theta \\
&\le \frac{L+2M_N}{L} \times \frac{C}{\alpha}|E|.
\end{aligned}$$

Since L can be taken as large as we like, we see that we have

$$\left|\left\{e^{i\theta} \in \mathbb{T} : T_N^*\chi_E\,(x) > \alpha\right\}\right| \le \frac{C}{\alpha}|E|,$$

as required.

It remains to show that if $\tilde{T}^*$ is not of weak type $(1, 1)$, then neither is T^*.

If $\tilde{T}^*$ is not of weak type (1,1), we know that given $C > 0$ we can find $\phi \in \ell^1(\mathbb{Z})$ such that ϕ has finite support and an $\alpha > 0$ such that

$$|\{k \in \mathbb{Z} : \tilde{T}^*\phi(k) > \alpha\}| \ge \frac{C}{\alpha}\|\phi\|_{\ell^1}.$$

Assume (without loss of generality) that ϕ has support in $[0, L]$ for some choice of L.

We know that the points $e^i, e^{2i}, \ldots, e^{iL}$ will all be disjoint, and consequently there is an arc $B = (e^{ia}, e^{ib})$ in $\mathbb{T}$ such that the arcs

$$(e^{ia}, e^{ib}), (e^{i(a+1)}, e^{i(b+1)}), \ldots, (e^{i(a+L)}, e^{i(b+L)})$$

are all disjoint.

Define $f \in L^1(\mathbb{T})$ by $f(e^{i(\theta+k)}) = \phi(k)$ if $0 \leq k \leq L$ and $e^{i\theta} \in (e^{ia}, e^{ib})$. Let f be zero otherwise. Note that $\|f\|_1 = |B| \|\phi\|_{\ell^1}$. We then have

$$\begin{aligned}|\{e^{i\theta} \in \mathbb{T} : T^* f(e^{i\theta}) > \alpha\}| &\geq |B| \times |\{k \in \mathbb{Z} : \tilde{T}^* \phi(k) > \alpha\}| \\ &\geq |B| \times \frac{C}{\alpha} \|\phi\|_{\ell_1} \\ &= \frac{C}{\alpha} \|f\|_1.\end{aligned}$$

Since C can be taken as large as desired, we see that T^* cannot be of weak type $(1,1)$. □

We note that Lemma 1 (and hence Theorem 3) can be generalized to the following setting.

Lemma 2. *Let (X, Σ, m) denote a complete non-atomic probability space and let $\tau : X \to X$ be a measurable invertible ergodic measure-preserving transformation. Let $\{w_k\}$, $\{\tilde{T}_k\}$ and $\tilde{T}^*$ be as in Lemma 1. Define the operators $\{T_n\}$ and T^* on $L^1(X, \Sigma, m)$ by $T_n f(x) = \sum_{k=-\infty}^{\infty} w_n(k) f(\tau^k x)$ and $T^* f(x) = \sup_n |T_n f(x)|$. If the maximal operator $\tilde{T}^*$ is of restricted weak type $(1,1)$ but not of weak type $(1,1)$, then the same is true for the associated maximal operator T^* on $L^1(X)$.*

The proof of this result follows the same argument as the proof of Lemma 1. Indeed, Lemma 1 is a special case of this more general result, with $x = e^{i\theta}$ and τ given by $\tau(x) = \tau(e^{i\theta}) = e^{i(\theta+1)}$. For the proof of the first part of this result, all we need to do is replace $e^{i\theta}$ by x, $e^{i(\theta+k)}$ by $\tau^k x$, and copy the argument. For the second half, we replace the disjoint arcs $(e^{ia}, e^{ib}), (e^{i(a+1)}, e^{i(b+1)}), \ldots, (e^{i(a+L)}, e^{i(b+L)})$ by disjoint sets of positive measure $B, \tau B, \ldots, \tau^L B$. The existence of such a set B is guaranteed by Rohlin's Lemma (see, e.g., [2]). The argument now follows as before.

We make a couple of closing remarks. Let T^* be the operator defined in the statement of Lemma 1 associated to the sequence $\{\omega_n\}$ of probability measures referred to in the statement of Theorem 2. Note that the operator T^* is clearly bounded on $L^\infty(\mathbb{T})$. Since it is of restricted weak type $(1,1)$, by the extension of the Marcinkiewicz interpolation theorem to the case of restricted weak-type endpoints (see, for example, [6]), we see that T^* has L^p bounds on the order of magnitude of $\frac{1}{p-1}$ for $1 < p < 2$. Hence, by the Yano extrapolation theorem [7], T^* maps $L \log L(\mathbb{T})$ boundedly into $L^1(\mathbb{T})$. So T^* provides an example of a sublinear translation-invariant operator acting on $L^1(\mathbb{T})$ that is of restricted weak type $(1,1)$, maps $L \log L(\mathbb{T})$ boundedly into $L^1(\mathbb{T})$, but is not of weak type $(1,1)$. This appears to be the first example of such an operator provided in the literature. We do note that in [4], letting Q denote the unit square $[0,1] \times [0,1]$ in $\mathbb{R}^2$, Robert Fefferman provided an example of a sublinear translation-invariant operator M_Ω^* that was bounded on $L^p(Q)$ for $1 < p \leq \infty$, that mapped $L \log L(Q)$ boundedly into $L^1(Q)$, but that was not of weak type $(1,1)$. This operator M_Ω^* was, however, not of restricted weak type $(1,1)$.

References

[1] M. Akcoglu, J. Baxter, A. Bellow, and R. L. Jones, *On restricted weak type* $(1,1)$*; the discrete case*, Israel J. Math. **124** (2001), 285-297. MR2002g:47060

[2] J. Brown, *Ergodic Theory and Topological Dynamics*, Academic Press, New York, 1976. MR58:11323

[3] A. P. Calderón, *Ergodic theory and translation invariant operators*, Proc. Nat. Acad. Sci. U.S.A. **59** (1968), 349-353. MR37:2939
[4] R. A. Fefferman, *A theory of entropy in Fourier analysis*, Adv. in Math. **30** (1978), 171-201. MR81g:42022
[5] K. H. Moon, *On restricted weak type* $(1,1)$, Proc. Amer. Math. Soc. **42** (1974), 148-152. MR49:5946
[6] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, NJ, 1971. MR46:4102
[7] S. Yano, *An extrapolation theorem*, J. Math. Soc. Japan **3** (1951), 296-305. MR14:41c

Department of Mathematics, Baylor University, Waco, Texas 76798
E-mail address: paul_hagelstein@baylor.edu

Department of Mathematics, DePaul University, Chicago, Illinois 60614
E-mail address: rjones@condor.depaul.edu